Maxima is a computer algebra system, implemented in Lisp.

Maxima is derived from the Macsyma system, developed at MIT in the years 1968 through 1982 as part of Project MAC. MIT turned over a copy of the Macsyma source code to the Department of Energy in 1982; that version is now known as DOE Macsyma. A copy of DOE Macsyma was maintained by Professor William F. Schelter of the University of Texas from 1982 until his death in 2001. In 1998, Schelter obtained permission from the Department of Energy to release the DOE Macsyma source code under the GNU Public License, and in 2000 he initiated the Maxima project at SourceForge to maintain and develop DOE Macsyma, now called Maxima.
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1 Introduction to Maxima

Start Maxima with the command "maxima". Maxima will display version information and a prompt. End each Maxima command with a semicolon. End the session with the command "quit;". Here's a sample session:

```
[wfs@chromium]$ maxima
Maxima 5.9.1 http://maxima.sourceforge.net
Using Lisp CMU Common Lisp 19a
Distributed under the GNU Public License. See the file COPYING.
Dedicated to the memory of William Schelter.
This is a development version of Maxima. The function bug_report() provides bug reporting information.
(%i1) factor(10!);

8 4 2
(%o1) 2 3 5 7

(%i2) expand ((x + y)^6);

6 5 2 4 3 3 4 2 5 6
(%o2) y + 6 x y + 15 x y + 20 x y + 15 x y + 6 x y + x

(%i3) factor (x^6 - 1);

2 2
(%o3) (x - 1) (x + 1) (x - x + 1) (x + x + 1)

(%i4) quit();
[wfs@chromium]$
```

Maxima can search the info pages. Use the `describe` command to show information about the command or all the commands and variables containing a string. The question mark `?` (exact search) and double question mark `??` (inexact search) are abbreviations for `describe`:

```
(%i1) ?? integ
0: Functions and Variables for Elliptic Integrals
1: Functions and Variables for Integration
2: Introduction to Elliptic Functions and Integrals
3: Introduction to Integration
4: askinteger (Functions and Variables for Simplification)
5: integerp (Functions and Variables for Miscellaneous Options)
6: integer_partitions (Functions and Variables for Sets)
7: integrate (Functions and Variables for Integration)
8: integrate_use_rootsof (Functions and Variables for Integration)
9: integration_constant_counter (Functions and Variables for Integration)
10: nonnegintegerp (Functions and Variables for linearalgebra)
Enter space-separated numbers, 'all' or 'none': 5 4

-- Function: integerp (<expr>)
  Returns 'true' if <expr> is a literal numeric integer, otherwise 'false'.
```
'integerp' returns false if its argument is a symbol, even if the argument is declared integer.

Examples:

```
(%i1) integerp (0); %o1 true
(%i2) integerp (1); %o2 true
(%i3) integerp (-17); %o3 true
(%i4) integerp (0.0); %o4 false
(%i5) integerp (1.0); %o5 false
(%i6) integerp (%pi); %o6 false
(%i7) integerp (n); %o7 false
(%i8) declare (n, integer); %o8 done
(%i9) integerp (n); %o9 false
```

-- Function: askinteger (<expr>, integer)
-- Function: askinteger (<expr>)
-- Function: askinteger (<expr>, even)
-- Function: askinteger (<expr>, odd)

'askinteger (<expr>, integer)' attempts to determine from the 'assume' database whether <expr> is an integer. 'askinteger' prompts the user if it cannot tell otherwise, and attempt to install the information in the database if possible. 'askinteger (<expr>)' is equivalent to 'askinteger (<expr>, integer)'.

'askinteger (<expr>, even)' and 'askinteger (<expr>, odd)' likewise attempt to determine if <expr> is an even integer or odd integer, respectively.

```
(%i1) true
```

To use a result in later calculations, you can assign it to a variable or refer to it by its automatically supplied label. In addition, % refers to the most recent calculated result:

```
(%i11) u: expand ((x + y)^6);
    6   5    2  4   3  3  4  2  5   6
   y + 6 x y + 15 x y + 20 x y + 15 x y + 6 x y + x
(%i12) diff (u, x);
    5  4  2  3  3  2  4  5
```
Chapter 1: Introduction to Maxima

(%i2) 6 y + 30 x y + 60 x y + 60 x y + 30 x y + 6 x
(%o2) factor (%o2);

(%o3) 6 (y + x)

Maxima knows about complex numbers and numerical constants:

(%i1) cos(%pi);
(%o1) - 1
(%i2) exp(%i*%pi);
(%o2) - 1

Maxima can do differential and integral calculus:

(%i1) u: expand ((x + y)^6);
(%o1) x^6 + 6 x^5 y + 15 x^4 y^2 + 20 x^3 y^3 + 15 x^2 y^4 + 6 x y^5 + y^6
(%i2) diff (%u, x);
(%o2) 6 y + 30 x y + 60 x y + 60 x y + 30 x y + 6 x
(%i3) integrate (1/(1 + x^3), x);
(%o3) \frac{2 x}{\sqrt{3}} - \frac{1}{\sqrt{3}} \arctan \left( \frac{\sqrt{3}}{x} \right) + \frac{\log(x - x + 1)}{3} + \frac{\log(x + 1)}{3}

Maxima can solve linear systems and cubic equations:

(%i1) linsolve ([3*x + 4*y = 7, 2*x + a*y = 13], [x, y]);
(%o1) \{x = \frac{7 a - 52}{3 a - 8}, y = \frac{25}{3 a - 8}\}
(%i2) solve (x^3 - 3*x^2 + 5*x = 15, x);
(%o2) \{x = -\sqrt{5} i, x = \sqrt{5} i, x = 3\}

Maxima can solve nonlinear sets of equations. Note that if you don’t want a result printed, you can finish your command with \$ instead of ;

(%i1) eq_1: x^2 + 3*x*y + y^2 = 0$
(%i2) eq_2: 3*x + y = 1$
(%i3) solve ([eq_1, eq_2]);
(%o3) \{x = \frac{-\sqrt{5} i}{2}, y = \frac{3 \sqrt{5} i}{2}, x = \frac{3 \sqrt{5} i}{2}, y = -\frac{3 \sqrt{5} i}{2}\}

Maxima can generate plots of one or more functions:
(%i1) plot2d (sin(x)/x, [x, -20, 20])$

(%i2) plot2d ([atan(x), erf(x), tanh(x)], [x, -5, 5], [y, -1.5, 2])$
(%i3) plot3d (sin(sqrt(x^2 + y^2))/sqrt(x^2 + y^2),
    [x, -12, 12], [y, -12, 12])$
2 Bug Detection and Reporting

2.1 Functions and Variables for Bug Detection and Reporting

\texttt{run\_testsuite ([options])}  \hspace{1cm} \textbf{[Function]}

Run the Maxima test suite. Tests producing the desired answer are considered “passes,” as are tests that do not produce the desired answer, but are marked as known bugs.

\texttt{run\_testsuite} takes the following optional keyword arguments

\begin{itemize}
  \item \texttt{display\_all} \hspace{1cm} Display all tests. Normally, the tests are not displayed, unless the test fails. (Defaults to \texttt{false}).
  \item \texttt{display\_known\_bugs} \hspace{1cm} Displays tests that are marked as known bugs. (Default is \texttt{false}).
  \item \texttt{tests} \hspace{1cm} This is a single test or a list of tests that should be run. Each test can be specified by either a string or a symbol. By default, all tests are run. The complete set of tests is specified by \texttt{testsuite\_files}.
  \item \texttt{time} \hspace{1cm} Display time information. If \texttt{true}, the time taken for each test file is displayed. If \texttt{all}, the time for each individual test is shown if \texttt{display\_all} is \texttt{true}. The default is \texttt{false}, so no timing information is shown.
  \item \texttt{share\_tests} \hspace{1cm} Load additional tests for the \texttt{share} directory. If \texttt{true}, these additional tests are run as a part of the testsuite. If \texttt{false}, no tests from the \texttt{share} directory are run. If \texttt{only}, only the tests from the \texttt{share} directory are run. Of course, the actual set of test that are run can be controlled by the \texttt{tests} option. The default is \texttt{false}.
\end{itemize}

For example \texttt{run\_testsuite(display\_known\_bugs = true, tests=[rtest5])} runs just test \texttt{rtest5} and displays the test that are marked as known bugs.

\texttt{run\_testsuite(display\_all = true, tests=["rtest1", rtest1a])} will run tests \texttt{rtest1} and \texttt{rtest2}, and displays each test.

\texttt{run\_testsuite} changes the Maxima environment. Typically a test script executes \texttt{kill} to establish a known environment (namely one without user-defined functions and variables) and then defines functions and variables appropriate to the test.

\texttt{run\_testsuite} returns \texttt{done}.

\texttt{testsuite\_files}  \hspace{1cm} \textbf{[Option variable]}

\texttt{testsuite\_files} is the set of tests to be run by \texttt{run\_testsuite}. It is a list of names of the files containing the tests to run. If some of the tests in a file are known to fail, then instead of listing the name of the file, a list containing the file name and the test numbers that fail is used.

For example, this is a part of the default set of tests:

\begin{verbatim}
["rtest13s", ["rtest14", 57, 63]]
\end{verbatim}
This specifies the testsuite consists of the files "rtest13s" and "rtest14", but "rtest14" contains two tests that are known to fail: 57 and 63.

share_testsuite_files is the set of tests from the share directory that is run as a part of the test suite by run_testsuite.

bug_report ()
Prints out Maxima and Lisp version numbers, and gives a link to the Maxima project bug report web page. The version information is the same as reported by build_info.

When a bug is reported, it is helpful to copy the Maxima and Lisp version information into the bug report.

bug_report returns an empty string "".

build_info ()
Returns a summary of the parameters of the Maxima build, as a Maxima structure (defined by defstruct). The fields of the structure are: version, timestamp, host, lisp_name, and lisp_version. When the pretty-printer is enabled (via display2d), the structure is displayed as a short table.

See also bug_report.

Examples:

(%i1) build_info ();
(%o1)
Maxima version: "5.36.1"
Maxima build date: "2015-06-02 11:26:48"
Host type: "x86_64-unknown-linux-gnu"
Lisp implementation type: "GNU Common Lisp (GCL)"
Lisp implementation version: "GCL 2.6.12"
(%i2) x : build_info ()$
(%i3) x@version;
(%o3) 5.36.1
(%i4) x@timestamp;
(%o4) 2015-06-02 11:26:48
(%i5) x@host;
(%o5) x86_64-unknown-linux-gnu
(%i6) x@lisp_name;
(%o6) GNU Common Lisp (GCL)
(%i7) x@lisp_version;
(%o7) GCL 2.6.12
(%i8) x;
(%o8)
Maxima version: "5.36.1"
Maxima build date: "2015-06-02 11:26:48"
Host type: "x86_64-unknown-linux-gnu"
Lisp implementation type: "GNU Common Lisp (GCL)"
Lisp implementation version: "GCL 2.6.12"
The Maxima version string can (here 5.36.1) can look very different:

\begin{verbatim}
(%i1) build_info();
(%o1)
Maxima version: "branch_5_37_base_331_g8322940_dirty"
Maxima build date: "2016-01-01 15:37:35"
Host type: "x86_64-unknown-linux-gnu"
Lisp implementation type: "CLISP"
Lisp implementation version: "2.49 (2010-07-07) (built 3605577779) (memory 3660647857)"
\end{verbatim}

In that case, Maxima was not build from a released sourcecode, but directly from the GIT-checkout of the sourcecode. In the example, the checkout is 331 commits after the latest GIT tag (usually a Maxima (major) release (5.37 in our example)) and the abbreviated commit hash of the last commit was "8322940".
3 Help

3.1 Documentation
The Maxima on-line user’s manual can be viewed in different forms. From the Maxima interactive prompt, the user’s manual is viewed as plain text by the \texttt{?} command (i.e., the \texttt{describe} function). The user’s manual is viewed as info hypertext by the \texttt{info} viewer program and as a web page by any ordinary web browser.

\texttt{example} displays examples for many Maxima functions. For example,

\begin{verbatim}
(\%i1) example (integrate);
\end{verbatim}

yields

\begin{verbatim}
(\%i2) test(f):=block([u],u:integrate(f,x),ratsimp(f-diff(u,x)))
(\%o2) test(f) := block([u], u : integrate(f, x),
\hspace{1cm} \text{ratsimp}(f - \text{diff}(u, x)))
(\%i3) test(sin(x))
(\%o3) 0
(\%i4) test(1/(x+1))
(\%o4) 0
(\%i5) test(1/(x^2+1))
(\%o5) 0
\end{verbatim}

and additional output.

3.2 Functions and Variables for Help

\texttt{apropos (string)}

[Function]

Searches for Maxima names which have \textit{string} appearing anywhere within them. Thus, \texttt{apropos (exp)} returns a list of all the flags and functions which have \textit{exp} as part of their names, such as \texttt{expand}, \texttt{exp}, and \texttt{exponentialize}. Thus if you can only remember part of the name of something you can use this command to find the rest of the name. Similarly, you could say \texttt{apropos (tr_)} to find a list of many of the switches relating to the translator, most of which begin with \texttt{tr_}.

\texttt{apropos(""')} returns a list with all Maxima names.

\texttt{apropos} returns the empty list \texttt{[]}, if no name is found.

Example:

Show all Maxima symbols which have "gamma" in the name:

\begin{verbatim}
(\%i1) apropos("gamma");
(\%o1) [%gamma, gamma, gammalim, gamma_expand, gamma_incomplete_lower,
\hspace{1cm} gamma_incomplete, gamma_incomplete_generalized,
\hspace{1cm} gamma_incomplete_regularized, Gamma, log_gamma, makegamma,
\hspace{1cm} prefer_gamma_incomplete, gamma-incomplete,
\hspace{1cm} gamma_incomplete_generalized_regularized]
\end{verbatim}

\texttt{demo (filename)}

[Function]

Evaluates Maxima expressions in \textit{filename} and displays the results. \texttt{demo} pauses after evaluating each expression and continues after the user enters a carriage return. (If
running in Xmaxima, demo may need to see a semicolon ; followed by a carriage return.)

demo searches the list of directories file_search_demo to find filename. If the file has the suffix dem, the suffix may be omitted. See also file_search.

demo evaluates its argument. demo returns the name of the demonstration file.

Example:

```
(%i1) demo ("disol");

batching /home/wfs/maxima/share/simplification/disol.dem

At the _ prompt, type ';' followed by enter to get next demo
(%i2) load(disol)


(%i3) exp1 : a (e (g + f) + b (d + c))
(%o3) a (e (g + f) + b (d + c))

(%i4) disolate(exp1, a, b, e)
(%o4) [d + c, g + f]

(%o5) a (%t4 e + %t5 b)
```

```
[Function]
```

describe

describe(string)
describe(string, exact)
describe(string, inexact)

describe(string) is equivalent to describe(string, exact).

describe(string, exact) finds an item with title equal (case-insensitive) to string, if there is any such item.

describe(string, inexact) finds all documented items which contain string in their titles. If there is more than one such item, Maxima asks the user to select an item or items to display.

At the interactive prompt, ? foo (with a space between ? and foo) is equivalent to describe("foo", exact), and ?? foo is equivalent to describe("foo", inexact).

describe("", inexact) yields a list of all topics documented in the on-line manual.

describe quotes its argument. describe returns true if some documentation is found, otherwise false.

See also Section 3.1 [Documentation], page 11.

Example:

```
(%i1) ?? integ
```
Chapter 3: Help

0: Functions and Variables for Elliptic Integrals
1: Functions and Variables for Integration
2: Introduction to Elliptic Functions and Integrals
3: Introduction to Integration
4: askinteger (Functions and Variables for Simplification)
5: integerp (Functions and Variables for Miscellaneous Options)
6: integer_partitions (Functions and Variables for Sets)
7: integrate (Functions and Variables for Integration)
8: integrate_use_rootsof (Functions and Variables for Integration)
9: integration_constant_counter (Functions and Variables for Integration)
10: nonnegintegerp (Functions and Variables for linearalgebra)

Enter space-separated numbers, 'all' or 'none': 7 8

-- Function: integrate (<expr>, <x>)
-- Function: integrate (<expr>, <x>, <a>, <b>)
Attempts to symbolically compute the integral of <expr> with respect to <x>. 'integrate (<expr>, <x>)' is an indefinite integral, while 'integrate (<expr>, <x>, <a>, <b>)' is a definite integral, [...]

-- Option variable: integrate_use_rootsof
Default value: 'false'

When 'integrate_use_rootsof' is 'true' and the denominator of a rational function cannot be factored, 'integrate' returns the integral in a form which is a sum over the roots (not yet known) of the denominator.

[...] In this example, items 7 and 8 were selected (output is shortened as indicated by [...]). All or none of the items could have been selected by entering all or none, which can be abbreviated a or n, respectively.

example [Function]
exmple (topic)
exmple ()
exmple (topic) displays some examples of topic, which is a symbol or a string. To get examples for operators like if, do, or lambda the argument must be a string, e.g. exmple ("do"). exmple is not case sensitive. Most topics are function names.
exmple () returns the list of all recognized topics.
The name of the file containing the examples is given by the global option variable manual_demo, which defaults to "manual.demo".
exmple quotes its argument. exmple returns done unless no examples are found or there is no argument, in which case exmple returns the list of all recognized topics.
Examples:
(%i1) example(append);
(%i2) append([y+x,0,-3.2],[2.5e+20,x])
(%o2) [y + x, 0, -3.2, 2.5e+20, x]
(%o2) done
(%i3) example("lambda");
(%i4) lambda([x,y,z],x^2+y^2+z^2)
(%o4) lambda([x, y, z], x^2 + y^2 + z^2)
(%i5) %(1,2,a)
(%o5) a + 5
(%i6) 1+2+a
(%o6) a + 3
(%o6) done

manual_demo

[Option variable]
Default value: "manual.demo"

manual_demo specifies the name of the file containing the examples for the function example. See example.
4 Command Line

4.1 Introduction to Command Line

4.2 Functions and Variables for Command Line

`__` [System variable]

`__` is the input expression currently being evaluated. That is, while an input expression `expr` is being evaluated, `__` is `expr`.

`__` is assigned the input expression before the input is simplified or evaluated. However, the value of `__` is simplified (but not evaluated) when it is displayed.

`__` is recognized by `batch` and `load`. In a file processed by `batch`, `__` has the same meaning as at the interactive prompt. In a file processed by `load`, `__` is bound to the input expression most recently entered at the interactive prompt or in a batch file; `__` is not bound to the input expressions in the file being processed. In particular, when `load (filename)` is called from the interactive prompt, `__` is bound to `load (filename)` while the file is being processed.

See also `_` and `%`.

Examples:

```plaintext
(%i1) print ("I was called as", __);
I was called as print(I was called as, __)
(%o1) print(I was called as, __)
(%i2) foo (__);
(%o2) foo(foo(__))
(%i3) g(x) := (print("Current input expression =", __), 0);
(%o3) g(x) := (print("Current input expression =", __), 0)
(%i4) [aa : 1, bb : 2, cc : 3];
(%o4) [1, 2, 3]
(%i5) (aa + bb + cc)/(dd + ee + g(x));
   cc + bb + aa
Current input expression = ------------------
   g(x) + ee + dd
     6
(%o5) -------
      ee + dd
```

`_` [System variable]

`_` is the most recent input expression (e.g., `%i1`, `%i2`, `%i3`, ...).

`_` is assigned the input expression before the input is simplified or evaluated. However, the value of `_` is simplified (but not evaluated) when it is displayed.

`_` is recognized by `batch` and `load`. In a file processed by `batch`, `_` has the same meaning as at the interactive prompt. In a file processed by `load`, `_` is bound to the input expression most recently evaluated at the interactive prompt or in a batch file; `_` is not bound to the input expressions in the file being processed.
See also \_\_ and \%.

Examples:

\%(i1) 13 + 29;
\%(o1) 42

\%(i2) :lisp $\_\_\_\_\_
((MPLUS) 13 29)
\%(o2) \_\_\_\_\_
42

\%(i3) sin (\%pi/2);
\%(o3) 1

\%(i4) :lisp $\_\_\_\_\_
((%SIN) ((MQUOTIENT) $\%PI 2))
\%(o4) \_\_\_\_\_
\%(i5) a: 13$
\%(i6) b: 29$
\%(i7) a + b;
\%(o7) 42

\%(i8) :lisp $\_\_\_\_\_
((MPLUS) $A $B)
\%(o8) \_\_\_\_\_
b + a

\%(i9) a + b;
\%(o9) 42

\%(i10) ev (_);
\%(o10) 42

\% [System variable]
\% is the output expression (e.g., \%o1, \%o2, \%o3, . . .) most recently computed by Maxima, whether or not it was displayed.

\% is recognized by batch and load. In a file processed by batch, \% has the same meaning as at the interactive prompt. In a file processed by load, \% is bound to the output expression most recently computed at the interactive prompt or in a batch file; \% is not bound to output expressions in the file being processed.

See also \_, \%, and %th.

\%% [System variable]
In compound statements, namely block, lambda, or (s_1, . . ., s_n), \%% is the value of the previous statement.

At the first statement in a compound statement, or outside of a compound statement, \%% is undefined.

\%% is recognized by batch and load, and it has the same meaning as at the interactive prompt.

See also \%.

Examples:
The following two examples yield the same result.

(\%i1) block (integrate (x^5, x), ev (\%\%, x=2) - ev (\%\%, x=1));
  21
(\%o1) --

(\%i2) block ([prev], prev: integrate (x^5, x),
  ev (prev, x=2) - ev (prev, x=1));
  21
(\%o2) --

A compound statement may comprise other compound statements. Whether a state-
ment be simple or compound, \% is the value of the previous statement.

(\%i3) block (block (a^n, \%*42), \%/6);

(\%o3) 7 a

Within a compound statement, the value of \% may be inspected at a break prompt,
which is opened by executing the break function. For example, entering \%; in the
following example yields 42.

(\%i4) block (a: 42, break ())$

Entering a Maxima break point. Type 'exit;' to resume.

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As prefix to a function or variable name, ? signifies that the name is a Lisp name, not a Maxima name. For example, ?round signifies the Lisp function ROUND. See Section 37.1 [Lisp and Maxima], page 617, for more on this point.

The notation ? word (a question mark followed a word, separated by whitespace) is equivalent to describe("word"). The question mark must occur at the beginning of an input line; otherwise it is not recognized as a request for documentation. See also describe.

The notation ?? word (?? followed a word, separated by whitespace) is equivalent to describe("word", inexact). The question mark must occur at the beginning of an input line; otherwise it is not recognized as a request for documentation. See also describe.

The dollar sign $ terminates an input expression, and the most recent output % and an output label, e.g. %o1, are assigned the result, but the result is not displayed. See also ;.

Example:

```
(%i1) 1 + 2 + 3 $
(%i2) %;
(%o2) 6
(%i3) %o1;
(%o3) 6
```

The semicolon ; terminates an input expression, and the resulting output is displayed. See also $.

Example:

```
(%i1) 1 + 2 + 3;
(%o1) 6
```

inchar

[Option variable]

Default value: %i

inchar is the prefix of the labels of expressions entered by the user. Maxima automatically constructs a label for each input expression by concatenating inchar and linenum.

inchar may be assigned any string or symbol, not necessarily a single character. Because Maxima internally takes into account only the first char of the prefix, the prefixes inchar, outchar, and linechar should have a different first char. Otherwise some commands like kill(inlabels) do not work as expected.

See also labels.

Example:

```
(%i1) inchar: "input";
(%o1) input
```
\( \text{(input2)} \text{ expand}((a+b)^3); \)

\[
\begin{align*}
3 & \quad 2 & 2 & 3 \\
\text{(\%o2)} & \quad b & + & 3 & a & b & + & 3 & a & b & + & a
\end{align*}
\]

\textbf{infolists}  
Default value: \[
[]
\]

\texttt{infolists} is a list of the names of all of the information lists in Maxima. These are:

- \texttt{labels} All bound \%i, \%o, and \%t labels.
- \texttt{values} All bound atoms which are user variables, not Maxima options or switches, created by \texttt{:} or \texttt{::} or functional binding.
- \texttt{functions} All user-defined functions, created by \texttt{:=} or \texttt{define}.
- \texttt{arrays} All declared and undeclared arrays, created by \texttt{::}, \texttt{::}, or \texttt{:=}.
- \texttt{macros} All user-defined macro functions, created by \texttt{::=}.
- \texttt{myoptions} All options ever reset by the user (whether or not they are later reset to their default values).
- \texttt{rules} All user-defined pattern matching and simplification rules, created by \texttt{tellsimp}, \texttt{tellsimpafter}, \texttt{defmatch}, or \texttt{defrule}.
- \texttt{aliases} All atoms which have a user-defined alias, created by the \texttt{alias}, \texttt{ordergreat}, \texttt{orderless} functions or by declaring the atom as a \texttt{noun} with \texttt{declare}.
- \texttt{dependencies} All atoms which have functional dependencies, created by the \texttt{depends}, \texttt{dependencies}, or \texttt{gradef} functions.
- \texttt{gradefs} All functions which have user-defined derivatives, created by the \texttt{gradef} function.
- \texttt{props} All atoms which have any property other than those mentioned above, such as properties established by \texttt{atvalue} or \texttt{matchdeclare}, etc., as well as properties established in the \texttt{declare} function.

\texttt{let_rule_packages}  
All user-defined \texttt{let} rule packages plus the special package \texttt{default_let_rule_package}. (\texttt{default_let_rule_package} is the name of the rule package used when one is not explicitly set by the user.)
kill

kill (a_1, ..., a_n)
kill (labels)
kill (inlabels, outlabels, linelabels)
kill (n)
kill ([m, n])
kill (values, functions, arrays, ...)
kill (all)
kill (allbut (a_1, ..., a_n))

Removes all bindings (value, function, array, or rule) from the arguments a_1, ..., a_n. An argument a_k may be a symbol or a single array element. When a_k is a single array element, kill unbinds that element without affecting any other elements of the array.

Several special arguments are recognized. Different kinds of arguments may be combined, e.g., kill (inlabels, functions, allbut (foo, bar)).

kill (labels) unbinds all input, output, and intermediate expression labels created so far. kill (inlabels) unbinds only input labels which begin with the current value of inchar. Likewise, kill (outlabels) unbinds only output labels which begin with the current value of outchar, and kill (linelabels) unbinds only intermediate expression labels which begin with the current value of linechar.

kill (n), where n is an integer, unbinds the n most recent input and output labels.
kill ([m, n]) unbinds input and output labels m through n.

kill (infolist), where infolist is any item in infolists (such as values, functions, or arrays) unbinds all items in infolist. See also infolists.

kill (all) unbinds all items on all infolists. kill (all) does not reset global variables to their default values; see reset on this point.

kill (allbut (a_1, ..., a_n)) unbinds all items on all infolists except for a_1, ..., a_n. kill (allbut (infolist)) unbinds all items except for the ones on infolist, where infolist is values, functions, arrays, etc.

The memory taken up by a bound property is not released until all symbols are unbound from it. In particular, to release the memory taken up by the value of a symbol, one unbinds the output label which shows the bound value, as well as unbinding the symbol itself.

kill quotes its arguments. The quote-quote operator '' defeats quotation.

kill (symbol) unbinds all properties of symbol. In contrast, the functions remvalue, remfunction, remarray, and remrule unbind a specific property.

kill always returns done, even if an argument has no binding.

labels (symbol)

Returns the list of input, output, or intermediate expression labels which begin with symbol. Typically symbol is the value of inchar, outchar, or linechar. If no labels begin with symbol, labels returns an empty list.

By default, Maxima displays the result of each user input expression, giving the result an output label. The output display is suppressed by terminating the input with $
(dollar sign) instead of ; (semicolon). An output label is constructed and bound to the result, but not displayed, and the label may be referenced in the same way as displayed output labels. See also %, %%, and %th.

Intermediate expression labels can be generated by some functions. The option variable programmode controls whether solve and some other functions generate intermediate expression labels instead of returning a list of expressions. Some other functions, such as ldisplay, always generate intermediate expression labels.

See also inchar, outchar, linechar, and infolists.

labels

The variable labels is the list of input, output, and intermediate expression labels, including all previous labels if inchar, outchar, or linechar were redefined.

linechar

Default value: %t

linechar is the prefix of the labels of intermediate expressions generated by Maxima. Maxima constructs a label for each intermediate expression (if displayed) by concatenating linechar and linenum.

linechar may be assigned any string or symbol, not necessarily a single character. Because Maxima internally takes into account only the first char of the prefix, the prefixes inchar, outchar, and linechar should have a different first char. Otherwise some commands like kill(inlabels) do not work as expected.

Intermediate expressions might or might not be displayed. See programmode and labels.

linenum

The line number of the current pair of input and output expressions.

myoptions

Default value: []

myoptions is the list of all options ever reset by the user, whether or not they get reset to their default value.

nolabels

Default value: false

When nolabels is true, input and output result labels (%i and %o, respectively) are displayed, but the labels are not bound to results, and the labels are not appended to the labels list. Since labels are not bound to results, garbage collection can recover the memory taken up by the results.

Otherwise input and output result labels are bound to results, and the labels are appended to the labels list.

Intermediate expression labels (%t) are not affected by nolabels; whether nolabels is true or false, intermediate expression labels are bound and appended to the labels list.

See also batch, load, and labels.
optionset

Default value: false

When optionset is true, Maxima prints out a message whenever a Maxima option is reset. This is useful if the user is doubtful of the spelling of some option and wants to make sure that the variable he assigned a value to was truly an option variable.

Example:

(%i1) optionset: true;
assignment: assigning to option optionset
(%o1) true
(%i2) gamma_expand: true;
assignment: assigning to option gamma_expand
(%o2) true

outchar

Default value: %o

outchar is the prefix of the labels of expressions computed by Maxima. Maxima automatically constructs a label for each computed expression by concatenating outchar and linenum.

outchar may be assigned any string or symbol, not necessarily a single character. Because Maxima internally takes into account only the first char of the prefix, the prefixes inchar, outchar and linechar should have a different first char. Otherwise some commands like kill(inlabels) do not work as expected.

See also labels.

Example:

(%i1) outchar: "output";
(output1) output
(%i2) expand((a+b)^3);
3 2 2 3
(%o2) b + 3 a b + 3 a b + a

playback

playback ()
playback (n)
playback ([m, n])
playback ([m])
playback (input)
playback (slow)
playback (time)
playback (grind)

Displays input, output, and intermediate expressions, without recomputing them. playback only displays the expressions bound to labels; any other output (such as text printed by print or describe, or error messages) is not displayed. See also labels.

playback quotes its arguments. The quote-quote operator '' defeats quotation. playback always returns done.
playback () (with no arguments) displays all input, output, and intermediate expressions generated so far. An output expression is displayed even if it was suppressed by the $ terminator when it was originally computed.

playback (n) displays the most recent n expressions. Each input, output, and intermediate expression counts as one.

playback ([m, n]) displays input, output, and intermediate expressions with numbers from m through n, inclusive.

playback ([m]) is equivalent to playback ([m, m]); this usually prints one pair of input and output expressions.

playback (input) displays all input expressions generated so far.

playback (slow) pauses between expressions and waits for the user to press enter. This behavior is similar to demo. playback (slow) is useful in conjunction with save or stringout when creating a secondary-storage file in order to pick out useful expressions.

playback (time) displays the computation time for each expression.

playback (grind) displays input expressions in the same format as the grind function. Output expressions are not affected by the grind option. See grind.

Arguments may be combined, e.g., playback ([5, 10], grind, time, slow).

prompt [Option variable]

Default value: _

prompt is the prompt symbol of the demo function, playback (slow) mode, and the Maxima break loop (as invoked by break).

quit () [Function]

Terminates the Maxima session. Note that the function must be invoked as quit(); or quit($), not quit by itself. To stop a lengthy computation, type control-C. The default action is to return to the Maxima prompt. If *debugger-hook* is nil, control-C opens the Lisp debugger. See also Chapter 38 [Debugging], page 633.

read (expr_1, ..., expr_n) [Function]

Prints expr_1, ..., expr_n, then reads one expression from the console and returns the evaluated expression. The expression is terminated with a semicolon ; or dollar sign $.

See also readonly

Example:

(%%i1) foo: 42$
(%%i2) foo: read ("foo is", foo, " -- enter new value.")$
foo is 42 -- enter new value.
(a+b)^3;
(%%i3) foo;

(%%o3) (b + a)
24

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readonly (expr_1, . . . , expr_n)

[Function]
Prints expr 1, . . . , expr n, then reads one expression from the console and returns the
expression (without evaluation). The expression is terminated with a ; (semicolon)
or $ (dollar sign).
See also read.
Examples:
(%i1) aa: 7$
(%i2) foo: readonly ("Enter an expression:");
Enter an expression:
2^aa;
aa
(%o2)
2
(%i3) foo: read ("Enter an expression:");
Enter an expression:
2^aa;
(%o3)
128

reset ()

[Function]
Resets many global variables and options, and some other variables, to their default
values.
reset processes the variables on the Lisp list *variable-initial-values*. The
Lisp macro defmvar puts variables on this list (among other actions). Many, but not
all, global variables and options are defined by defmvar, and some variables defined
by defmvar are not global variables or options.

[Option variable]
Default value: false
When showtime is true, the computation time and elapsed time is printed with each
output expression.
The computation time is always recorded, so time and playback can display the
computation time even when showtime is false.
See also timer.

showtime

to_lisp ()

[Function]
Enters the Lisp system under Maxima. (to-maxima) returns to Maxima.
Example:
Define a function and enter the Lisp system under Maxima. The definition is inspected
on the property list, then the function definition is extracted, factored and stored in
the variable $result. The variable can be used in Maxima after returning to Maxima.
(%i1) f(x):=x^2+x;
2
(%o1)
f(x) := x + x
(%i2) to_lisp();
Type (to-maxima) to restart, ($quit) to quit Maxima.
MAXIMA> (symbol-plist '$f)
(MPROPS (NIL MEXPR ((LAMBDA) ((MLIST) $X)


\((\text{MPLUS} \ (\text{MEXPT} \ X \ 2) \ X))\)

\text{MAXIMA> (setq $result ($factor (caddr (mget '$f 'mexpr))))}
\((\text{MTIMES SIMP FACTORED}) \ X \ ((\text{MPLUS SIMP IRREDUCIBLE}) \ 1 \ X))\)

\text{MAXIMA> (to-maxima)}
\text{Returning to Maxima}
\(\%o2\) true
\(\%i3\) result; \(\%o3\) \(X \ (X + 1)\)

\text{values} \quad \text{[System variable]}

Initial value: []

\text{values} is a list of all bound user variables (not Maxima options or switches). The list comprises symbols bound by :, or ::.

If the value of a variable is removed with the commands \text{kill}, \text{remove}, or \text{remvalue} the variable is deleted from \text{values}.

See \text{functions} for a list of user defined functions.

Examples:
First, \text{values} shows the symbols \(a\), \(b\), and \(c\), but not \(d\), it is not bound to a value, and not the user function \(f\). The values are removed from the variables. \text{values} is the empty list.

\(\%i1\) [\(a:99\), \(b:: a-90\), \(c:a-b\), \(d\), \(f(x):=x^2\)];
\(\%o1\) [99, 9, 90, \(d\), \(f(x) := x \)]
\(\%i2\) values;
\(\%o2\) [\(a\), \(b\), \(c\)]
\(\%i3\) [\text{kill} \(a\), \text{remove} \(b\), \text{value}, \text{remvalue} \(c\)];
\(\%o3\) [\text{done}, \text{done}, [\(c\)]]
\(\%i4\) values;
\(\%o4\) []

\subsection*{4.3 Functions and Variables for Display}

\text{%edispflag} \quad \text{[Option variable]}

Default value: false

When \text{%edispflag} is true, Maxima displays \%e to a negative exponent as a quotient. For example, \%e^{-x} is displayed as \(1/\%e^{-x}\). See also \text{exptdispflag}.

Example:
\(\%i1\) \%e^{-10}; \quad \%o1\) \(-10\)
\(\%i2\) \text{%edispflag:true} $
\(\%i3\) \%e^{-10}; \quad \%o3\) \(\frac{1}{10}\)
\(\%o3\) \(\%e\)
absboxchar [Option variable]

Default value: !

absboxchar is the character used to draw absolute value signs around expressions which are more than one line tall.

Example:

(%i1) abs((x^3+1));

! 3 !

(%o1) !x + 1!

disp (expr_1, expr_2, ...) [Function]

is like display but only the value of the arguments are displayed rather than equations. This is useful for complicated arguments which don't have names or where only the value of the argument is of interest and not the name.

See also ldisp and print.

Example:

(%i1) b[1,2]:x-x^2$

(%i2) x:123$

(%i3) disp(x, b[1,2], sin(1.0));

123

2

x - x

0.8414709848078965

(%o3) done

display (expr_1, expr_2, ... ) [Function]

Displays equations whose left side is expr_i unevaluated, and whose right side is the value of the expression centered on the line. This function is useful in blocks and for statements in order to have intermediate results displayed. The arguments to display are usually atoms, subscripted variables, or function calls.

See also ldisplay, disp, and ldisp.

Example:

(%i1) b[1,2]:x-x^2$

(%i2) x:123$

(%i3) display(x, b[1,2], sin(1.0));

x = 123

2

b = x - x

1, 2

sin(1.0) = 0.8414709848078965

(%o3) done
**display2d**

[Option variable]

Default value: **true**

When `display2d` is **false**, the console display is a string (1-dimensional) form rather than a display (2-dimensional) form.

See also **leftjust** to switch between a left justified and a centered display of equations.

Example:

```
(%i1) x/(x^2+1);
       x
(%o1) -------
       2
     x + 1

(%i2) display2d:false$
(%i3) x/(x^2+1);
(%o3) x/(x^2+1)
```

**display_format_internal**

[Option variable]

Default value: **false**

When `display_format_internal` is **true**, expressions are displayed without being transformed in ways that hide the internal mathematical representation. The display then corresponds to what `inpart` returns rather than `part`.

Examples:

<table>
<thead>
<tr>
<th>User</th>
<th>part</th>
<th>inpart</th>
</tr>
</thead>
<tbody>
<tr>
<td>a-b;</td>
<td>a - b</td>
<td>a + (-1) b</td>
</tr>
<tr>
<td>a/b;</td>
<td>a/b</td>
<td>a b</td>
</tr>
<tr>
<td>sqrt(x);</td>
<td>sqrt(x)</td>
<td>x</td>
</tr>
<tr>
<td>X*4/3;</td>
<td>---</td>
<td>- X</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>3</td>
</tr>
</tbody>
</table>

**dispterms (expr)**

[Function]

Displays `expr` in parts one below the other. That is, first the operator of `expr` is displayed, then each term in a sum, or factor in a product, or part of a more general expression is displayed separately. This is useful if `expr` is too large to be otherwise displayed. For example if `P1`, `P2`, ... are very large expressions then the display program may run out of storage space in trying to display `P1 + P2 + ...` all at once. However, `dispterms (P1 + P2 + ...)` displays `P1`, then below it `P2`, etc. When not using `dispterms`, if an exponential expression is too wide to be displayed as `A^B` it appears as `expt (A, B)` (or as `ncexpt (A, B) in the case of `A^-B`).

Example:

```
(%i1) dispterms(2*a*sin(x)+%e^x);
```
+ 
2 a sin(x)

\[ x \]
\%e

(%o1) done

expt (a, b)  
ncexpt (a, b) 

If an exponential expression is too wide to be displayed as \( a^b \) it appears as \texttt{expt (a, b)} or as \texttt{ncexpt (a, b)} in the case of \( a^{-b} \).

\texttt{expt} and \texttt{ncexpt} are not recognized in input.

exptdispflag

Default value: true

When \texttt{exptdispflag} is true, Maxima displays expressions with negative exponents using quotients. See also \%edispsflag.

Example:

(%i1) exptdispflag:true;
(%o1) true
(%i2) 10^-x;
(%o2) \frac{1}{10^x}

(%i3) exptdispflag:false;
(%o3) false
(%i4) 10^-x;
(%o4) 10^{-x}

grind (expr)

The function \texttt{grind} prints \texttt{expr} to the console in a form suitable for input to Maxima. \texttt{grind} always returns \texttt{done}.

When \texttt{expr} is the name of a function or macro, \texttt{grind} prints the function or macro definition instead of just the name.

See also \texttt{string}, which returns a string instead of printing its output. \texttt{grind} attempts to print the expression in a manner which makes it slightly easier to read than the output of \texttt{string}.

\texttt{grind} evaluates its argument.

Examples:

(%i1) aa + 1729;
(%o1) aa + 1729
(%i2) grind (%);
aa+1729$
(%o2) done
(%i3) [aa, 1729, aa + 1729];
(%o3) [aa, 1729, aa + 1729]
(%i4) grind (%);
[aa,1729,aa+1729]$
(%o4) done
(%i5) matrix ([aa, 17], [29, bb]);
[ aa 17 ]
[   ]
[ 29 bb ]
(%i6) grind (%);
matrix([aa,17],[29,bb])$
(%o6) done
(%i7) set (aa, 17, 29, bb);
(%o7) {17, 29, aa, bb}
(%i8) grind (%);
{17,29,aa,bb}$
(%o8) done
(%i9) exp (aa / (bb + 17)^29);
\[ \frac{aa}{(bb + 17)^{29}} \]
(%o9) %e
(%i10) grind (%);
%e^{(aa/(bb+17)^{29})}$
(%o10) done
(%i11) expr: expand ((aa + bb)^10);
\[ bb^{10} + 10 aa bb^{9} + 45 aa^{2} bb^{8} + 120 aa^{3} bb^{7} + 210 aa^{4} bb^{6} + 252 aa^{5} bb^{5} + 210 aa^{6} bb^{4} + 120 aa^{7} bb^{3} + 45 aa^{8} bb^{2} + 10 aa^{9} bb + aa^{10} \]
(%i12) grind (expr);
\[ bb^{10} + 10 aa bb^{9} + 45 aa^{2} bb^{8} + 120 aa^{3} bb^{7} + 210 aa^{4} bb^{6} + 252 aa^{5} bb^{5} + 210 aa^{6} bb^{4} + 120 aa^{7} bb^{3} + 45 aa^{8} bb^{2} + 10 aa^{9} bb + aa^{10} \]
(%i13) string (expr);
\[ bb^{10} + 10 aa bb^{9} + 45 aa^{2} bb^{8} + 120 aa^{3} bb^{7} + 210 aa^{4} bb^{6} + 252 aa^{5} bb^{5} + 210 aa^{6} bb^{4} + 120 aa^{7} bb^{3} + 45 aa^{8} bb^{2} + 10 aa^{9} bb + aa^{10} \]
(%i14) cholesky(A) := block([n: length(A), L: copymatrix(A), p: makelist(0, i, 1, length(A))],
   for i thru n do for j : i thru n do
   (x : L[i, j], x : x - sum(L[j, k] * L[i, k], k, 1, i - 1),
    if i = j then p[i] : 1 / sqrt(x) else L[j, i] : x * p[i]),
   for i thru n do L[i, i] : 1 / p[i],
   for i thru n do for j : i + 1 thru n do L[i, j] : 0, L)$
define: warning: redefining the built-in function cholesky
(%i15) grind (cholesky);
cholesky(A) := block([n: length(A), L: copymatrix(A), p: makelist(0, i, 1, length(A))],
   for i thru n do (for j from i thru n do
   (x : L[i, j], x : x - sum(L[j, k] * L[i, k], k, 1, i - 1),
    if i = j then p[i] : 1 / sqrt(x) else L[j, i] : x * p[i])),
   for i thru n do L[i, i] : 1 / p[i],
   for i thru n do (for j from i + 1 thru n do L[i, j] : 0), L)
(%o15) done
(%i16) string (fundef (cholesky));
(\%o16) cholesky(A) := block([n:length(A), L:copymatrix(A), p:makelist(0, i, 1, length(A))],
   for i thru n do (for j from i thru n do
   (x:L[i, j], x:x-\sum(L[j, k]*L[i, k], k, 1, i-1),
    if i = j then p[i]:1/sqrt(x) else L[j, i]:x*p[i])),
   for i thru n do L[i, i]:1/p[i],
   for i thru n do (for j from i+1 thru n do L[i, j]:0), L)

\textbf{grind} [Option variable]

When the variable \texttt{grind} is \texttt{true}, the output of \texttt{string} and \texttt{stringout} has the same format as that of \texttt{grind}; otherwise no attempt is made to specially format the output of those functions. The default value of the variable \texttt{grind} is \texttt{false}.

\texttt{grind} can also be specified as an argument of \texttt{playback}. When \texttt{grind} is present, \texttt{playback} prints input expressions in the same format as the \texttt{grind} function. Otherwise, no attempt is made to specially format input expressions.

\textbf{ibase} [Option variable]

Default value: 10

\texttt{ibase} is the base for integers read by Maxima.

\texttt{ibase} may be assigned any integer between 2 and 36 (decimal), inclusive. When \texttt{ibase} is greater than 10, the numerals comprise the decimal numerals 0 through 9 plus letters of the alphabet A, B, C, \ldots, as needed to make \texttt{ibase} digits in all. Letters are interpreted as digits only if the first digit is 0 through 9.

Uppercase and lowercase letters are not distinguished. The numerals for base 36, the largest acceptable base, comprise 0 through 9 and A through Z.

Whatever the value of \texttt{ibase}, when an integer is terminated by a decimal point, it is interpreted in base 10.
See also `obase`.

Examples:

**ibase** less than 10 (for example binary numbers).

```
(%i1) ibase : 2 $
(%i2) obase;
(%o2) 10
(%i3) 1111111111111111;
(%o3) 65535
```

**ibase** greater than 10. Letters are interpreted as digits only if the first digit is 0 through 9 which means that hexadecimal numbers might need to be prepended by a 0.

```
(%i1) ibase : 16 $
(%i2) obase;
(%o2) 10
(%i3) 1000;
(%o3) 4096
(%i4) abcd;
(%o4) abcd
(%i5) symbolp (abcd);
(%o5) true
(%i6) 0abcd;
(%o6) 43981
(%i7) symbolp (0abcd);
(%o7) false
```

When an integer is terminated by a decimal point, it is interpreted in base 10.

```
(%i1) ibase : 36 $
(%i2) obase;
(%o2) 10
(%i3) 1234;
(%o3) 49360
(%i4) 1234.;
(%o4) 1234
```

**ldisp** (expr_1, ..., expr_n) [Function]

Displays expressions expr_1, ..., expr_n to the console as printed output. **ldisp** assigns an intermediate expression label to each argument and returns the list of labels.

See also **disp**, **display**, and **ldisplay**.

Examples:

```
(%i1) e: (a+b)^3;
3
(%o1) (b + a)
(%i2) f: expand (e);
3 2 2 3
(%o2) b + 3 a b + 3 a b + a
```
(%i3) ldisp (e, f);

(%t3)

3
(b + a)

(%t4)

3   2 2 3
b + 3 a b + 3 a b + a

(%o4) [%t3, %t4]

(%i4) %t3;

3
(b + a)

(%o4)

(%i5) %t4;

3 2 2 3
b + 3 a b + 3 a b + a

(%o5)

ldisplay (expr_1, ..., expr_n)

[Function]
Displays expressions expr_1, ..., expr_n to the console as printed output. Each
expression is printed as an equation of the form lhs = rhs in which lhs is one of the
arguments of ldisplay and rhs is its value. Typically each argument is a variable.
ldisp assigns an intermediate expression label to each equation and returns the list
of labels.

See also display, disp, and ldisp.

Examples:

(%i11) e: (a+b)^3;

3
(b + a)

(%i12) f: expand (e);

3   2 2 3
b + 3 a b + 3 a b + a

(%i13) ldisplay (e, f);

3
(b + a)

(%t3)

3   2 2 3
f = b + 3 a b + 3 a b + a

(%o4) [%t3, %t4]

(%i4) %t3;

3
(b + a)

(%o4)

(%i5) %t4;

3 2 2 3
f = b + 3 a b + 3 a b + a

(%o5)

leftjust

[Option variable]

Default value: false
When `leftjust` is `true`, equations in 2D-display are drawn left justified rather than centered. See also `display2d` to switch between 1D- and 2D-display.

Example:

```plaintext
(%i1) expand((x+1)^3);
   3  2
(%o1) x + 3 x + 3 x + 1
(%i2) leftjust:true$
(%i3) expand((x+1)^3);
   3  2
(%o3) x + 3 x + 3 x + 1
```

`linel`  
Default value: 79  
`linel` is the assumed width (in characters) of the console display for the purpose of displaying expressions. `linel` may be assigned any value by the user, although very small or very large values may be impractical. Text printed by built-in Maxima functions, such as error messages and the output of `describe`, is not affected by `linel`.

`lispdisp`  
Default value: `false`  
When `lispdisp` is `true`, Lisp symbols are displayed with a leading question mark `?`. Otherwise, Lisp symbols are displayed with no leading mark. This has the same effect for 1-d and 2-d display.

Examples:

```plaintext
(%i1) lispdisp: false$
(%i2) ?foo + ?bar;
   foo + bar
(%i3) lispdisp: true$
(%i4) ?foo + ?bar;
   ?foo + ?bar
```

`negsumdispflag`  
Default value: `true`  
When `negsumdispflag` is `true`, `x - y` displays as `x - y` instead of as `- y + x`. Setting it to `false` causes the special check in display for the difference of two expressions to not be done. One application is that thus `a + %i*b` and `a - %i*b` may both be displayed the same way.

`obase`  
Default value: 10  
`obase` is the base for integers displayed by Maxima. `obase` may be assigned any integer between 2 and 36 (decimal), inclusive. When `obase` is greater than 10, the numerals comprise the decimal numerals 0 through 9 plus capital letters of the alphabet A, B, C, . . . , as needed. A leading 0 digit is
displayed if the leading digit is otherwise a letter. The numerals for base 36, the largest acceptable base, comprise 0 through 9, and A through Z.

See also `ibase`.

Examples:

```maxima
(%i1) obase : 2;
(%o1) 10
(%i2) 2^8 - 1;
(%o2) 11111111
(%i3) obase : 8;
(%o3) 10
(%i4) 8^8 - 1;
(%o4) 77777777
(%i5) obase : 16;
(%o5) 10
(%i6) 16^8 - 1;
(%o6) 0FFFFFFFF
(%i7) obase : 36;
(%o7) 10
(%i8) 36^8 - 1;
(%o8) 0ZZZZZZZZ
```

**pfeformat**  
[Option variable]

Default value: `false`

When `pfeformat` is `true`, a ratio of integers is displayed with the solidus (forward slash) character, and an integer denominator `n` is displayed as a leading multiplicative term `1/n`.

Examples:

```maxima
(%i1) pfeformat: false$
(%i2) 2^16/7^3;
(%o2) 65536/343
(%i3) (a+b)/8;
(%o3) b + a/8
(%i4) pfeformat: true$
(%i5) 2^16/7^3;
(%o5) 65536/343
(%i6) (a+b)/8;
(%o6) 1/8 (b + a)
```

**powerdisp**  
[Option variable]

Default value: `false`

When `powerdisp` is `true`, a sum is displayed with its terms in order of increasing power. Thus a polynomial is displayed as a truncated power series, with the constant term first and the highest power last.
By default, terms of a sum are displayed in order of decreasing power.

Example:

\[
\begin{align*}
(\%i1) & \text{ powerdisp: true; } \\
(\%o1) & \text{ true } \\
(\%i2) & x^2 + x^3 + x^4; \\
(\%o2) & x + x + x \\
(\%i3) & \text{ powerdisp: false; } \\
(\%o3) & \text{ false } \\
(\%i4) & x^2 + x^3 + x^4; \\
(\%o4) & x + x + x
\end{align*}
\]

**print (expr_1, ..., expr_n)**  
[Function]  
Evaluates and displays `expr_1, ..., expr_n` one after another, from left to right, starting at the left edge of the console display.  
The value returned by `print` is the value of its last argument. `print` does not generate intermediate expression labels.  
See also `display`, `disp`, `ldisplay`, and `ldisp`. Those functions display one expression per line, while `print` attempts to display two or more expressions per line.  
To display the contents of a file, see `printfile`.  

Examples:

\[
\begin{align*}
(\%i1) & \text{ r: print ("(a+b)^3 is", expand ((a+b)^3), "log (a^10/b) is", radcan (log (a^10/b)))$} \\
& \text{ 3 2 2 3} \\
(a+b)^3 & \text{ is } b + 3a b + 3 a b + a \log (a^{10}/b) \text{ is} \\
& \text{ 10 log(a) - log(b)} \\
(\%i2) & r; \\
(\%o2) & \text{ 10 log(a) - log(b)} \\
(\%i3) & \text{ disp ("(a+b)^3 is", expand ((a+b)^3), "log (a^10/b) is", radcan (log (a^10/b)))$} \\
& \text{ 3 2 2 3} \\
(a+b)^3 & \text{ is } b + 3a b + 3 a b + a \log (a^{10}/b) \text{ is} \\
& \text{ 10 log(a) - log(b)}
\end{align*}
\]

**sqrtdispflag**  
[Option variable]  
Default value: true  
When `sqrtdispflag` is false, causes `sqrt` to display with exponent 1/2.
**stardisp**  
[Option variable]  
Default value: `false`  
When `stardisp` is `true`, multiplication is displayed with an asterisk `*` between operands.

**ttyoff**  
[Option variable]  
Default value: `false`  
When `ttyoff` is `true`, output expressions are not displayed. Output expressions are still computed and assigned labels. See `labels`.

Text printed by built-in Maxima functions, such as error messages and the output of `describe`, is not affected by `ttyoff`. 
5 Data Types and Structures

5.1 Numbers

5.1.1 Introduction to Numbers

Complex numbers

A complex expression is specified in Maxima by adding the real part of the expression to \( %i \) times the imaginary part. Thus the roots of the equation \( x^2 - 4x + 13 = 0 \) are \( 2 + 3%i \) and \( 2 - 3%i \). Note that simplification of products of complex expressions can be effected by expanding the product. Simplification of quotients, roots, and other functions of complex expressions can usually be accomplished by using the \texttt{realpart}, \texttt{imagpart}, \texttt{rectform}, \texttt{polarform}, \texttt{abs}, \texttt{carg} functions.

5.1.2 Functions and Variables for Numbers

\texttt{bfloat (expr)}

Converts all numbers and functions of numbers in \texttt{expr} to bigfloat numbers. The number of significant digits in the resulting bigfloats is specified by the global variable \texttt{fpprec}.

When \texttt{float2bf} is \texttt{false} a warning message is printed when a floating point number is converted into a bigfloat number (since this may lead to loss of precision).

\texttt{bfloatp (expr)}

Returns \texttt{true} if \texttt{expr} is a bigfloat number, otherwise \texttt{false}.

\texttt{bftorat}

Default value: false

\texttt{bftorat} controls the conversion of \texttt{bfloats} to rational numbers. When \texttt{bftorat} is \texttt{false}, \texttt{ratepsilon} will be used to control the conversion (this results in relatively small rational numbers). When \texttt{bftorat} is \texttt{true}, the rational number generated will accurately represent the bfloat.

Note: \texttt{bftorat} has no effect on the transformation to rational numbers with the function \texttt{rationalize}.

Example:

\begin{verbatim}
(%i1) ratepsilon:1e-4;
     1.e-4
(%o1)                               1.e-4
(%i2) rat(bfloat(11111/111111)), bftorat:false;
   'rat' replaced 9.9990999991B-2 by 1/10 = 1.0B-1
     1
   --
(%o2)/R/                               10

(%i3) rat(bfloat(11111/111111)), bftorat:true;
   'rat' replaced 9.9990999991B-2 by 11111/111111 = 9.9990999991B-2
     111111
(%o3)/R/   ------
          111111
\end{verbatim}
bftrunc
[Option variable]
Default value: true

bftrunc causes trailing zeroes in non-zero bigfloat numbers not to be displayed. Thus, if bftrunc is false, bfloat (1) displays as 1.000000000000000B0. Otherwise, this is displayed as 1.0B0.

evenp (expr)
[Function]
Returns true if expr is a literal even integer, otherwise false.
evenp returns false if expr is a symbol, even if expr is declared even.

float (expr)
[Function]
Converts integers, rational numbers and bigfloats in expr to floating point numbers. It is also an evflag, float causes non-integral rational numbers and bigfloat numbers to be converted to floating point.

float2bf
[Option variable]
Default value: true

When float2bf is false, a warning message is printed when a floating point number is converted into a bigfloat number (since this may lead to loss of precision).

floatnump (expr)
[Function]
Returns true if expr is a floating point number, otherwise false.

fpprec
[Option variable]
Default value: 16

fpprec is the number of significant digits for arithmetic on bigfloat numbers. fpprec does not affect computations on ordinary floating point numbers.

See also bfloat and fpprintprec.

fpprintprec
[Option variable]
Default value: 0

fpprintprec is the number of digits to print when printing an ordinary float or bigfloat number.

For ordinary floating point numbers, when fpprintprec has a value between 2 and 16 (inclusive), the number of digits printed is equal to fpprintprec. Otherwise, fpprintprec is 0, or greater than 16, and the number of digits printed is 16.

For bigfloat numbers, when fpprintprec has a value between 2 and fpprec (inclusive), the number of digits printed is equal to fpprintprec. Otherwise, fpprintprec is 0, or greater than fpprec, and the number of digits printed is equal to fpprec.

For both ordinary floats and bigfloats, trailing zero digits are suppressed. The actual number of digits printed is less than fpprintprec if there are trailing zero digits. fpprintprec cannot be 1.

integerp (expr)
[Function]
Returns true if expr is a literal numeric integer, otherwise false.

integerp returns false if expr is a symbol, even if expr is declared integer.
Examples:

(%i1) integerp (0);
(%o1) true
(%i2) integerp (1);
(%o2) true
(%i3) integerp (-17);
(%o3) true
(%i4) integerp (0.0);
(%o4) false
(%i5) integerp (1.0);
(%o5) false
(%i6) integerp (%pi);
(%o6) false
(%i7) integerp (n);
(%o7) false
(%i8) declare (n, integer);
(%o8) done
(%i9) integerp (n);
(%o9) false

m1pbranch

[Option variable]

Default value: false

m1pbranch is the principal branch for -1 to a power. Quantities such as (-1)^(1/3) (that is, an "odd" rational exponent) and (-1)^(1/4) (that is, an "even" rational exponent) are handled as follows:

\[
\begin{align*}
\text{domain: real} & \\
(-1)^{1/3} : & -1 \\
(-1)^{1/4} : & (-1)^{1/4}
\end{align*}
\]

\[
\begin{align*}
\text{domain: complex} & \\
\text{m1pbranch: false} & \text{m1pbranch: true} \\
(-1)^{1/3} & 1/2+%i*sqrt(3)/2 \\
(-1)^{1/4} & sqrt(2)/2+%i*sqrt(2)/2
\end{align*}
\]

nonnegintegerp (n)

[Function]

Return true if and only if \(n \geq 0\) and \(n\) is an integer.

numberp (expr)

[Function]

Returns true if expr is a literal integer, rational number, floating point number, or bigfloat, otherwise false.

numberp returns false if expr is a symbol, even if expr is a symbolic number such as \(%pi\) or \(%i\), or declared to be even, odd, integer, rational, irrational, real, imaginary, or complex.

Examples:

(%i11) numberp (42);
(%o1)   true
(%i2)   numberp (-13/19);               true
(%o2)   true
(%i3)   numberp (3.14159);             true
(%o3)   true
(%i4)   numberp (-1729b-4);            true
(%o4)   true
(%i5)   map (numberp, [e, pi, i, phi, inf, minf]);
(%o5)   [false, false, false, false, false, false]
(%i6)   declare (a, even, b, odd, c, integer, d, rational,
             e, irrational, f, real, g, imaginary, h, complex);
(%o6) done
(%i7)   map (numberp, [a, b, c, d, e, f, g, h]);
(%o7)   [false, false, false, false, false, false, false, false]

[Option variable]
numer

numer causes some mathematical functions (including exponentiation) with numerical
arguments to be evaluated in floating point. It causes variables in expr which have
been given numerals to be replaced by their values. It also sets the float switch on.
See also %enumer.

Examples:
(%i1)   [sqrt(2), sin(1), 1/(1+sqrt(3))];
    1
(%o1)   [sqrt(2), sin(1), 1/(1+sqrt(3))]
          sqrt(3) + 1
(%i2)   [sqrt(2), sin(1), 1/(1+sqrt(3))],numer;
(%o2)   [1.414213562373095, 0.8414709848078965, 0.3660254037844387]

[Option variable]
numer_pbranch

Default value: false

The option variable numer_pbranch controls the numerical evaluation of the power of
a negative integer, rational, or floating point number. When numer_pbranch is true
and the exponent is a floating point number or the option variable numer is true
too, Maxima evaluates the numerical result using the principal branch. Otherwise a
simplified, but not an evaluated result is returned.

Examples:
(%i1)   (-2)^0.75;
    0.75
(%o1)   (- 2)
(%i2)   (-2)^0.75,numer_pbranch:true;
(%o2)   1.189207115002721 %i - 1.189207115002721
(%i3)   (-2)^(3/4);
    3/4  3/4
(%o3)   (- 1) 2
(%i4)   (-2)^(3/4),numer;
    0.75
(%o4)   1.681792830507429 (- 1)
numeral \( x_1, expr_1, \ldots, var_n, expr_n \)  
Declares the variables \( x_1, \ldots, x_n \) to have numeric values equal to \( expr_1, \ldots, expr_n \). The numeric value is evaluated and substituted for the variable in any expressions in which the variable occurs if the \texttt{numer} flag is \texttt{true}. See also \texttt{ev}.

The expressions \( expr_1, \ldots, expr_n \) can be any expressions, not necessarily numeric.

\texttt{oddp (expr)}  
Returns \texttt{true} if \( expr \) is a literal odd integer, otherwise \texttt{false}.

\texttt{ratepsilon}  
Default value: 2.0e-15

\texttt{ratepsilon} is the tolerance used in the conversion of floating point numbers to rational numbers, when the option variable \texttt{bftorat} has the value \texttt{false}. See \texttt{bftorat} for an example.

\texttt{rationalize (expr)}  
Convert all double floats and big floats in the Maxima expression \( expr \) to their exact rational equivalents. If you are not familiar with the binary representation of floating point numbers, you might be surprised that \texttt{rationalize (0.1)} does not equal 1/10. This behavior isn’t special to Maxima – the number 1/10 has a repeating, not a terminating, binary representation.

\texttt{(%i1) rationalize (0.5);}
\texttt{1}
\texttt{(%o1) -}
\texttt{2}
\texttt{(%i2) rationalize (0.1);}
\texttt{3602879701896397}
\texttt{(%o2) ------------------}
\texttt{36028797018963968}
\texttt{(%i3) fpprec : 5$}
\texttt{(%i4) rationalize (0.1b0);}
\texttt{209715}
\texttt{(%o4) -------}
\texttt{2097152}
\texttt{(%i5) fpprec : 20$}
\texttt{(%i6) rationalize (0.1b0);}
\texttt{236118324143482260685}
\texttt{(%o6) ------------------}
\texttt{2361183241434822606848}
\texttt{(%i7) rationalize (sin (0.1*x + 5.6));}
\texttt{3602879701896397 x 3152519739159347}
\texttt{(%o7) sin(------------------ + ------------------)}
\texttt{36028797018963968 562949953421312}
ratnump (expr)

Returns true if expr is a literal integer or ratio of literal integers, otherwise false.
5.2 Strings

5.2.1 Introduction to Strings

Strings (quoted character sequences) are enclosed in double quote marks " for input, and displayed with or without the quote marks, depending on the global variable stringdisp.

Strings may contain any characters, including embedded tab, newline, and carriage return characters. The sequence \" is recognized as a literal double quote, and \ \ as a literal backslash. When backslash appears at the end of a line, the backslash and the line termination (either newline or carriage return and newline) are ignored, so that the string continues with the next line. No other special combinations of backslash with another character are recognized; when backslash appears before any character other than ", \, or a line termination, the backslash is ignored. There is no way to represent a special character (such as tab, newline, or carriage return) except by embedding the literal character in the string.

There is no character type in Maxima; a single character is represented as a one-character string.

The stringproc add-on package contains many functions for working with strings.

Examples:

(%i1) s_1 : "This is a string."
(%o1) This is a string.
(%i2) s_2 : "Embedded \"double quotes\" and backslash \ characters."
(%o2) Embedded "double quotes" and backslash \ characters.
(%i3) s_3 : "Embedded line termination
(%o3) Embedded line termination
in this string.
(%i4) in this string."
(%i5) s_4 : "Ignore the \false
(%i6) line termination \true
(%i7) characters in \This is a string.
(%i8) this string.";
(%o8) "This is a string."
(%i9) stringdisp : false;

5.2.2 Functions and Variables for Strings

concat (arg_1, arg_2, ...) [Function]
Concatenates its arguments. The arguments must evaluate to atoms. The return value is a symbol if the first argument is a symbol and a string otherwise.

concat evaluates its arguments. The single quote ' prevents evaluation.
See also sconcat and eval_string.

(%i10) y: 7$
(%i11) z: 88$
(%i3) concat (y, z/2); 744
(%o3) y44
(%i4) concat ('y, z/2); y44
(%o4) y44

A symbol constructed by \texttt{concat} may be assigned a value and appear in expressions. The :: (double colon) assignment operator evaluates its left-hand side.

(%i5) a: concat ('y, z/2); y44
(%o5) y44
(%i6) a:: 123; 123
(%o6) 123
(%i7) y44; 123
(%o7) 123
(%i8) b^a; y44
(%o8) y44

Note that although \texttt{concat} (1, 2) looks like a number, it is a string.

(%i9) concat (1, 2) + 3; 12 + 3
(%o9) 12 + 3

\texttt{sconcat (arg\_1, arg\_2, \ldots)} \texttt{[Function]}
Concatenates its arguments into a string. Unlike \texttt{concat}, the arguments do not need to be atoms.

See also \texttt{concat} and \texttt{eval_string}.

(%i10) sconcat ("xx[", 3, "]":, expand ((x+y)^3)); xx[3]:y^3+3*x*y^2+3*x^2*y+x^3

\texttt{string (expr)} \texttt{[Function]}
Converts \texttt{expr} to Maxima’s linear notation just as if it had been typed in.

The return value of \texttt{string} is a string, and thus it cannot be used in a computation.

\texttt{stringdisp} \texttt{[Option variable]}
Default value: false

When \texttt{stringdisp} is true, strings are displayed enclosed in double quote marks. Otherwise, quote marks are not displayed.\texttt{stringdisp} is always true when displaying a function definition.

Examples:

(%i1) stringdisp: false$
(%i2) "This is an example string."; This is an example string.
(%o2) This is an example string.
(%i3) foo () :=
   print ("This is a string in a function definition.");
(%o3) foo() :=
   print("This is a string in a function definition.")
(%i4) stringdisp: true$
(%i5) "This is an example string."
(%o5) "This is an example string."
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5.3 Constants

5.3.1 Functions and Variables for Constants

%e

%e represents the base of the natural logarithm, also known as Euler’s number. The numeric value of %e is the double-precision floating-point value 2.718281828459045d0.

%i

%i represents the imaginary unit, sqrt(-1).

false

false represents the Boolean constant of the same name. Maxima implements false by the value NIL in Lisp.

%gamma

The Euler-Mascheroni constant, 0.5772156649015329 ....

ind

ind represents a bounded, indefinite result. See also limit. Example:

(%i1) limit (sin(1/x), x, 0);
(%o1) ind

inf

inf represents real positive infinity.

infinity

infinity represents complex infinity.

minf

minf represents real minus (i.e., negative) infinity.

%phi

%phi represents the so-called golden mean, (1 + sqrt(5))/2. The numeric value of %phi is the double-precision floating-point value 1.618033988749895d0. fibtophi expresses Fibonacci numbers fib(n) in terms of %phi. By default, Maxima does not know the algebraic properties of %phi. After evaluating tellrat(%phi^2 - %phi - 1) and algebraic: true, ratsimp can simplify some expressions containing %phi.

Examples:

fibtophi expresses Fibonacci numbers fib(n) in terms of %phi.

(%i1) fibtophi (fib (n));

(%i1) fibtophi (fib (n));

(%o1) %phi - (1 - %phi)

(%o1) %phi - (1 - %phi)

(%o1) %phi - (1 - %phi)

(%o1) 2 %phi - 1

(%o1) 2 %phi - 1

(%o1) 2 %phi - 1
By default, Maxima does not know the algebraic properties of $\phi$. After evaluating `tellrat (\phi^2 - \phi - 1)` and `algebraic: true`, `ratsimp` can simplify some expressions containing $\phi$.

```
(%i1) e : expand ((\phi^2 - \phi - 1) * (A + 1));

2
(\phi A - \phi A - \phi - \phi - 1)
(%o1)

(%i2) ratsimp (e);

2
(\phi - \phi - 1) A + \phi - \phi - 1
(%o2)

(%i3) tellrat (\phi^2 - \phi - 1);

[\phi - \phi - 1]
(%o3)

(%i4) algebraic : true;

true
(%o4)

(%i5) ratsimp (e);

0
(%o5)
```

$\pi$ [Constant]

$\pi$ represents the ratio of the perimeter of a circle to its diameter. The numeric value of $\pi$ is the double-precision floating-point value 3.141592653589793d0.

$\text{true}$ [Constant]

$\text{true}$ represents the Boolean constant of the same name. Maxima implements $\text{true}$ by the value T in Lisp.

$\text{und}$ [Constant]

$\text{und}$ represents an undefined result.

See also `limit`.

Example:

```
(%i1) limit (x*sin(x), x, inf);

und
(%o1)
```

$\text{zeroa}$ [Constant]

$\text{zeroa}$ represents an infinitesimal above zero. $\text{zeroa}$ can be used in expressions. `limit` simplifies expressions which contain infinitesimals.
See also \texttt{zerob} and \texttt{limit}.

Example:

\texttt{limit} simplifies expressions which contain infinitesimals:

\begin{verbatim}
(%i1) limit(zeroa);
(%o1) 0
(%i2) limit(x+zeroa);
(%o2) x
\end{verbatim}

\texttt{zerob} [Constant]

\texttt{zerob} represents an infinitesimal below zero. \texttt{zerob} can be used in expressions. \texttt{limit} simplifies expressions which contain infinitesimals.

See also \texttt{zeroa} and \texttt{limit}. 
5.4 Lists

5.4.1 Introduction to Lists
Lists are the basic building block for Maxima and Lisp. All data types other than arrays, hash tables and numbers are represented as Lisp lists. These Lisp lists have the form

\[(\texttt{MPLUS}) \, \$A \, 2)\]

to indicate an expression \(a+2\). At Maxima level one would see the infix notation \(a+2\). Maxima also has lists which are printed as

\[\texttt{[1, 2, 7, x+y]}\]

for a list with 4 elements. Internally this corresponds to a Lisp list of the form

\[(\texttt{MLIST}) \, 1 \, 2 \, 7 \, (\texttt{MPLUS}) \, \$X \, \$Y)\]

The flag which denotes the type field of the Maxima expression is a list itself, since after it has been through the simplifier the list would become

\[(\texttt{MLIST SIMP}) \, 1 \, 2 \, 7 \, (\texttt{MPLUS SIMP}) \, \$X \, \$Y)\]

5.4.2 Functions and Variables for Lists

[ ] mark the beginning and end, respectively, of a list. [ and ] also enclose the subscripts of a list, array, hash array, or array function. Note that other than for arrays accessing the \(n\)th element of a list may need an amount of time that is roughly proportional to \(n\), See Section 5.4.3 [Performance considerations for Lists], page 66.

Examples:

\[
\begin{align*}
\%i1) & \ x: [a, b, c]; \\
\%o1) & \ [a, b, c] \\
\%i2) & \ x[3]; \\
\%o2) & \ c \\
\%i3) & \ \text{array} (y, \text{fixnum}, 3); \\
\%o3) & \ y \\
\%i4) & \ y[2]: \%pi; \\
\%o4) & \ %pi \\
\%i5) & \ y[2]; \\
\%o5) & \ %pi \\
\%i6) & \ z[\text{'foo}]: \text{'bar}; \\
\%o6) & \ \text{bar} \\
\%i7) & \ z[\text{'foo}]; \\
\%o7) & \ \text{bar} \\
\%i8) & \ g[k] := 1/(k^2+1); \\
\%o8) & \ g := \frac{1}{k^2+1}
\end{align*}
\]
append (list_1, ..., list_n)  
Returns a single list of the elements of list_1 followed by the elements of list_2, ...
append also works on general expressions, e.g. append (f(a,b), f(c,d,e)); yields f(a,b,c,d,e).

See also addrow, addcol and join.
Do example(append); for an example.

assoc

assoc (key, list, default)
assoc (key, list)
This function searches for key in the left hand side of the input list. The list argument
should be a list, each of whose elements is an expression with exactly two parts. Most
usually, the elements of list are themselves lists, each with two elements.

The assoc function iterates along list, checking the first part of each element for
equality with key. If an element is found where the comparison is true, assoc returns
the second part of that element. If there is no such element in the list, assoc returns
either false or default, if given.

For example, in the expression assoc (y, [[x,1], [y,2], [z,3]]), the assoc function searches for x in the left hand side of the list [[y,1],[x,2]] and finds it at the
second term, returning 2. In assoc (z, [[x,1], [z,2], [z,3]]), the search stops
at the first term starting with z and returns 2. In assoc(x, [[y,1]]), there is no
matching element, so assoc returns false.

(%i1) assoc (y, [[x,1], [y,2],[z,3]]);
(%o1) 2
(%i2) assoc (z, [[x,1], [z,2], [z,3]]);
(%o2) 2
(%i3) assoc (x, [[y,1]]);
(%o3) false

cons

cons (expr, list)
cons (expr_1, expr_2)
cons (expr, list) returns a new list constructed of the element expr as its first
element, followed by the elements of list. This is analogous to the Lisp language
construction operation "cons".

The Maxima function cons can also be used where the second argument is other than a
list and this might be useful. In this case, cons (expr_1, expr_2) returns an expres-
sion with same operator as expr_2 but with argument cons(expr_1, args(expr_2)).

Examples:

(%i1) cons(a, [b, c, d]);
(%o1) [a, b, c, d]
In general, \emph{cons} applied to a nonlist doesn’t make sense. For instance, \emph{cons}(a, b^c) results in an illegal expression, since `^' cannot take three arguments.

When \emph{inflag} is true, \emph{cons} operates on the internal structure of an expression, otherwise \emph{cons} operates on the displayed form. Especially when \emph{inflag} is true, \emph{cons} applied to a nonlist sometimes gives a surprising result; for example:

\begin{verbatim}
(%i1) cons(a,-a), inflag : true;
     2
(%o1)       - a
(%i2) cons(a,-a), inflag : false;
(%o2)        0
\end{verbatim}

\textbf{copylist} \texttt{(list)}

Returns a copy of the list \texttt{list}.

\textbf{create_list} \texttt{(form, x_1, list_1, \ldots, x_n, list_n)}

Create a list by evaluating \texttt{form} with \texttt{x_1} bound to each element of \texttt{list_1}, and for each such binding bind \texttt{x_2} to each element of \texttt{list_2}, \ldots. The number of elements in the result will be the product of the number of elements in each list. Each variable \texttt{x_i} must actually be a symbol – it will not be evaluated. The list arguments will be evaluated once at the beginning of the iteration.

\begin{verbatim}
(%i1) create_list (x^i, i, [1, 3, 7]);
     3 7
(%o1)      [x, x , x ]
\end{verbatim}

With a double iteration:

\begin{verbatim}
(%i1) create_list ([i, j], i, [a, b], j, [e, f, h]);
(%o1)      [[a, e], [a, f], [a, h], [b, e], [b, f], [b, h]]
\end{verbatim}

Instead of \texttt{list_i} two args may be supplied each of which should evaluate to a number. These will be the inclusive lower and upper bounds for the iteration.

\begin{verbatim}
(%i1) create_list ([i, j], i, [1, 2, 3], j, 1, i);
(%o1)      [[1, 1], [2, 1], [2, 2], [3, 1], [3, 2], [3, 3]]
\end{verbatim}

Note that the limits or list for the \texttt{j} variable can depend on the current value of \texttt{i}.

\textbf{delete} \texttt{(expr_1, expr_2)}

\textbf{delete} \texttt{(expr_1, expr_2, n)}

\texttt{delete} \texttt{(expr_1, expr_2)} removes from \texttt{expr_2} any arguments of its top-level operator which are the same (as determined by `"=") as \texttt{expr_1}. Note that `"=" tests for formal equality, not equivalence. Note also that arguments of subexpressions are not affected.

\texttt{expr_1} may be an atom or a non-atomic expression. \texttt{expr_2} may be any non-atomic expression. \texttt{delete} returns a new expression; it does not modify \texttt{expr_2}.

\texttt{delete} \texttt{(expr_1, expr_2, n)} removes from \texttt{expr_2} the first \texttt{n} arguments of the top-level operator which are the same as \texttt{expr_1}. If there are fewer than \texttt{n} such arguments, then all such arguments are removed.
Examples:

Removing elements from a list.

```lisp
(%i1) delete (y, [w, x, y, z, z, y, x, w]);
(%o1) [w, x, z, z, x, w]
```

Removing terms from a sum.

```lisp
(%i1) delete (sin(x), x + sin(x) + y);
(%o1) y + x
```

Removing factors from a product.

```lisp
(%i1) delete (u - x, (u - w)*(u - x)*(u - y)*(u - z));
(%o1) (u - w) (u - y) (u - z)
```

Removing arguments from an arbitrary expression.

```lisp
(%i1) delete (a, foo (a, b, c, d, a));
(%o1) foo(b, c, d)
```

Limit the number of removed arguments.

```lisp
(%i1) delete (a, foo (a, b, a, c, d, a), 2);
(%o1) foo(b, c, d, a)
```

Whether arguments are the same as `expr_1` is determined by `"="`. Arguments which are equal but not `"="` are not removed.

```lisp
(%i1) [is (equal (0, 0)), is (equal (0, 0.0)), is (equal (0, 0b0))];
(%o1) [true, true, true]
```

```lisp
(%i2) [is (0 = 0), is (0 = 0.0), is (0 = 0b0)];
(%o2) [true, false, false]
```

```lisp
(%i3) delete (0, [0, 0.0, 0b0]);
(%o3) [0.0, 0.0b0]
```

```lisp
(%i4) is (equal ((x + y)*(x - y), x^2 - y^2));
(%o4) true
```

```lisp
(%i5) is ((x + y)*(x - y) = x^2 - y^2);
(%o5) false
```

```lisp
(%i6) delete ((x + y)*(x - y), [(x + y)*(x - y), x^2 - y^2]);
(%o6) [x - y ]
```

---

### eighth (expr)

Returns the 8'th item of expression or list `expr`. See `first` for more details.

### endcons

>`endcons (expr, list)`

>`endcons (expr_1, expr_2)`

`endcons (expr, list)` returns a new list constructed of the elements of `list` followed by `expr`. The Maxima function `endcons` can also be used where the second argument is other than a list and this might be useful. In this case, `endcons (expr_1, expr_2)` returns an expression with same operator as `expr_2` but with argument `endcons(expr_1, args(expr_2))`. Examples:

```lisp
(%i1) endcons(a,[b,c,d]);
(%o1) [b, c, d, a]
```
In general, \texttt{endcons} applied to a nonlist doesn't make sense. For instance, \texttt{endcons(a,b\textasciicircum{}c)} results in an illegal expression, since \textasciicircum{} cannot take three arguments.

When \texttt{inflag} is true, \texttt{endcons} operates on the internal structure of an expression, otherwise \texttt{endcons} operates on the displayed form. Especially when \texttt{inflag} is true, \texttt{endcons} applied to a nonlist sometimes gives a surprising result; for example

\begin{verbatim}
(%i1) endcons(a,-a), inflag : true;
   2
(%o1) - a
(%i2) endcons(a,-a), inflag : false;
(%o2) 0
\end{verbatim}

\textbf{fifth (expr)}

Returns the 5'th item of expression or list \texttt{expr}. See \texttt{first} for more details.

\textbf{first (expr)}

Returns the first part of \texttt{expr} which may result in the first element of a list, the first row of a matrix, the first term of a sum, etc.:

\begin{verbatim}
(%i1) matrix([1,2],[3,4]);
   [ 1  2 ]
   [     ]
   [ 3  4 ]
(%o1)

(%i2) first(%);
(%o2) [1,2]

(%i3) first(%);
(%o3) 1

(%i4) first(a\times{}b/c+d+e/x);
   a b
   ---
   c
(%o4)

(%i5) first(a=b/c+d+e/x);
   a
(%o5)
\end{verbatim}

Note that \texttt{first} and its related functions, \texttt{rest} and \texttt{last}, work on the form of \texttt{expr} which is displayed not the form which is typed on input. If the variable \texttt{inflag} is set to \texttt{true} however, these functions will look at the internal form of \texttt{expr}. One reason why this may make a difference is that the simplifier re-orders expressions:

\begin{verbatim}
(%i1) x+y;
  y+1
(%o1)

(%i2) first(x+y),inflag : true;
  x
(%o2)

(%i3) first(x+y),inflag : false;
  y
(%o3)
\end{verbatim}

The functions \texttt{second} \ldots \texttt{tenth} yield the second through the tenth part of their input argument.
See also `firstn` and `part`.

`firstn(expr, count)`

Returns the first `count` arguments of `expr`, if `expr` has at least `count` arguments. Returns `expr` if `expr` has less than `count` arguments.

`expr` may be any nonatomic expression. When `expr` is something other than a list, `firstn` returns an expression which has the same operator as `expr`. `count` must be a nonnegative integer.

`firstn` honors the global flag `inflag`, which governs whether the internal form of an expression is processed (when `inflag` is true) or the displayed form (when `inflag` is false).

Note that `firstn(expr, 1)`, which returns a nonatomic expression containing the first argument, is not the same as `first(expr)`, which returns the first argument by itself.

See also `lastn` and `rest`.

Examples:

`firstn` returns the first `count` elements of `expr`, if `expr` has at least `count` elements.

```maxima
(%i1) mylist : [1, a, 2, b, 3, x, 4 - y, 2*z + sin(u)];
(%o1) [1, a, 2, b, 3, x, 4 - y, 2 z + sin(u)]
(%i2) firstn (mylist, 0);
(%o2) []
(%i3) firstn (mylist, 1);
(%o3) [1]
(%i4) firstn (mylist, 2);
(%o4) [1, a]
(%i5) firstn (mylist, 7);
(%o5) [1, a, 2, b, 3, x, 4 - y]
```

`firstn` returns `expr` if `expr` has less than `count` elements.

```maxima
(%i1) mylist : [1, a, 2, b, 3, x, 4 - y, 2*z + sin(u)];
(%o1) [1, a, 2, b, 3, x, 4 - y, 2 z + sin(u)]
(%i2) firstn (mylist, 100);
(%o2) [1, a, 2, b, 3, x, 4 - y, 2 z + sin(u)]
```

`expr` may be any nonatomic expression.

```maxima
(%i1) myfoo : foo(1, a, 2, b, 3, x, 4 - y, 2*z + sin(u));
(%o1) foo(1, a, 2, b, 3, x, 4 - y, 2 z + sin(u))
(%i2) firstn (myfoo, 4);
(%o2) foo(1, a, 2, b)
(%i3) mybar : bar[m, n](1, a, 2, b, 3, x, 4 - y, 2*z + sin(u));
(%o3) bar (1, a, 2, b, 3, x, 4 - y, 2 z + sin(u))
    m, n
(%i4) firstn (mybar, 4);
(%o4) bar (1, a, 2, b)
    m, n
(%i5) mymatrix : genmatrix (lambda ([i, j], 10*i + j), 10, 4) $
```
(%i6) firstn (mymatrix, 3);
   [ 11 12 13 14 ]
   [ ]
(%o6)     [ 21 22 23 24 ]
   [ ]
   [ 31 32 33 34 ]

firstn honors the global flag inflag.
(%i1) myexpr : a + b + c + d + e;
(%o1)          e + d + c + b + a
(%i2) firstn (myexpr, 3), inflag=true;
(%o2)       c + b + a
(%i3) firstn (myexpr, 3), inflag=false;
(%o3)      e + d + c

Note that \texttt{firstn(expr, 1)} is not the same as \texttt{first(expr)}.
(%i1) firstn ([w, x, y, z], 1);
(%o1)        [w]
(%i2) first ([w, x, y, z]);
(%o2)        w

\textbf{fourth(expr)}

Returns the 4'\textsuperscript{th} item of expression or list expr. See \texttt{first} for more details.

\textbf{join(l, m)}

Creates a new list containing the elements of lists \texttt{l} and \texttt{m}, interspersed. The result has elements \texttt{[l[1], m[1], l[2], m[2], ...]}. The lists \texttt{l} and \texttt{m} may contain any type of elements.

If the lists are different lengths, \texttt{join} ignores elements of the longer list.

Maxima complains if \texttt{l} or \texttt{m} is not a list.

See also \texttt{append}.

Examples:

(%i1) L1: [a, sin(b), c!, d - 1];
(%o1)                      [a, sin(b), c!, d - 1]
(%i2) join (L1, [1, 2, 3, 4]);
(%o2)                  [a, 1, sin(b), 2, c!, 3, d - 1, 4]
(%i3) join (L1, [aa, bb, cc, dd, ee, ff]);
(%o3)              [a, aa, sin(b), bb, c!, cc, d - 1, dd]

\textbf{last(expr)}

Returns the last part (term, row, element, etc.) of the expr.

See also \texttt{lastn}.

\textbf{lastn(expr, count)}

Returns the last \texttt{count} arguments of expr, if expr has at least \texttt{count} arguments. Returns expr if expr has less than \texttt{count} arguments.

expr may be any nonatomic expression. When expr is something other than a list, \texttt{lastn} returns an expression which has the same operator as expr. \texttt{count} must be a nonnegative integer.
lastn honors the global flag inflag, which governs whether the internal form of an expression is processed (when inflag is true) or the displayed form (when inflag is false).

Note that lastn(expr, 1), which returns a nonatomic expression containing the last argument, is not the same as last(expr), which returns the last argument by itself. See also firstn and rest.

Examples:

lastn returns the last count elements of expr, if expr has at least count elements.

```
(%i1) mylist : [1, a, 2, b, 3, x, 4 - y, 2*z + sin(u)];
(%o1) [1, a, 2, b, 3, x, 4 - y, 2 z + sin(u)]
(%i2) lastn (mylist, 0);
(%o2) []
(%i3) lastn (mylist, 1);
(%o3) [2 z + sin(u)]
(%i4) lastn (mylist, 2);
(%o4) [4 - y, 2 z + sin(u)]
(%i5) lastn (mylist, 7);
(%o5) [a, 2, b, 3, x, 4 - y, 2 z + sin(u)]
```

lastn returns expr if expr has less than count elements.

```
(%i1) mylist : [1, a, 2, b, 3, x, 4 - y, 2*z + sin(u)];
(%o1) [1, a, 2, b, 3, x, 4 - y, 2 z + sin(u)]
(%i2) lastn (mylist, 100);
(%o2) [1, a, 2, b, 3, x, 4 - y, 2 z + sin(u)]
```

expr may be any nonatomic expression.

```
(%i1) myfoo : foo(1, a, 2, b, 3, x, 4 - y, 2*z + sin(u));
(%o1) foo(1, a, 2, b, 3, x, 4 - y, 2 z + sin(u))
(%i2) lastn (myfoo, 4);
(%o2) foo(3, x, 4 - y, 2 z + sin(u))
(%i3) mybar : bar[m, n](1, a, 2, b, 3, x, 4 - y, 2*z + sin(u));
(%o3) bar (1, a, 2, b, 3, x, 4 - y, 2 z + sin(u))
(%i4) lastn (mybar, 4);
(%o4) bar (3, x, 4 - y, 2 z + sin(u))
(%i5) mymatrix : genmatrix (lambda ([i, j], 10*i + j), 10, 4) $
(%i6) lastn (mymatrix, 3);
(%o6) [ 81 82 83 84 ]
    [              ]
    [              ]
    [ 91 92 93 94 ]
    [              ]
    [ 101 102 103 104 ]
```

lastn honors the global flag inflag.

```
(%i1) myexpr : a + b + c + d + e;
(%o1) e + d + c + b + a
```
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(%i2) lastn (myexpr, 3), inflag=true;
(%o2) e + d + c
(%i3) lastn (myexpr, 3), inflag=false;
(%o3) c + b + a

Note that lastn(expr, 1) is not the same as last(expr).

(%i1) lastn ([w, x, y, z], 1);
(%o1) [z]
(%i2) last ([w, x, y, z]);
(%o2) z

length (expr)  [Function]
Returns (by default) the number of parts in the external (displayed) form of expr. For lists this is the number of elements, for matrices it is the number of rows, and for sums it is the number of terms (see dispform).

The length command is affected by the inflag switch. So, e.g. length(a/(b*c)); gives 2 if inflag is false (Assuming exptdispflag is true), but 3 if inflag is true (the internal representation is essentially a*b^(-1)*c^(-1)).

Determining a list’s length typically needs an amount of time proportional to the number of elements in the list. If the length of a list is used inside a loop it therefore might drastically increase the performance if the length is calculated outside the loop instead.

listarith  [Option variable]
Default value: true
If false causes any arithmetic operations with lists to be suppressed; when true, list-matrix operations are contagious causing lists to be converted to matrices yielding a result which is always a matrix. However, list-list operations should return lists.

listp (expr)  [Function]
Returns true if expr is a list else false.

lreduce  [Function]

lreduce (F, s)
lreduce (F, s, s_0)
Extends the binary function F to an n-ary function by composition, where s is a list. lreduce(F, s) returns F(... F(F(s_1, s_2), s_3), ... s_n). When the optional argument s_0 is present, the result is equivalent to lreduce(F, cons(s_0, s)).

The function F is first applied to the leftmost list elements, thus the name "lreduce".

See also rreduce, xreduce, and tree_reduce.

Examples:
lreduce without the optional argument.

(%i1) lreduce (f, [1, 2, 3]);
(%o1) f(f(1, 2), 3)
(%i2) lreduce (f, [1, 2, 3, 4]);
(%o2) f(f(f(1, 2), 3), 4)
lreduce with the optional argument.

(\%i1) lreduce (f, [1, 2, 3], 4);
(\%o1) \(f(f(f(4, 1), 2), 3)\)

lreduce applied to built-in binary operators. / is the division operator.

(\%i1) lreduce ("-", args ({a, b, c, d}));
(\%o1) \((a)\)
(\%i2) lreduce ("/", args ({a, b, c, d}));
(\%o2) \(\frac{a}{b c d}\)

makelist

makelist ()
makelist (expr, n)
makelist (expr, i, i_max)
makelist (expr, i, i_0, i_max)
makelist (expr, i, i_0, i_max, step)
makelist (expr, x, list)

The first form, makelist (), creates an empty list. The second form, makelist (expr), creates a list with expr as its single element. makelist (expr, n) creates a list of n elements generated from expr.

The most general form, makelist (expr, i, i_0, i_max, step), returns the list of elements obtained when ev (expr, i=j) is applied to the elements j of the sequence: i_0, i_0 + step, i_0 + 2*step, ..., with |j| less than or equal to |i_max|.

The increment step can be a number (positive or negative) or an expression. If it is omitted, the default value 1 will be used. If both i_0 and step are omitted, they will both have a default value of 1.

makelist (expr, x, list) returns a list, the j'th element of which is equal to ev (expr, x=list[j]) for j equal to 1 through length (list).

Examples:

(\%i1) makelist (concat (x,i), i, 6);
(\%o1) \([x1, x2, x3, x4, x5, x6]\)
(\%i2) makelist (x=y, y, [a, b, c]);
(\%o2) \([x = a, x = b, x = c]\)
(\%i3) makelist (x^2, x, 3, 2*\%pi, 2);
(\%o3) \([9, 25]\)
(\%i4) makelist (random(6), 4);
(\%o4) \([2, 0, 2, 5]\)
(\%i5) flatten (makelist (makelist (i^2, 3), i, 4));
(\%o5) \([1, 1, 1, 4, 4, 4, 9, 9, 9, 9, 16, 16, 16]\)
(\%i6) flatten (makelist (makelist (i^2, i, 3), 4));
(\%o6) \([1, 4, 9, 1, 4, 9, 1, 4, 9, 1, 4, 9]\)
member (expr_1, expr_2)

Returns true if is(expr_1 = a) for some element a in args(expr_2), otherwise returns false.

expr_2 is typically a list, in which case args(expr_2) = expr_2 and is(expr_1 = a) for some element a in expr_2 is the test.

member does not inspect parts of the arguments of expr_2, so it may return false even if expr_1 is a part of some argument of expr_2.

See also elementp.

Examples:

(%i1) member (8, [8, 8.0, 8b0]);
   (%o1) true
(%i2) member (8, [8.0, 8b0]);
   (%o2) false
(%i3) member (b, [a, b, c]);
   (%o3) true
(%i4) member (b, [[a, b], [b, c]]);
   (%o4) false
(%i5) member ([b, c], [[a, b], [b, c]]);
   (%o5) true
(%i6) F (1, 1/2, 1/4, 1/8);
   1 1 1
   (%o6) F(1, -, -, -)
      2 4 8
(%i7) member (1/8, %);
   (%o7) true
(%i8) member ("ab", ["aa", "ab", sin(1), a + b]);
   (%o8) true

ninth (expr)

Returns the 9'th item of expression or list expr. See first for more details.

pop (list)

pop removes and returns the first element from the list list. The argument list must be a mapatom that is bound to a nonempty list. If the argument list is not bound to a nonempty list, Maxima signals an error. For examples, see push.

push (item, list)

push prepends the item item to the list list and returns a copy of the new list. The second argument list must be a mapatom that is bound to a list. The first argument item can be any Maxima symbol or expression. If the argument list is not bound to a list, Maxima signals an error.

To remove the first item from a list, see pop.

Examples:

(%i1) ll: [];
   (%o1) []
(%i2) push (x, ll);
   (%o2) [x]
(%i3) push (x^2+y, ll);
   2
(%o3)     [y + x , x]
(%i4) a: push ("string", ll);
   2
(%o4)    [string, y + x , x]
(%i5) pop (ll);                       string
(%o5)                  
(%i6) pop (ll);                       2
(%o6)                       y + x
(%i7) pop (ll);                         x
(%i8) ll;                                 []
(%i9) a;                                      2
(%o9)    [string, y + x , x]

rest

rest (expr, n)
rest (expr)

Returns expr with its first n elements removed if n is positive and its last - n elements removed if n is negative. If n is 1 it may be omitted. The first argument expr may be a list, matrix, or other expression. When expr is an atom, rest signals an error; when expr is an empty list and partswitch is false, rest signals an error. When expr is an empty list and partswitch is true, rest returns end.

Applying rest to expression such as f(a,b,c) returns f(b,c). In general, applying rest to an nonlist doesn’t make sense. For example, because ‘~’ requires two arguments, rest(a~b) results in an error message. The functions args and op may be useful as well, since args(a~b) returns [a,b] and op(a~b) returns ~.

See also firstn and lastn.

(%i10) rest(a+b+c);
   1
(%o10)                    b+a
(%i11) rest(a+b+c,2);
   1
(%o11)                      a
(%i12) rest(a+b+c,-2);
   1
(%o12)                    c

reverse (list)

Reverses the order of the members of the list (not the members themselves). reverse also works on general expressions, e.g. reverse(a=b); gives b=a.

rreduce

rreduce (F, s)
rreduce (F, s, s_{n + 1})

Extends the binary function F to an n-ary function by composition, where s is a list.
\textbf{rreduce}(F, s) \text{ returns } F(s_{\cdot 1}, \ldots F(s_{\cdot n-2}, F(s_{\cdot n-1}, s_n))). \text{ When the optional argument } s_{\cdot n+1} \text{ is present, the result is equivalent to } \textbf{rreduce}(F, \text{endcons}(s_{\cdot n+1}, s)).

The function \textit{F} is first applied to the \textit{rightmost} list elements, thus the name "rreduce".

See also \textbf{lreduce}, \textbf{tree\_reduce}, and \textbf{xreduce}.

Examples:

\textbf{rreduce} without the optional argument.

\begin{verbatim}
(%i1) rreduce (f, [1, 2, 3]);
\end{verbatim}
\begin{verbatim}
(%o1) f(1, f(2, 3))
\end{verbatim}

\begin{verbatim}
(%i2) rreduce (f, [1, 2, 3, 4]);
\end{verbatim}
\begin{verbatim}
(%o2) f(1, f(2, f(3, 4)))
\end{verbatim}

\textbf{rreduce} with the optional argument.

\begin{verbatim}
(%i1) rreduce (f, [1, 2, 3], 4);
\end{verbatim}
\begin{verbatim}
(%o1) f(1, f(2, f(3, 4)))
\end{verbatim}

\textbf{rreduce} applied to built-in binary operators. / is the division operator.

\begin{verbatim}
(%i1) rreduce ("^", args ([a, b, c, d]));
\end{verbatim}
\begin{verbatim}
d
\end{verbatim}
\begin{verbatim}
c
\end{verbatim}
\begin{verbatim}
b
\end{verbatim}
\begin{verbatim}
a
\end{verbatim}
\begin{verbatim}
(%o1) a
\end{verbatim}

\begin{verbatim}
(%i2) rreduce ("/", args ([a, b, c, d]));
\end{verbatim}
\begin{verbatim}
a
\end{verbatim}
\begin{verbatim}
c
\end{verbatim}
\begin{verbatim}
---
\end{verbatim}
\begin{verbatim}
b
\end{verbatim}
\begin{verbatim}
d
\end{verbatim}
\begin{verbatim}
(%o2) ---
\end{verbatim}

\textbf{second} (\texttt{expr}) \hspace{1cm} \textbf{[Function]}

Returns the 2'nd item of expression or list \texttt{expr}. See \textbf{first} for more details.

\textbf{seventh} (\texttt{expr}) \hspace{1cm} \textbf{[Function]}

Returns the 7'th item of expression or list \texttt{expr}. See \textbf{first} for more details.

\textbf{sixth} (\texttt{expr}) \hspace{1cm} \textbf{[Function]}

Returns the 6'th item of expression or list \texttt{expr}. See \textbf{first} for more details.

\textbf{sort} \hspace{1cm} \textbf{[Function]}

\begin{verbatim}
sort (L, P)
\end{verbatim}
\begin{verbatim}
sort (L)
\end{verbatim}

\textbf{sort}(L, P) sorts a \textit{L} according to a predicate \textit{P} of two arguments which defines a strict weak order on the elements of \textit{L}. If \textit{P(a, b)} is \textit{true}, then \textit{a} appears before \textit{b} in the result. If neither \textit{P(a, b)} nor \textit{P(b, a)} are \textit{true}, then \textit{a} and \textit{b} are equivalent, and appear in the result in the same order as in the input. That is, sort is a stable sort.

If \textit{P(a, b)} and \textit{P(b, a)} are both \textit{true} for some elements of \textit{L}, then \textit{P} is not a valid sort predicate, and the result is undefined. If \textit{P(a, b)} is something other than \textit{true} or \textit{false}, \textbf{sort} signals an error.
The predicate may be specified as the name of a function or binary infix operator, or as a `lambda` expression. If specified as the name of an operator, the name must be enclosed in double quotes.

The sorted list is returned as a new object; the argument \( L \) is not modified.

\[ \text{sort}(L) \text{ is equivalent to sort}(L, \text{orderlessp}). \]

The default sorting order is ascending, as determined by `orderlessp`. The predicate `ordergreatp` sorts a list in descending order.

All Maxima atoms and expressions are comparable under `orderlessp` and `ordergreatp`.

Operators `<` and `>` order numbers, constants, and constant expressions by magnitude. Note that `orderlessp` and `ordergreatp` do not order numbers, constants, and constant expressions by magnitude.

`ordermagnitudep` orders numbers, constants, and constant expressions the same as `<`, and all other elements the same as `orderlessp`.

Examples:

`sort` sorts a list according to a predicate of two arguments which defines a strict weak order on the elements of the list.

\[
(\%i1) \quad \text{sort ([1, a, b, 2, 3, c], 'orderlessp);}
\]
\[
(\%o1) \quad [1, 2, 3, a, b, c]
\]

\[
(\%i2) \quad \text{sort ([1, a, b, 2, 3, c], 'ordergreatp);}
\]
\[
(\%o2) \quad [c, b, a, 3, 2, 1]
\]

The predicate may be specified as the name of a function or binary infix operator, or as a `lambda` expression. If specified as the name of an operator, the name must be enclosed in double quotes.

\[
(\%i1) \quad L : [[1, x], [3, y], [4, w], [2, z]]; \quad (\%o1) \quad [[1, x], [3, y], [4, w], [2, z]]
\]

\[
(\%i2) \quad \text{foo (a, b) := a[1] > b[1];}
\]
\[
(\%o2) \quad \text{foo(a, b) := a > b}
\]

\[
(\%i3) \quad \text{sort (L, 'foo);} \quad (\%o3) \quad [[4, w], [3, y], [2, z], [1, x]]
\]

\[
(\%i4) \quad \text{infix (">>()};
\]
\[
(\%o4) \quad >>
\]

\[
(\%i5) \quad a >> b := a[1] > b[1]; \quad (\%o5) \quad (a >> b) := a > b
\]

\[
(\%i6) \quad \text{sort (L, ">>()};
\]
\[
(\%o6) \quad [[4, w], [3, y], [2, z], [1, x]]
\]

\[
(\%i7) \quad \text{sort (L, lambda ([a, b], a[1] > b[1]));}
\]
\[
(\%o7) \quad [[4, w], [3, y], [2, z], [1, x]]
\]

\[ \text{sort}(L) \text{ is equivalent to sort}(L, \text{orderlessp}). \]

\[
(\%i1) \quad L : [a, 2*b, -5, 7, 1 + %e, %pi]; \quad (\%o1) \quad [a, 2 b, - 5, 7, \%e + 1, \%pi]
\]
The default sorting order is ascending, as determined by orderlessp. The predicate ordergreatp sorts a list in descending order.

All Maxima atoms and expressions are comparable under orderlessp and ordergreatp.

Operators < and > order numbers, constants, and constant expressions by magnitude. Note that orderlessp and ordergreatp do not order numbers, constants, and constant expressions by magnitude.

ordermagnitudep orders numbers, constants, and constant expressions the same as <, and all other elements the same as orderlessp.
sort (L, orderlessp);  
[0, 1, 1.0, 2, 3, sin(1), 1.0b0, %e, %i, %i + 1, inf, minf, 2 x]

sublist (list, p)
Returns the list of elements of list for which the predicate p returns true.

Example:
L: [1, 2, 3, 4, 5, 6];
sublist (L, evenp);
[2, 4, 6]

sublist_indices (L, P)
Returns the indices of the elements x of the list L for which the predicate maybe(P(x)) returns true; this excludes unknown as well as false. P may be the name of a function or a lambda expression. L must be a literal list.

Examples:
sublist_indices ('[a, b, b, c, 1, 2, b, 3, b], lambda ([x], x='b));
[2, 3, 7, 9]

sublist_indices ('[a, b, b, c, 1, 2, b, 3, b], symbolp);
[1, 2, 3, 4, 7, 9]

sublist_indices ([1 > 0, 1 < 0, 2 < 1, 2 > 1, 2 > 0], identity);
[1, 4, 5]

assume (x < -1);
[x < - 1]

map (maybe, [x > 0, x < 0, x < -2]);
[false, true, unknown]

sublist_indices ([x > 0, x < 0, x < -2], identity);
[2]

tenth (expr)
Returns the 10'th item of expression or list expr. See first for more details.

third (expr)
Returns the 3'rd item of expression or list expr. See first for more details.

tree_reduce

tree_reduce (F, s)
tree_reduce (F, s, s_0)
Extends the binary function F to an n-ary function by composition, where s is a set or list.
tree_reduce is equivalent to the following: Apply F to successive pairs of elements to form a new list [F(s_1, s_2), F(s_3, s_4), ...], carrying the final element unchanged if there are an odd number of elements. Then repeat until the list is reduced to a single element, which is the return value.
When the optional argument $s_0$ is present, the result is equivalent $\text{tree\_reduce}(F, \text{cons}(s_0, s))$.

For addition of floating point numbers, $\text{tree\_reduce}$ may return a sum that has a smaller rounding error than either $\text{rreduce}$ or $\text{lreduce}$.

The elements of $s$ and the partial results may be arranged in a minimum-depth binary tree, thus the name "tree\_reduce".

Examples:

$\text{tree\_reduce}$ applied to a list with an even number of elements.

$$\begin{align*}
(\%i1) \quad & \text{tree\_reduce} (f, [a, b, c, d]); \\
(\%o1) \quad & f(f(a, b), f(c, d))
\end{align*}$$

$\text{tree\_reduce}$ applied to a list with an odd number of elements.

$$\begin{align*}
(\%i1) \quad & \text{tree\_reduce} (f, [a, b, c, d, e]); \\
(\%o1) \quad & f(f(f(a, b), f(c, d)), e)
\end{align*}$$

unique $(L)$

Returns the unique elements of the list $L$.

When all the elements of $L$ are unique, unique returns a shallow copy of $L$, not $L$ itself.

If $L$ is not a list, unique returns $L$.

Example:

$$\begin{align*}
(\%i1) \quad & \text{unique ([1, \%pi, a + b, 2, 1, \%e, \%pi, a + b, [1]])); \\
(\%o1) \quad & [1, 2, \%e, \%pi, [1], b + a]
\end{align*}$$

xreduce

$xreduce (F, s)$

$xreduce (F, s, s_0)$

Extends the function $F$ to an n-ary function by composition, or, if $F$ is already n-ary, applies $F$ to $s$. When $F$ is not n-ary, $xreduce$ is the same as $\text{lreduce}$. The argument $s$ is a list.

Functions known to be n-ary include addition $+$, multiplication $\ast$, and $\text{append}$. Functions may also be declared n-ary by $\text{declare}(F, \text{nary})$. For these functions, $xreduce$ is expected to be faster than either $\text{rreduce}$ or $\text{lreduce}$.

When the optional argument $s_0$ is present, the result is equivalent to $xreduce(s, \text{cons}(s_0, s))$.

Floating point addition is not exactly associative; be that as it may, $xreduce$ applies Maxima’s n-ary addition when $s$ contains floating point numbers.

Examples:

$xreduce$ applied to a function known to be n-ary. $F$ is called once, with all arguments.

$$\begin{align*}
(\%i1) \quad & \text{declare} (F, \text{nary}); \\
(\%o1) \quad & \text{done} \\
(\%i2) \quad & F ([L]) := L; \\
(\%o2) \quad & F([L]) := L \\
(\%i3) \quad & xreduce (F, [a, b, c, d, e]); \\
(\%o3) \quad & [a, b, c, d, e]
\end{align*}$$
xreduce applied to a function not known to be n-ary. G is called several times, with two arguments each time.

\[
\begin{align*}
(\%i1) & \quad G([L]) := L; \\
(\%o1) & \quad G([L]) := L \\
(\%i2) & \quad xreduce (G, [a, b, c, d, e]); \\
(\%o2) & \quad [[[a, b], c], d], e] \\
(\%i3) & \quad lreduce (G, [a, b, c, d, e]); \\
(\%o3) & \quad [[[a, b], c], d], e]
\end{align*}
\]

### 5.4.3 Performance considerations for Lists

Lists provide efficient ways of appending and removing elements. They can be created without knowing their final dimensions. Lisp provides efficient means of copying and handling lists. Also nested lists do not need to be strictly rectangular. These advantages over declared arrays come with the drawback that the amount of time needed for accessing a random element within a list may be roughly proportional to the element's distance from its beginning. Efficient traversal of lists is still possible, though, by using the list as a stack or a fifo:

\[
\begin{align*}
(\%i1) & \quad 1: [Test, 1, 2, 3, 4]; \\
(\%o1) & \quad [Test, 1, 2, 3, 4] \\
(\%i2) & \quad \text{while } 1 \neq [] \text{ do} \\
& \quad \quad \text{disp(pop}(1)); \\
& \quad \quad \text{disp}(1) \\
& \quad \quad \text{disp}() \\
& \quad \quad \text{disp}(1) \\
& \quad \quad \text{disp}(1) \\
(\%o2) & \quad \text{done}
\end{align*}
\]

Another even faster example would be:

\[
\begin{align*}
(\%i1) & \quad 1: [Test, 1, 2, 3, 4]; \\
(\%o1) & \quad [Test, 1, 2, 3, 4] \\
(\%i2) & \quad \text{for } i \text{ in } 1 \text{ do} \\
& \quad \quad \text{disp(pop}(1)); \\
& \quad \quad \text{disp}(1) \\
& \quad \quad \text{disp}(1) \\
& \quad \quad \text{disp}(1) \\
& \quad \quad \text{disp}(1) \\
(\%o2) & \quad \text{done}
\end{align*}
\]
Beginning traversal with the last element of a list is possible after reversing the list using `reverse()`. If the elements of a long list need to be processed in a different order, performance might be increased by converting the list into a declared array first.

It is also to note that the ending condition of `for` loops is tested for every iteration which means that the result of a `length` should be cached if it is used in the ending condition:

```lisp
(%i1) l: makelist(i, i, 1, 100000)
(%i2) length(l);
(%o2) 100000
(%i3) x: 1;
(%o3) 1
(%i4) for i: 1 thru length do
    x: x + 1$
(%i5) x;
(%o5) 100001
```
5.5 Arrays

5.5.1 Functions and Variables for Arrays

array

array (name, dim_1, ..., dim_n)
array (name, type, dim_1, ..., dim_n)
array ([name_1, ..., name_m], dim_1, ..., dim_n)

Creates an n-dimensional array. n may be less than or equal to 5. The subscripts for the i'th dimension are the integers running from 0 to dim_i.

array (name, dim_1, ..., dim_n) creates a general array.

array (name, type, dim_1, ..., dim_n) creates an array, with elements of a specified type. type can be fixnum for integers of limited size or flonum for floating-point numbers.

array ([name_1, ..., name_m], dim_1, ..., dim_n) creates m arrays, all of the same dimensions.

If the user assigns to a subscripted variable before declaring the corresponding array, an undeclared array is created. Undeclared arrays, otherwise known as hashed arrays (because hash coding is done on the subscripts), are more general than declared arrays. The user does not declare their maximum size, and they grow dynamically by hashing as more elements are assigned values. The subscripts of undeclared arrays need not even be numbers. However, unless an array is rather sparse, it is probably more efficient to declare it when possible than to leave it undeclared. The array function can be used to transform an undeclared array into a declared array.

arrayapply (A, [i_1, ..., i_n])

Evaluates A [i_1, ..., i_n], where A is an array and i_1, ..., i_n are integers.

This is reminiscent of apply, except the first argument is an array instead of a function.

arrayinfo (A)

Returns information about the array A. The argument A may be a declared array, an undeclared (hashed) array, an array function, or a subscripted function.

For declared arrays, arrayinfo returns a list comprising the atom declared, the number of dimensions, and the size of each dimension. The elements of the array, both bound and unbound, are returned by listarray.

For undeclared arrays (hashed arrays), arrayinfo returns a list comprising the atom hashed, the number of subscripts, and the subscripts of every element which has a value. The values are returned by listarray.

For array functions, arrayinfo returns a list comprising the atom hashed, the number of subscripts, and any subscript values for which there are stored function values. The stored function values are returned by listarray.

For subscripted functions, arrayinfo returns a list comprising the atom hashed, the number of subscripts, and any subscript values for which there are lambda expressions. The lambda expressions are returned by listarray.

See also listarray.
Examples:

arrayinfo and listarray applied to a declared array.

```lisp
(%i1) array (aa, 2, 3);
(%o1) aa
(%i2) aa [2, 3] : %pi;
(%o2) %pi
(%i3) aa [1, 2] : %e;
(%o3) %e
(%i4) arrayinfo (aa);
(%o4) [declared, 2, [2, 3]]
(%i5) listarray (aa);
(%o5) [#####, #####, #####, #####, #####, #####, %e, #####, #####, #####, #####, %pi]
```

arrayinfo and listarray applied to an undeclared (hashed) array.

```lisp
(%i1) bb [FOO] : (a + b)^2;
2
(%o1) (b + a)
(%i2) bb [BAR] : (c - d)^3;
3
(%i3) arrayinfo (bb);
(%o3) [hashed, 1, [BAR], [FOO]]
(%i4) listarray (bb);
(%o4) [(c - d) , (b + a) ]
```

arrayinfo and listarray applied to an array function.

```lisp
(%i1) cc [x, y] := y / x;
y
(%o1) cc := -
x, y
(%i2) cc [u, v];
(v
(%i3) cc [4, z];
(z
(%i4) arrayinfo (cc);
(%o4) [hashed, 2, [4, z], [u, v]]
(%i5) listarray (cc);
(%o5) [[- , - ]
4 u
```

arrayinfo and listarray applied to a subscripted function.
(\%i1) dd [x] (y) := y ^ x;

(\%o1) \hspace{1cm} dd (y) := y^x

(\%i2) dd [a + b];

(\%o2) lambda([y], y )

(\%i3) dd [v - u];

(\%o3) lambda([y], y )

(\%i4) arrayinfo (dd);

(\%o4) [hashed, 1, [b + a], [v - u]]

(\%i5) listarray (dd);

(\%o5) [lambda([y], y ), lambda([y], y )]

arraymake (A, [i_1, \ldots, i_n])

[Function]

Returns the expression $A[i_1, \ldots, i_n]$. The result is an unevaluated array reference.

arraymake is reminiscent of funmake, except the return value is an unevaluated array reference instead of an unevaluated function call.

Examples:

(\%i1) arraymake (A, [1]);

(\%o1) A

(\%i2) arraymake (A, [k]);

(\%o2) A

(\%i3) arraymake (A, [i, j, 3]);

(\%o3) A

(\%i4) array (A, fixnum, 10);

(\%o4) A

(\%i5) fillarray (A, makelist (i^2, i, 1, 11));

(\%o5) A

(\%i6) arraymake (A, [5]);

(\%o6) A

(\%i7) ''%;

(\%o7) 36

(\%i8) L : [a, b, c, d, e];

(\%o8) [a, b, c, d, e]

(\%i9) arraymake ('L, [n]);

(\%o9) L

(\%i10) ''%, n = 3;

(\%o10) c
Chapter 5: Data Types and Structures

arrays  
Default value: []
arrays is a list of arrays that have been allocated. These comprise arrays declared by array, hashed arrays constructed by implicit definition (assigning something to an array element), and array functions defined by := and define. Arrays defined by make_array are not included.
See also array, arrayapply, arrayinfo, arraymake, fillarray, listarray, and rearray.
Examples:

(array (aa, 5, 7);
  aa
(b [F001] : (a + b)^2;
  2
  (b + a)
(cc [x] := x/100;
  x
  cc := ---
  x 100
(dd : make_array ('any, 7);
  {Array: #(NIL NIL NIL NIL NIL NIL NIL)}
(arrays;
  [aa, bb, cc]
arraysetapply (A, [i_1, ..., i_n], x)  
Assigns x to A[i_1, ..., i_n], where A is an array and i_1, ..., i_n are integers.
arraysetapply evaluates its arguments.
fillarray (A, B)  
Fills array A from B, which is a list or an array.
If a specific type was declared for A when it was created, it can only be filled with elements of that same type; it is an error if an attempt is made to copy an element of a different type.
If the dimensions of the arrays A and B are different, A is filled in row-major order. If there are not enough elements in B the last element is used to fill out the rest of A. If there are too many, the remaining ones are ignored.
fillarray returns its first argument.
Examples:

Create an array of 9 elements and fill it from a list.

```lisp
(%i1) array (a1, fixnum, 8);
(%o1) a1
(%i2) listarray (a1);
(%o2) [0, 0, 0, 0, 0, 0, 0, 0]
(%i3) fillarray (a1, [1, 2, 3, 4, 5, 6, 7, 8, 9]);
(%o3) a1
(%i4) listarray (a1);
(%o4) [1, 2, 3, 4, 5, 6, 7, 8, 9]
```

When there are too few elements to fill the array, the last element is repeated. When there are too many elements, the extra elements are ignored.

```lisp
(%i1) a2 : make_array (fixnum, 8);
(%o1) {Array: #(0 0 0 0 0 0 0 0)}
(%i2) fillarray (a2, [1, 2, 3, 4, 5]);
(%o2) {Array: #(1 2 3 4 5 5 5 5)}
(%i3) fillarray (a2, [4]);
(%o3) {Array: #(4 4 4 4 4 4 4 4)}
(%i4) fillarray (a2, makelist (i, i, 1, 100));
(%o4) {Array: #(1 2 3 4 5 6 7 8)}
```

Multiple-dimension arrays are filled in row-major order.

```lisp
(%i1) a3 : make_array (fixnum, 2, 5);
(%o1) {Array: #2A((0 0 0 0 0) (0 0 0 0 0))}
(%i2) fillarray (a3, [1, 2, 3, 4, 5, 6, 7, 8, 9, 10]);
(%o2) {Array: #2A((1 2 3 4 5) (6 7 8 9 10))}
(%i3) a4 : make_array (fixnum, 5, 2);
(%o3) {Array: #2A((0 0) (0 0) (0 0) (0 0))}
(%i4) fillarray (a4, a3);
(%o4) {Array: #2A((1 2) (3 4) (5 6) (7 8) (9 10))}
```

**listarray**

Returns a list of the elements of the array A. The argument A may be a declared array, an undeclared (hashed) array, an array function, or a subscripted function.

Elements are listed in row-major order. That is, elements are sorted according to the first index, then according to the second index, and so on. The sorting order of index values is the same as the order established by `orderless`.

For undeclared arrays, array functions, and subscripted functions, the elements correspond to the index values returned by `arrayinfo`.

Unbound elements of declared general arrays (that is, not `fixnum` and not `flonum`) are returned as `#####`. Unbound elements of declared `fixnum` or `flonum` arrays are returned as 0 or 0.0, respectively. Unbound elements of undeclared arrays, array functions, and subscripted functions are not returned.

Examples:

`listarray` and `arrayinfo` applied to a declared array.
(%i1) array (aa, 2, 3);
(%o1) aa
(%i2) aa [2, 3] : %pi;
(%o2) %pi
(%i3) aa [1, 2] : %e;
(%o3) %e
(%i4) listarray (aa);
(%o4) [#####, #####, #####, #####, #####, #####, %e, #####, #####, #####, %pi]
(%i5) arrayinfo (aa);
(%o5) [declared, 2, [2, 3]]

listarray and arrayinfo applied to an undeclared (hashed) array.

(%i1) bb [FOO] : (a + b)^2;
(%o1) (b + a)
(%i2) bb [BAR] : (c - d)^3;
(%o2) (c - d)
(%i3) listarray (bb);
(%o3) [(c - d), (b + a)]
(%i4) arrayinfo (bb);
(%o4) [hashed, 1, [BAR], [FOO]]

listarray and arrayinfo applied to an array function.

(%i1) cc [x, y] := y / x;
(%o1) cc := -
    x, y x
(%i2) cc [u, v];
(%o2) -
    u
(%i3) cc [4, z];
(%o3) -
    4
(%i4) listarray (cc);
(%o4) [z, v]
    [-, -]
    4 u
(%i5) arrayinfo (cc);
(%o5) [hashed, 2, [4, z], [u, v]]

listarray and arrayinfo applied to a subscripted function.
(%i1) dd [x] (y) := y ^ x;
     x
(%o1) dd (y) := y
     x
(%i2) dd [a + b];
     b + a
(%o2) lambda([y], y )
(%i3) dd [v - u];
     v - u
(%o3) lambda([y], y )
(%i4) listarray (dd);
     b + a   v - u
(%o4) [lambda([y], y ), lambda([y], y )]
(%i5) arrayinfo (dd);
     [hashed, 1, [b + a], [v - u]]

make_array (type, dim_1, ..., dim_n)                           [Function]
Creates and returns a Lisp array. type may be any, flonum, fixnum, hashed or
functional. There are n indices, and the i'th index runs from 0 to dim_i − 1.

The advantage of make_array over array is that the return value doesn’t have a
name, and once a pointer to it goes away, it will also go away. For example, if y:
make_array (...) then y points to an object which takes up space, but after y:
false, y no longer points to that object, so the object can be garbage collected.

Examples:

(%i1) A1 : make_array (fixnum, 10);
(%o1) {Array: #(0 0 0 0 0 0 0 0 0 0)}
(%o2) 1729
(%i3) A1;
(%o3) {Array: #(0 0 0 0 0 0 0 1729 0 0)}
(%i4) A2 : make_array (flonum, 10);
(%o4) {Array: #(0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0)}
(%o5) 2.718281828
(%i6) A2;
(%o6) {Array: #(0.0 0.0 2.718281828 0.0 0.0 0.0 0.0 0.0 0.0 0.0)}
(%i7) A3 : make_array (any, 10);
(%o7) {Array: #(NIL NIL NIL NIL NIL NIL NIL NIL NIL NIL)}
(%i8) A3 [4] : x - y - z;
(%o8) - z - y + x
(%i9) A3;
(%o9) {Array: #(NIL NIL NIL NIL ((MPLUS SIMP) $X ((MTIMES SIMP) -1 $Y) ((MTIMES SIMP) -1 $Z)) NIL NIL NIL NIL NIL)}
Chapter 5: Data Types and Structures

(%i10) A4 : make_array (fixnum, 2, 3, 5);
(%o10) {Array: #3A(((0 0 0 0 0) (0 0 0 0 0) (0 0 0 0 0)) ((0 0 \
   0 0 0) (0 0 0 0 0) (0 0 0 0 0)))}
(%i11) fillarray (A4, makelist (i, i, 1, 2*3*5));
(%o11) {Array: #3A(((1 2 3 4 5) (6 7 8 9 10) (11 12 13 14 15))
   ((16 17 18 19 20) (21 22 23 24 25) (26 27 28 29 30)))}
(%i12) A4 [0, 2, 1];
(%o12) 12

rearray (A, dim_1, ... , dim_n)
Changes the dimensions of an array. The new array will be filled with the elements of the old one in row-major order. If the old array was too small, the remaining elements are filled with false, 0.0 or 0, depending on the type of the array. The type of the array cannot be changed.

remarray
remarray (A_1, ... , A_n)
remarray (all)
Removes arrays and array associated functions and frees the storage occupied. The arguments may be declared arrays, undeclared (hashed) arrays, array functions, and subscripted functions.
remarray (all) removes all items in the global list arrays.
It may be necessary to use this function if it is desired to redefine the values in a hashed array.
remarray returns the list of arrays removed.
remarray quotes its arguments.

subvar (x, i)
Evaluated the subscripted expression x[i].
subvar evaluates its arguments.

arraymake (x, [i]) constructs the expression x[i], but does not evaluate it.

Examples:
(%i11) x : foo $
(%i12) i : 3 $
(%i13) subvar (x, i);
(%o13) foo
3
(%i14) foo : [aa, bb, cc, dd, ee]$
(%i15) subvar (x, i);
(%o15) cc
(%i16) arraymake (x, [i]);
(%o16) foo
3
(%i17) '',';
(%o17) cc
**subvarp (expr)**  

Returns `true` if `expr` is a subscripted variable, for example `a[i]`.

**use_fast_arrays**  

Default value: `false`  

When `use_fast_arrays` is `true`, undeclared arrays and arrays declared by `array` are values instead of properties, and undeclared arrays are implemented as Lisp hash tables.  

When `use_fast_arrays` is `false`, undeclared arrays and arrays declared by `array` are properties, and undeclared arrays are implemented with Maxima’s own hash table implementation.  

Arrays created by `make_array` are not affected by `use_fast_arrays`.  

See also `translate_fast_arrays`.

**translate_fast_arrays**  

Default value: `false`  

When `translate_fast_arrays` is `true`, the Maxima-to-Lisp translator generates code that assumes arrays are values instead of properties, as if `use_fast_arrays` were `true`.  

When `translate_fast_arrays` is `false`, the Maxima-to-Lisp translator generates code that assumes arrays are properties, as if `use_fast_arrays` were `false`. 
5.6 Structures

5.6.1 Introduction to Structures

Maxima provides a simple data aggregate called a structure. A structure is an expression in which arguments are identified by name (the field name) and the expression as a whole is identified by its operator (the structure name). A field value can be any expression.

A structure is defined by the `defstruct` function; the global variable `structures` is the list of user-defined structures. The function `new` creates instances of structures. The `@` operator refers to fields. `kill(S)` removes the structure definition for `S`, and `kill(x@a)` unbinds the field `a` of the structure instance `x`.

In the pretty-printing console display (with `display2d` equal to `true`), structure instances are displayed with the value of each field represented as an equation, with the field name on the left-hand side and the value on the right-hand side. (The equation is only a display construct; only the value is actually stored.) In 1-dimensional display (via `grind` or with `display2d` equal to `false`), structure instances are displayed without the field names.

There is no way to use a field name as a function name, although a field value can be a lambda expression. Nor can the values of fields be restricted to certain types; any field can be assigned any kind of expression. There is no way to make some fields accessible or inaccessible in different contexts; all fields are always visible.

5.6.2 Functions and Variables for Structures

`structures` [Global variable]

`structures` is the list of user-defined structures defined by `defstruct`.

`defstruct` [Function]

```
defstruct (S(a_1, . . ., a_n))
defstruct (S(a_1 = v_1, . . ., a_n = v_n))
```

Define a structure, which is a list of named fields `a_1, . . ., a_n` associated with a symbol `S`. An instance of a structure is just an expression which has operator `S` and exactly `n` arguments. `new(S)` creates a new instance of structure `S`.

An argument which is just a symbol `a` specifies the name of a field. An argument which is an equation `a = v` specifies the field name `a` and its default value `v`. The default value can be any expression.

`defstruct` puts `S` on the list of user-defined structures, `structures`.

`kill(S)` removes `S` from the list of user-defined structures, and removes the structure definition.

Examples:

```
(%i1) defstruct (foo (a, b, c));
(%o1) [foo(a, b, c)]
(%i2) structures;
(%o2) [foo(a, b, c)]
(%i3) new (foo);
(%o3) foo(a, b, c)
(%i4) defstruct (bar (v, w, x = 123, y = %pi));
```
\[
\text{(%o4)} \quad [\text{bar}(v, w, x = 123, y = \pi)]
\]

\[
\text{(%i5)} \quad \text{structures;}
\]

\[
\text{(%o5)} \quad [\text{foo}(a, b, c), \text{bar}(v, w, x = 123, y = \pi)]
\]

\[
\text{(%i6)} \quad \text{new (bar)};
\]

\[
\text{(%o6)} \quad \text{bar}(v, w, x = 123, y = \pi)
\]

\[
\text{(%i7)} \quad \text{kill (foo)};
\]

\[
\text{(%o7)} \quad \text{done}
\]

\[
\text{(%i8)} \quad \text{structures;}
\]

\[
\text{(%o8)} \quad [\text{bar}(v, w, x = 123, y = \pi)]
\]

\[
\text{new}
\]

\[
\text{new (S)}
\]

\[
\text{new (S(v_1, ..., v_n))}
\]

\text{new} creates new instances of structures.

\text{new(S)} creates a new instance of structure \( S \) in which each field is assigned its default value, if any, or no value at all if no default was specified in the structure definition.

\text{new(S(v_1, ..., v_n))} creates a new instance of \( S \) in which fields are assigned the values \( v_1, ..., v_n \).

Examples:

\[
\text{(%i1)} \quad \text{defstruct (foo (w, x = %e, y = 42, z));}
\]

\[
\text{(%o1)} \quad [\text{foo}(w, x = \pi, y = 42, z)]
\]

\[
\text{(%i2)} \quad \text{new (foo)};
\]

\[
\text{(%o2)} \quad \text{foo}(w, x = \pi, y = 42, z)
\]

\[
\text{(%i3)} \quad \text{new (foo (1, 2, 4, 8))};
\]

\[
\text{(%o3)} \quad \text{foo}(w = 1, x = 2, y = 4, z = 8)
\]

\[
\emptyset
\]

\( \emptyset \) is the structure field access operator. The expression \( x@a \) refers to the value of field \( a \) of the structure instance \( x \). The field name is not evaluated.

If the field \( a \) in \( x \) has not been assigned a value, \( x@a \) evaluates to itself.

\text{kill(x@a)} removes the value of field \( a \) in \( x \).

Examples:

\[
\text{(%i1)} \quad \text{defstruct (foo (x, y, z));}
\]

\[
\text{(%o1)} \quad [\text{foo}(x, y, z)]
\]

\[
\text{(%i2)} \quad u : \text{new (foo (123, a - b, %pi))};
\]

\[
\text{(%o2)} \quad \text{foo}(x = 123, y = a - b, z = \pi)
\]

\[
\text{(%i3)} \quad u@z;
\]

\[
\text{(%o3)} \quad \pi
\]

\[
\text{(%i4)} \quad u@z : \%e;
\]

\[
\text{(%o4)} \quad \%e
\]

\[
\text{(%i5)} \quad u;
\]

\[
\text{(%o5)} \quad \text{foo}(x = 123, y = a - b, z = \pi)
\]

\[
\text{(%i6)} \quad \text{kill (u@z)};
\]

\[
\text{(%o6)} \quad \text{done}
\]

\[
\text{(%i7)} \quad u;
\]

\[
\text{(%o7)} \quad \text{foo}(x = 123, y = a - b, z)
\]

\[
\text{new}
\]

\[
\text{new (S)}
\]

\[
\text{new (S(v_1, ..., v_n))}
\]

\text{new} creates new instances of structures.

\text{new(S)} creates a new instance of structure \( S \) in which each field is assigned its default value, if any, or no value at all if no default was specified in the structure definition.

\text{new(S(v_1, ..., v_n))} creates a new instance of \( S \) in which fields are assigned the values \( v_1, ..., v_n \).

Examples:

\[
\text{(%i1)} \quad \text{defstruct (foo (w, x = %e, y = 42, z));}
\]

\[
\text{(%o1)} \quad [\text{foo}(w, x = %e, y = 42, z)]
\]

\[
\text{(%i2)} \quad \text{new (foo)};
\]

\[
\text{(%o2)} \quad \text{foo}(w, x = %e, y = 42, z)
\]

\[
\text{(%i3)} \quad \text{new (foo (1, 2, 4, 8))};
\]

\[
\text{(%o3)} \quad \text{foo}(w = 1, x = 2, y = 4, z = 8)
\]

\[
\emptyset
\]

\( \emptyset \) is the structure field access operator. The expression \( x@a \) refers to the value of field \( a \) of the structure instance \( x \). The field name is not evaluated.

If the field \( a \) in \( x \) has not been assigned a value, \( x@a \) evaluates to itself.

\text{kill(x@a)} removes the value of field \( a \) in \( x \).

Examples:

\[
\text{(%i1)} \quad \text{defstruct (foo (x, y, z));}
\]

\[
\text{(%o1)} \quad [\text{foo}(x, y, z)]
\]

\[
\text{(%i2)} \quad u : \text{new (foo (123, a - b, %pi))};
\]

\[
\text{(%o2)} \quad \text{foo}(x = 123, y = a - b, z = \pi)
\]

\[
\text{(%i3)} \quad u@z;
\]

\[
\text{(%o3)} \quad \pi
\]

\[
\text{(%i4)} \quad u@z : \%e;
\]

\[
\text{(%o4)} \quad \%e
\]

\[
\text{(%i5)} \quad u;
\]

\[
\text{(%o5)} \quad \text{foo}(x = 123, y = a - b, z = \pi)
\]

\[
\text{(%i6)} \quad \text{kill (u@z)};
\]

\[
\text{(%o6)} \quad \text{done}
\]

\[
\text{(%i7)} \quad u;
\]

\[
\text{(%o7)} \quad \text{foo}(x = 123, y = a - b, z)
\]
(\%i8) u@z;
\%o8) u@z

The field name is not evaluated.

(\%i1) defstruct (bar (g, h));
\%o1) [bar(g, h)]
(\%i2) x : new (bar);
\%o2) bar(g, h)
(\%i3) x@h : 42;
\%o3) 42
(\%i4) h : 123;
\%o4) 123
(\%i5) x@h;
\%o5) 42
(\%i6) x@h : 19;
\%o6) 19
(\%i7) x;
\%o7) bar(g, h = 19)
(\%i8) h;
\%o8) 123
6 Expressions

6.1 Introduction to Expressions

There are a number of reserved words which should not be used as variable names. Their use would cause a possibly cryptic syntax error.

<table>
<thead>
<tr>
<th>integrate</th>
<th>next</th>
<th>from</th>
<th>diff</th>
</tr>
</thead>
<tbody>
<tr>
<td>in</td>
<td>at</td>
<td>limit</td>
<td>sum</td>
</tr>
<tr>
<td>for</td>
<td>and</td>
<td>elseif</td>
<td>then</td>
</tr>
<tr>
<td>else</td>
<td>do</td>
<td>or</td>
<td>if</td>
</tr>
<tr>
<td>unless</td>
<td>product</td>
<td>while</td>
<td>thru</td>
</tr>
<tr>
<td>step</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Most things in Maxima are expressions. A sequence of expressions can be made into an expression by separating them by commas and putting parentheses around them. This is similar to the C comma expression.

\[(%i1) \ x: 3\$\]
\[(%i2) \ (x: x+1, x: x^2);\]
\[(%o2) \ 16\]
\[(%i3) \ (if (x > 17) then 2 else 4);\]
\[(%o3) \ 4\]
\[(%i4) \ (if (x > 17) then x: 2 else y: 4, y+x);\]
\[(%o4) \ 20\]

Even loops in Maxima are expressions, although the value they return is the not too useful done.

\[(%i1) \ y: (x: 1, for i from 1 thru 10 do (x: x*i))\$\]
\[(%i2) \ y;\]
\[(%o2) \ done\]

Whereas what you really want is probably to include a third term in the comma expression which actually gives back the value.

\[(%i3) \ y: (x: 1, for i from 1 thru 10 do (x: x*i), x)\$\]
\[(%i4) \ y;\]
\[(%o4) \ 3628800\]

6.2 Nouns and Verbs

Maxima distinguishes between operators which are "nouns" and operators which are "verbs". A verb is an operator which can be executed. A noun is an operator which appears as a symbol in an expression, without being executed. By default, function names are verbs. A verb can be changed into a noun by quoting the function name or applying the nounify function. A noun can be changed into a verb by applying the verbify function. The evaluation flag nouns causes ev to evaluate nouns in an expression.

The verb form is distinguished by a leading dollar sign \$ on the corresponding Lisp symbol. In contrast, the noun form is distinguished by a leading percent sign % on the corresponding Lisp symbol. Some nouns have special display properties, such as 'integrate and 'derivative (returned by diff), but most do not. By default, the noun and verb
forms of a function are identical when displayed. The global flag \texttt{noundisp} causes Maxima to display nouns with a leading quote mark \texttt{'}.

See also \texttt{noun}, \texttt{nouns}, \texttt{nounify}, and \texttt{verbify}.

Examples:

```lisp
(%i1) foo (x) := x^2;
 2
(%o1) foo(x) := x
(%i2) foo (42);
(%o2) 1764
(%i3) 'foo (42);
(%o3) foo(42)
(%i4) 'foo (42), nouns;
(%o4) 1764
(%i5) declare (bar, noun);
(%o5) done
(%i6) bar (x) := x/17;
   x
(%o6) bar(x) := --
    17
(%i7) bar (52);
(%o7) bar(52)
(%i8) bar (52), nouns;
(%o8) bar(52)
(%i9) integrate (1/x, x, 1, 42);
(%o9) log(42)
(%i10) 'integrate (1/x, x, 1, 42);
 42
/ [ 1
 1 - dx
] x
/ 1
(%o10)
(%i11) ev (%), nouns);
(%o11) log(42)
```

### 6.3 Identifiers

Maxima identifiers may comprise alphabetic characters, plus the numerals 0 through 9, plus any special character preceded by the backslash \textbackslash character.

A numeral may be the first character of an identifier if it is preceded by a backslash. Numerals which are the second or later characters need not be preceded by a backslash.

Characters may be declared alphabetic by the \texttt{declare} function. If so declared, they need not be preceded by a backslash in an identifier. The alphabetic characters are initially \texttt{A} through \texttt{Z}, \texttt{a} through \texttt{z}, \texttt{\%}, and \texttt{\_}.
Maxima is case-sensitive. The identifiers foo, FOO, and Foo are distinct. See Section 37.1 [Lisp and Maxima], page 617, for more on this point.

A Maxima identifier is a Lisp symbol which begins with a dollar sign $. Any other Lisp symbol is preceded by a question mark ? when it appears in Maxima. See Section 37.1 [Lisp and Maxima], page 617, for more on this point.

Examples:

```
(%i1) %an_ordinary_identifier42;
(%o1) %an_ordinary_identifier42
(%i2) embedded\ spaces\ in\ an\ identifier;
(%o2) embedded spaces in an identifier
(%i3) symbolp (%);
(%o3) true
(%i4) [foo+bar, foo\+bar];
(%o4) [foo + bar, foo+bar]
(%i5) [1729, \1729];
(%o5) [1729, 1729]
(%i6) [symbolp (foo\+bar), symbolp (\1729)];
(%o6) [true, true]
(%i7) [is (foo\+bar = foo+bar), is (\1729 = 1729)];
(%o7) [false, false]
(%i8) baz\¬quux;
(%o8) baz¬quux
(%i9) declare ("\¬", alphabetic);
(%o9) done
(%i10) baz¬quux;
(%o10) baz¬quux
(%i11) [is (foo = FOO), is (FOO = Foo), is (Foo = foo)];
(%o11) [false, false, false]
(%i12) :lisp (defvar *my-lisp-variable* '$foo)
*MY-LISP-VARIABLE*
(%i12) ?*my\¬lisp\¬variable\*;
(%o12) foo
```

### 6.4 Inequality

Maxima has the inequality operators <, <=, >=, >, #, and notequal. See if for a description of conditional expressions.

### 6.5 Functions and Variables for Expressions

**alias** (*new_name_1*, *old_name_1*, ..., *new_name_n*, *old_name_n*)  
[Function]  
provides an alternate name for a (user or system) function, variable, array, etc. Any even number of arguments may be used.

**aliases**  
[System variable]  
Default value: []
aliases is the list of atoms which have a user defined alias (set up by the alias, ordergreat, orderless functions or by declaring the atom a noun with declare.)

allbut [Keyword]
works with the part commands (i.e. part, inpart, substpart, substinpert, dpart, and ipart). For example,

\begin{verbatim}
(%i1) expr : e + d + c + b + a;
     \text{e + d + c + b + a}
(%i2) part (expr, [2, 5]);
     \text{d + a}
\end{verbatim}

while

\begin{verbatim}
(%i1) expr : e + d + c + b + a;
     \text{e + d + c + b + a}
(%i2) part (expr, allbut (2, 5));
     \text{e + c + b}
\end{verbatim}

allbut is also recognized by kill.

\begin{verbatim}
(%i1) [aa : 11, bb : 22, cc : 33, dd : 44, ee : 55];
     \text{[11, 22, 33, 44, 55]}
(%i2) kill (allbut (cc, dd));
     \text{done}
(%i1) [aa, bb, cc, dd];
     \text{[aa, bb, 33, 44]}
\end{verbatim}

kill(allbut(a_1, a_2, ...)) has the effect of kill(all) except that it does not kill the symbols a_1, a_2, ...

args (expr) [Function]
Returns the list of arguments of expr, which may be any kind of expression other than an atom. Only the arguments of the top-level operator are extracted; subexpressions of expr appear as elements or subexpressions of elements of the list of arguments.

The order of the items in the list may depend on the global flag inflag.

args (expr) is equivalent to substpart ("[", expr, 0). See also substpart, apply, funmake, and op.

How to convert a matrix to a nested list:
\begin{verbatim}
(%i1) M:matrix([1,2],[3,4]);
     \begin{bmatrix}
     1 & 2 \\
     3 & 4
     \end{bmatrix}
(%i2) args(M);
     \[[1, 2], [3, 4]]
\end{verbatim}

Since maxima internally treats a sum of n terms as a summation command with n arguments args() can extract the list of terms in a sum:
\begin{verbatim}
(%i1) a+b+c;
     \text{c + b + a}
(%i2) args(%);
     \[c, b, a]
\end{verbatim}
atom (expr)  [Function]
Returns true if expr is atomic (i.e. a number, name or string) else false. Thus
atom(5) is true while atom(a[1]) and atom(sin(x)) are false (assuming a[1]
and x are unbound).

box

box (expr)
box (expr, a)
Returns expr enclosed in a box. The return value is an expression with box as the
operator and expr as the argument. A box is drawn on the display when display2d
is true.

box (expr, a) encloses expr in a box labelled by the symbol a. The label is truncated
if it is longer than the width of the box.

box evaluates its argument. However, a boxed expression does not evaluate to its
content, so boxed expressions are effectively excluded from computations. rembox
removes the box again.

boxchar is the character used to draw the box in box and in the dpart and lpart
functions.

See also rembox, dpart and lpart.

Examples:

(%i1) box (a^2 + b^2);
                           2  2
" b + a "
(%o1) (c - d) + 1522756
(%i2) a : 1234;
(%o2) 1234
(%i3) b : c - d;
(%o3) c - d
(%i4) box (a^2 + b^2);
                           2
"(c - d) + 1522756"
(%o4) 1729 - box (1729);
(%o5) boxchar: "-";
(%o6) 1729 - "1729"
(%o7) -
(%i8) box (sin(x) + cos(y)); 

---------------------

(%o8) -cos(y) + sin(x)---------------------

[Option variable]

boxchar
Default value: "

boxchar is the character used to draw the box in the box and in the dpart and lpart functions.

All boxes in an expression are drawn with the current value of boxchar; the drawing character is not stored with the box expression.

collapse (expr) [Function]
Collapses expr by causing all of its common (i.e., equal) subexpressions to share (i.e., use the same cells), thereby saving space. (collapse is a subroutine used by the optimize command.) Thus, calling collapse may be useful after loading in a save file. You can collapse several expressions together by using collapse ([expr_1, ..., expr_n]). Similarly, you can collapse the elements of the array A by doing collapse (listarray ('A)).

copy (e) [Function]
Return a copy of the Maxima expression e. Although e can be any Maxima expression, the copy function is the most useful when e is either a list or a matrix; consider:

(%i1) m : [1,[2,3]]$
(%i2) mm : m$
(%i3) mm[2][1] : x$
(%i4) m;
(%o4) [1, [x, 3]]
(%i5) mm;
(%o5) [1, [x, 3]]

Let's try the same experiment, but this time let mm be a copy of m

(%i1) m : [1,[2,3]]$
(%i2) mm : copy(m)$
(%i3) mm[2][1] : x$
(%i4) m;
(%o4) [1, [2, 3]]
(%i5) mm;
(%o5) [1, [x, 3]]

This time, the assignment to mm does not change the value of m.

disolate (expr, x_1, ..., x_n) [Function]
is similar to isolate (expr, x) except that it enables the user to isolate more than one variable simultaneously. This might be useful, for example, if one were attempting to change variables in a multiple integration, and that variable change involved two or more of the integration variables. This function is autoloaded from simplification/disol.mac. A demo is available by demo("disol")$.
dispform

dispform (expr)
dispform (expr, all)

Returns the external representation of expr.

dispform(expr) returns the external representation with respect to the main (top-level) operator. dispform(expr, all) returns the external representation with respect to all operators in expr.

See also part, inpart, and inflag.

Examples:

The internal representation of \(-x\) is "negative one times \(x\)" while the external representation is "minus \(x\).

\[
\begin{align*}
\text{(\%i1) } & \text{- x;} \\
\text{(\%o1) } & - \text{ x} \\
\text{(\%i2) } & \text?=format (true, "\text{-S\%-}"\text{, }\%); \\
& ((\text{MTIMES SIMP}) \text{-1 }\%X) \\
\text{(\%o2) } & \text{false} \\
\text{(\%i3) } & \text{dispform (- x);} \\
\text{(\%o3) } & - \text{ x} \\
\text{(\%i4) } & \text?=format (true, "\text{-S\%-}"\text{, }\%); \\
& ((\text{MMINUS SIMP}) \%X) \\
\text{(\%o4) } & \text{false}
\end{align*}
\]

The internal representation of \(\sqrt{x}\) is "\(x\) to the power 1/2" while the external representation is "square root of \(x\)".

\[
\begin{align*}
\text{(\%i1) } & \text{sqrt (x);} \\
\text{(\%o1) } & \text{sqrt(x)} \\
\text{(\%i2) } & \text?=format (true, "\text{-S\%-}"\text{, }\%); \\
& ((\text{MEXPT SIMP}) \%X ((\text{RAT SIMP}) 1 2)) \\
\text{(\%o2) } & \text{false} \\
\text{(\%i3) } & \text{dispform (sqrt (x));} \\
\text{(\%o3) } & \text{sqrt(x)} \\
\text{(\%i4) } & \text?=format (true, "\text{-S\%-}"\text{, }\%); \\
& ((\text{\%SQRT SIMP}) \%X) \\
\text{(\%o4) } & \text{false}
\end{align*}
\]

Use of the optional argument all.

\[
\begin{align*}
\text{(\%i1) } & \text{expr : sin (sqrt (x));} \\
\text{(\%o1) } & \text{sin(sqrt(x))} \\
\text{(\%i2) } & \text{freeof (sqrt, expr);} \\
\text{(\%o2) } & \text{true} \\
\text{(\%i3) } & \text{freeof (sqrt, dispform (expr));} \\
\text{(\%o3) } & \text{true} \\
\text{(\%i4) } & \text{freeof (sqrt, dispform (expr, all));} \\
\text{(\%o4) } & \text{false}
\end{align*}
\]
**dpart (expr, n_1, ..., n_k)**

Selects the same subexpression as `part`, but instead of just returning that subexpression as its value, it returns the whole expression with the selected subexpression displayed inside a box. The box is actually part of the expression.

```
(%i1) dpart (x+y/z^2, 1, 2, 1);
```

```
  y
---- + x
  2
"z"
```

**exptisolate**

[Option variable]

Default value: `false`

`exptisolate`, when `true`, causes `isolate (expr, var)` to examine exponents of atoms (such as `%e`) which contain `var`.

**exptsubst**

[Option variable]

Default value: `false`

`exptsubst`, when `true`, permits substitutions such as `y` for `%e^x` in `%e^(a*x)`.

```
(%i1) %e^(a*x);
```

```
a x
```

```
(%o1) %e
```

```
(%i2) exptsubst;
```

```
(%o2) false
```

```
(%i3) subst(y, %e^x, %e^(a*x));
```

```
a x
```

```
(%o3) %e
```

```
(%i4) exptsubst: not exptsubst;
```

```
(%o4) true
```

```
(%i5) subst(y, %e^x, %e^(a*x));
```

```
ya
```

**freeof (x_1, ..., x_n, expr)**

[Function]

`freeof (x_1, expr)` returns `true` if no subexpression of `expr` is equal to `x_1` or if `x_1` occurs only as a dummy variable in `expr`, or if `x_1` is neither the noun nor verb form of any operator in `expr`, and returns `false` otherwise.

`freeof (x_1, ..., x_n, expr)` is equivalent to `freeof (x_1, expr) and ... and freeof (x_n, expr)`.

The arguments `x_1, ..., x_n` may be names of functions and variables, subscripted names, operators (enclosed in double quotes), or general expressions. `freeof` evaluates its arguments.

`freeof` operates only on `expr` as it stands (after simplification and evaluation) and does not attempt to determine if some equivalent expression would give a different result. In particular, simplification may yield an equivalent but different expression which comprises some different elements than the original form of `expr`. 
A variable is a dummy variable in an expression if it has no binding outside of the expression. Dummy variables recognized by \texttt{freeof} are the index of a sum or product, the limit variable in \texttt{limit}, the integration variable in the definite integral form of \texttt{integrate}, the original variable in \texttt{laplace}, formal variables in \texttt{at} expressions, and arguments in \texttt{lambda} expressions.

The indefinite form of \texttt{integrate} is not free of its variable of integration.

Examples:

Arguments are names of functions, variables, subscripted names, operators, and expressions. \texttt{freeof (a, b, expr)} is equivalent to \texttt{freeof (a, expr)} and \texttt{freeof (b, expr)}.

\begin{verbatim}
(%i1) expr: z^3 * cos(a[1]) * b^(c+d);
       
       3 d + c
cos(a ) b z
       1

(%i2) freeof (z, expr);
       false

(%i3) freeof (cos, expr);
       true

(%i4) freeof (a[1], expr);
       false

(%i5) freeof (cos (a[1]), expr);
       false

(%i6) freeof (b^(c+d), expr);
       false

(%i7) freeof ("^", expr);
       false

(%i8) freeof (w, sin, a[2], sin (a[2]), b*(c+d), expr);
       true
\end{verbatim}

\texttt{freeof} evaluates its arguments.

\begin{verbatim}
(%i1) expr: (a+b)^5$

(%i2) c: a$

(%i3) freeof (c, expr);
       false

(%i4) freeof (a+b, %);
       true

(%i5) freeof (a+b, expr);
       false

(%i6) exp (x);

x
\end{verbatim}

\texttt{freeof} does not consider equivalent expressions. Simplification may yield an equivalent but different expression.

\begin{verbatim}
(%i1) expr: (a+b)^5$

(%i2) expand (expr);
       5 4 2 3 3 2 4 5
       b + 5 a b + 10 a b + 10 a b + 5 a b + a

(%i3) freeof (a+b, %);
       true

(%i4) freeof (a+b, expr);
       false

(%i5) exp (x);
       x
\end{verbatim}
A summation or definite integral is free of its dummy variable. An indefinite integral is not free of its variable of integration.

(%i1) freeof (i, 'sum (f(i), i, 0, n));
(%o1) true
(%i2) freeof (x, 'integrate (x^2, x, 0, 1));
(%o2) true
(%i3) freeof (x, 'integrate (x^2, x));
(%o3) false

inflag
[Option variable]
Default value: false
When inflag is true, functions for part extraction inspect the internal form of expr.

Note that the simplifier re-orders expressions. Thus first (x + y) returns x if inflag is true and y if inflag is false. (first (y + x) gives the same results.) Also, setting inflag to true and calling part or substpart is the same as calling inpart or substinpart.

Functions affected by the setting of inflag are: part, substpart, first, rest, last, length, the for ... in construct, map, fullmap, maplist, reveal and pickapart.

inpart (expr, n_1, ..., n_k)
[Function]
is similar to part but works on the internal representation of the expression rather than the displayed form and thus may be faster since no formatting is done. Care should be taken with respect to the order of subexpressions in sums and products (since the order of variables in the internal form is often different from that in the displayed form) and in dealing with unary minus, subtraction, and division (since these operators are removed from the expression). part (x+y, 0) or inpart (x+y, 0) yield +, though in order to refer to the operator it must be enclosed in "s. For example ... if inpart (%o9,0) = "+" then ....

Examples:

(%i1) x + y + w*z;
(%o1) w z + y + x
(%i2) inpart (%o1, 3, 2);
(%o2) z
(%i3) part (%o1, 3, 2);
(%o3) z
(%i4) 'limit (f(x)^g(x+1), x, 0, minus);
limit f(x)
x -> 0-
(%i5) inpart (%o4, 1, 2);
(%o5) g(x + 1)
isolate (expr, x)  [Function]
Returns expr with subexpressions which are sums and which do not contain var
replaced by intermediate expression labels (these being atomic symbols like %t1, %t2,
...). This is often useful to avoid unnecessary expansion of subexpressions which
don’t contain the variable of interest. Since the intermediate labels are bound to the
subexpressions they can all be substituted back by evaluating the expression in which
they occur.

exptisolate (default value: false) if true will cause isolate to examine exponents
of atoms (like %e) which contain var.
isolate_wrt_times if true, then isolate will also isolate with respect to products.
See isolate_wrt_times. See also disolate.

Do example (isolate) for examples.

isolate_wrt_times  [Option variable]
Default value: false
When isolate_wrt_times is true, isolate will also isolate with respect to products.
E.g. compare both settings of the switch on

(%i1) isolate_wrt_times: true$
(%i2) isolate (expand ((a+b+c)^2), c);

(%t2) 2 a

(%t3) 2 b

(%t4) b + 2 a b + a

(%o4) c + %t3 c + %t2 c + %t4
(%i4) isolate_wrt_times: false$
(%i5) isolate (expand ((a+b+c)^2), c);

(%o5) c + 2 b c + 2 a c + %t4

listconstvars  [Option variable]
Default value: false
When listconstvars is true the list returned by listofvars contains constant
variables, such as %e, %pi, %i or any variables declared as constant that occur in
expr. A variable is declared as constant type via declare, and constantp returns
true for all variables declared as constant. The default is to omit constant variables
from listofvars return value.

listdummyvars  [Option variable]
Default value: true
When listdummyvars is false, "dummy variables" in the expression will not be
included in the list returned by listofvars. (The meaning of "dummy variables" is
as given in \texttt{freeof}. "Dummy variables" are mathematical things like the index of a sum or product, the limit variable, and the definite integration variable.)

Example:

```lisp
(\%i1) listdummyvars: true$
(\%i2) listofvars ('sum(f(i), i, 0, n));
(\%o2) [i, n]
(\%i3) listdummyvars: false$
(\%i4) listofvars ('sum(f(i), i, 0, n));
(\%o4) [n]
```

\textbf{listofvars (expr)}

Returns a list of the variables in \texttt{expr}.

\textbf{listconstvars} if \texttt{true} causes \texttt{listofvars} to include \%e, \%pi, \%i, and any variables declared constant in the list it returns if they appear in \texttt{expr}. The default is to omit these.

See also the option variable \texttt{listdummyvars} to exclude or include "dummy variables" in the list of variables.

```lisp
(\%i1) listofvars (f (x[1]+y) / g^(2+a));
(\%o1) [g, a, x, y]
```

\textbf{lfreeof (list, expr)}

For each member \textit{m} of \texttt{list}, calls \texttt{freeof} \texttt{(m, expr)}. It returns \texttt{false} if any call to \texttt{freeof} does and \texttt{true} otherwise.

Example:

```lisp
(\%i1) lfreeof ([ a, x], x^2+b);
(\%o1) false
(\%i2) lfreeof ([ b, x], x^2+b);
(\%o2) false
(\%i3) lfreeof ([ a, y], x^2+b);
(\%o3) true
```

\textbf{lpart (label, expr, n_1, \ldots, n_k)}

is similar to \texttt{dpart} but uses a labelled box. A labelled box is similar to the one produced by \texttt{dpart} but it has a name in the top line.

\textbf{mainvar}

You may declare variables to be \texttt{mainvar}. The ordering scale for atoms is essentially: numbers < constants (e.g., \%e, \%pi) < scalars < other variables < mainvars. E.g., compare \texttt{expand ((X+Y)^4)} with \texttt{(declare (x, mainvar), expand ((x+y)^4))}. (Note: Care should be taken if you elect to use the above feature. E.g., if you subtract an expression in which \texttt{x} is a \texttt{mainvar} from one in which \texttt{x} isn’t a \texttt{mainvar}, resimplification e.g. with \texttt{ev (expr, simp)} may be necessary if cancellation is to occur. Also, if you save an expression in which \texttt{x} is a \texttt{mainvar}, you probably should also save \texttt{x}.)

\textbf{noun}

\texttt{noun} is one of the options of the \texttt{declare} command. It makes a function so declared a "noun", meaning that it won’t be evaluated automatically.
Example:

```lisp
(%i1) factor (12345678);
(2)
(%o1) 2 3 47 14593
(%i2) declare (factor, noun);
(%o2) done
(%i3) factor (12345678);
(%o3) factor(12345678)
(%i4) ''%, nouns;
(2)
(%o4) 2 3 47 14593
```

`noundisp` [Option variable]

Default value: `false`

When `noundisp` is `true`, nouns display with a single quote. This switch is always `true` when displaying function definitions.

`nounify (f)` [Function]

Returns the noun form of the function name `f`. This is needed if one wishes to refer to the name of a verb function as if it were a noun. Note that some verb functions will return their noun forms if they can't be evaluated for certain arguments. This is also the form returned if a function call is preceded by a quote.

See also `verbify`.

`nterms (expr)` [Function]

Returns the number of terms that `expr` would have if it were fully expanded out and no cancellations or combination of terms occurred. Note that expressions like `sin (expr)`, `sqrt (expr)`, `exp (expr)`, etc. count as just one term regardless of how many terms `expr` has (if it is a sum).

`op (expr)` [Function]

Returns the main operator of the expression `expr`. `op (expr)` is equivalent to `part (expr, 0)`.

`op` returns a string if the main operator is a built-in or user-defined prefix, binary or n-ary infix, postfix, matchfix, or nofix operator. Otherwise, if `expr` is a subscripted function expression, `op` returns the subscripted function; in this case the return value is not an atom. Otherwise, `expr` is an array function or ordinary function expression, and `op` returns a symbol.

`op` observes the value of the global flag `inflag`.

`op` evaluates its argument.

See also `args`.

Examples:

```
(%i1) stringdisp: true$
(%i2) op (a * b * c);
(2)
(%o2) "*
(%i3) op (a * b + c);
(%o3) "+
```
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(%i4) op ('sin (a + b));
(%o4) sin
(%i5) op (a!);
(%o5) "!
(%i6) op (-a);
(%o6) "-
(%i7) op ([a, b, c]);
(%o7) "[
(%i8) op ('(if a > b then c else d));
(%o8) "if"
(%i9) op ('foo (a));
(%o9) foo
(%i10) prefix (foo);
(%o10) "foo"
(%i11) op (foo a);
(%o11) "foo"
(%i12) op (F [x, y] (a, b, c));
(%o12) F
   x, y
(%i13) op (G [u, v, w]);
(%o13) G

operatorp [Function]
operatorp (expr, op)
operatorp (expr, [op_1, . . ., op_n])
operatorp (expr, op) returns true if op is equal to the operator of expr.
operatorp (expr, [op_1, . . ., op_n]) returns true if some element op_1, . . ., op_n
is equal to the operator of expr.

opsubst [Option variable]
Default value: true
When opsubst is false, subst does not attempt to substitute into the operator of
an expression. E.g., (opsubst: false, subst (x^2, r, r+r[0])) will work.

(%i11) r+r[0];
(%o11) r + r
0
(%i12) opsubst;
(%o12) true
(%i13) subst (x^2, r, r+r[0]);
2 2
   x + (x )
0
(%i14) opsubst: not opsubst;
(%o14) false
(\%i5) subst (x^2, r, r+r[0]);
(\%o5)
\[ 2 \]
\[ x + r \]

optimize (expr)  
Returns an expression that produces the same value and side effects as expr but does so more efficiently by avoiding the recomputation of common subexpressions. optimize also has the side effect of "collapsing" its argument so that all common subexpressions are shared. Do example (optimize) for examples.

optimprefix  
Default value: \%  
optimprefix is the prefix used for generated symbols by the optimize command.

ordergreatp (expr_1, expr_2)  
orderlessp (expr_1, expr_2)

ordereatp returns true if expr_1 succeeds expr_2 in the canonical ordering of Maxima expressions, and false otherwise.

orderlessp returns true if expr_1 precedes expr_2 in the canonical ordering of Maxima expressions, and false otherwise.

All Maxima atoms and expressions are comparable under ordereatp and orderlessp, although there are isolated examples of expressions for which these predicates are not transitive; that is a bug.

The canonical ordering of atoms (symbols, literal numbers, and strings) is the following.
(integers and floats) precede (bigfloats) precede (declared constants) precede (strings) precede (declared scalars) precede (first argument to orderless) precedes . . . precedes (last argument to orderless) precedes (other symbols) precede (declared main variables)

For non-atomic expressions, the canonical ordering is derived from the ordering for atoms. For the built-in + * and - operators, the ordering is not easily summarized.
For other built-in operators and all other functions and operators, expressions are ordered by their arguments (beginning with the first argument), then by the name of the operator or function. In the case of subscripted expressions, the subscripted symbol is considered the operator and the subscript is considered an argument.

The canonical ordering of expressions is modified by the functions \texttt{ordergreat} and \texttt{orderless}, and the \texttt{mainvar}, \texttt{constant}, and \texttt{scalar} declarations.

See also \texttt{sort}.

Examples:

Ordering ordinary symbols and constants. Note that \%pi is not ordered according to its numerical value.

\begin{verbatim}
(%i1) stringdisp : true;
(%o1) true
(%i2) sort ([%pi, 3b0, 3.0, x, X, "foo", 3, a, 4, "bar", 4.0, 4b0]);
(%o2) [3, 3.0, 4, 4.0, 3.0b0, 4.0b0, %pi, "bar", "foo", X, a, x]
\end{verbatim}

Effect of \texttt{ordergreat} and \texttt{orderless} functions.

\begin{verbatim}
(%i1) sort ([M, H, K, T, E, W, G, A, P, J, S]);
(%i2) ordergreat (S, J);
(%o2) done
(%i3) orderless (M, H);
(%o3) done
(%i4) sort ([M, H, K, T, E, W, G, A, P, J, S]);
\end{verbatim}

Effect of \texttt{mainvar}, \texttt{constant}, and \texttt{scalar} declarations.

\begin{verbatim}
(%i1) sort ([aa, foo, bar, bb, baz, quux, cc, dd, A1, B1, C1]);
(%o1) [A1, B1, C1, aa, bar, baz, bb, cc, dd, foo, quux]
(%i2) declare (aa, mainvar);
(%o2) done
(%i3) declare ([baz, quux], constant);
(%o3) done
(%i4) declare ([A1, B1], scalar);
(%o4) done
(%i5) sort ([aa, foo, bar, bb, baz, quux, cc, dd, A1, B1, C1]);
(%o5) [baz, quux, A1, B1, C1, bar, bb, cc, dd, foo, aa]
\end{verbatim}

Ordering non-atomic expressions.

\begin{verbatim}
(%i1) sort ([1, 2, n, f(1), f(2), f(2, 1), g(1), g(1, 2), g(n),
          f(n, 1)]);
(%o1) [1, 2, f(1), g(1), g(1, 2), f(2), f(2, 1), n, g(n),
          f(n, 1)]
(%i2) sort ([foo(1), X[1], X[k], foo(k), 1, k]);
(%o2) [1, X, foo(1), k, X, foo(k)]
\end{verbatim}
part (expr, n_1, ..., n_k)  [Function]

Returns parts of the displayed form of expr. It obtains the part of expr as specified by the indices n_1, ..., n_k. First part n_1 of expr is obtained, then part n_2 of that, etc. The result is part n_k of ... part n_2 of part n_1 of expr. If no indices are specified expr is returned.

part can be used to obtain an element of a list, a row of a matrix, etc.

If the last argument to a part function is a list of indices then several subexpressions are picked out, each one corresponding to an index of the list. Thus part (x + y + z, [1, 3]) is z+x.

piece holds the last expression selected when using the part functions. It is set during the execution of the function and thus may be referred to in the function itself as shown below.

If partswitch is set to true then end is returned when a selected part of an expression doesn’t exist, otherwise an error message is given.

See also inpart, substpart, substinpert, dpart, and lpart.

Examples:

(%i1) part(z+2*y+a,2);
(%o1) 2 y
(%i2) part(z+2*y+a, [1,3]);
(%o2) z + a
(%i3) part(z+2*y+a,2,1);
(%o3) 2

example (part) displays additional examples.

partition (expr, x)  [Function]

Returns a list of two expressions. They are (1) the factors of expr (if it is a product), the terms of expr (if it is a sum), or the list (if it is a list) which don’t contain x and, (2) the factors, terms, or list which do.

Examples:

(%i1) partition (2*a*x*f(x), x);
(%o1) [2 a, x f(x)]
(%i2) partition (a+b, x);
(%o2) [b + a, 0]
(%i3) partition ([a, b, f(a), c], a);
(%o3) [[b, c], [a, f(a)]]

partswitch  [Option variable]

Default value: false

When partswitch is true, end is returned when a selected part of an expression doesn’t exist, otherwise an error message is given.

pickapart (expr, n)  [Function]

Assigns intermediate expression labels to subexpressions of expr at depth n, an integer. Subexpressions at greater or lesser depths are not assigned labels. pickapart returns an expression in terms of intermediate expressions equivalent to the original expression expr.
See also \texttt{part}, \texttt{dpart}, \texttt{lpart}, \texttt{inpart}, and \texttt{reveal}.

Examples:

\begin{verbatim}
(%i1) expr: (a+b)/2 + sin(x^2)/3 - log(1 + sqrt(x+1));
 2
   sin(x ) b + a
(%o1) - log(sqrt(x + 1) + 1) + ----- + -----  
       3  2

(%i2) pickapart (expr, 0);
 2
   sin(x ) b + a
(%t2) - log(sqrt(x + 1) + 1) + ----- + -----  
       3  2

(%o2) %t2

(%i3) pickapart (expr, 1);

(%t3) - log(sqrt(x + 1) + 1)

(%t4)  
   2
   sin(x )
----
 3

(%t5) 
   b + a
----
 2

(%o5) %t5 + %t4 + %t3

(%i5) pickapart (expr, 2);

(%t6) log(sqrt(x + 1) + 1)

(%t7)  
   2
   sin(x )

(%t8) 
   b + a

(%o8) --- + --- - %t6
    2  3

(%i8) pickapart (expr, 3);

(%t9) sqrt(x + 1) + 1
\end{verbatim}
Chapter 6: Expressions

(\%t10) \quad \frac{2}{x}

(\%o10) \quad \frac{b + a}{2} \sin(\%t10) - \log(\%t9) + \frac{3}{2}

(\%i10) \text{pickapart (expr, 4);}\

(\%t11) \quad \sqrt{x + 1}

(\%o11) \quad \frac{2}{\sin(x)} \quad \frac{b + a}{3} + \frac{\sin(x)}{2} - \log(\%t11 + 1)

(\%i11) \text{pickapart (expr, 5);}\

(\%t12) \quad x + 1

(\%o12) \quad \frac{2}{\sin(x)} \quad \frac{b + a}{3} + \frac{\sin(x)}{2} - \log(\sqrt{x + 1} + 1)

(\%i12) \text{pickapart (expr, 6);}\

\text{piece} \quad \text{[System variable]}

Holds the last expression selected when using the \text{part} functions. It is set during the execution of the function and thus may be referred to in the function itself.

\text{psubst} \quad \text{[Function]}

\text{psubst} (list, expr) \\
\text{psubst} (a, b, expr)

\text{psubst}(a, b, expr) \text{ is similar to subst. See subst.}

In distinction from subst the function \text{psubst} makes parallel substitutions, if the first argument list is a list of equations.

See also \text{sublis} for making parallel substitutions.

Example:

The first example shows parallel substitution with \text{psubst}. The second example shows the result for the function \text{subst}, which does a serial substitution.

(\%i11) \text{psubst ([a\^2=b, b=a], sin(a\^2) + sin(b))}\
(\%o11) \quad \sin(b) + \sin(a)

(\%i12) \text{subst ([a\^2=b, b=a], sin(a\^2) + sin(b))}\
(\%o12) \quad 2 \sin(a)
rembox

rembox (expr, unlabelled)
rembox (expr, label)
rembox (expr)

Removes boxes from expr.

rembox (expr, unlabelled) removes all unlabelled boxes from expr.
rembox (expr, label) removes only boxes bearing label.
rembox (expr) removes all boxes, labelled and unlabelled.

Boxes are drawn by the box, dpart, and lpart functions.

Examples:

(%i1) expr: (a*d - b*c)/h^2 + sin(%pi*x);
     a d - b c
(%o1)            sin(%pi x) + ---------
             2
             h

(%i2) dpart (dpart (expr, 1, 1), 2, 2);
dpart: fell off the end.
    -- an error. To debug this try: debugmode(true);

(%i3) expr2: lpart (BAR, lpart (FOO, %, 1), 2);

BAR***********
   FOO*********** "a d - b c"
 (%o3) "sin(%pi x)" + "---------"
       *********** " 2 "
       " h "
       ***********

(%i4) rembox (expr2, unlabelled);

BAR***********
   FOO*********** "a d - b c"
 (%o4) "sin(%pi x)" + "---------"
       *********** " 2 "
       " h "
       ***********

(%i5) rembox (expr2, FOO);

BAR***********
   "a d - b c"
(%o5) sin(%pi x) + "---------"
       " 2 "
       " h "
       ***********

(%i6) rembox (expr2, BAR);

F00*********** a d - b c
(%o6) "sin(%pi x)" + "---------"
       2
       h
Chapter 6: Expressions

(\%i7) rembox (expr2);  
\hspace{1cm} a d - b c  
(\%o7) \hspace{1cm} \sin(\%pi \ x) + \frac{---------}{2} = \frac{2}{h}  

reveal (expr, depth)  
[Function]  
Replaces parts of expr at the specified integer depth with descriptive summaries.  
- Sums and differences are replaced by \texttt{Sum(n)} where \(n\) is the number of operands of the sum.  
- Products are replaced by \texttt{Product(n)} where \(n\) is the number of operands of the product.  
- Exponentials are replaced by \texttt{Expt}.  
- Quotients are replaced by \texttt{Quotient}.  
- Unary negation is replaced by \texttt{Negterm}.  
- Lists are replaced by \texttt{List(n)} where \(n\) is the number of elements of the list.  

When depth is greater than or equal to the maximum depth of expr, \texttt{reveal (expr, depth)} returns expr unmodified.  
\texttt{reveal} evaluates its arguments. \texttt{reveal} returns the summarized expression.  
Example:  
(\%i1) e: expand ((a - b)^2)/expand ((exp(a) + exp(b))^2);  
\hspace{1cm} \frac{2^2 b - 2 a b + a}{b + a^2 b^2 a^2}  
(\%o1) \hspace{1cm} \frac{\frac{b + a}{2 a} \frac{b + a}{2 b} \frac{b + a}{2 a} \frac{2 \%e + \%e + \%e}{Product(2) + Expt + Expt}}{Expt + Negterm + Expt}  
(\%i2) reveal (e, 1);  
(\%o2) Quotient  
(\%i3) reveal (e, 2);  
(\%o3) \texttt{Sum(3)}\hspace{1cm}\texttt{Sum(3)}  
(\%i4) reveal (e, 3);  
(\%o4) \texttt{Expt + Negterm + Expt}  
(\%i5) reveal (e, 4);  
(\%o5) \texttt{Product(2) + Product(2)}\hspace{1cm}\texttt{Product(2)}\hspace{1cm}\texttt{Product(2)}\hspace{1cm}\frac{2 \%e + \%e + \%e}{Product(2) + Expt + Expt}  
(\%i6) reveal (e, 5);  
(\%o6) \frac{2^2 b - 2 a b + a}{b + a^2 b^2 a^2}
sublis (list, expr) [Function]
Makes multiple parallel substitutions into an expression. list is a list of equations. The left hand side of the equations must be an atom.
The variable sublis_apply_lambda controls simplification after sublis.
See also psubst for making parallel substitutions.
Example:
(%i1) sublis ([a=b, b=a], sin(a) + cos(b));
(%o1) sin(b) + cos(a)

sublis_apply_lambda [Option variable]
Default value: true
Controls whether lambda’s substituted are applied in simplification after sublis is used or whether you have to do an ev to get things to apply. true means do the application.

subnumsimp [Option variable]
Default value: false
If true then the functions subst and psubst can substitute a subscripted variable f[x] with a number, when only the symbol f is given.
See also subst.
(%i1) subst(100,g,g[x]+2);
subst: cannot substitute 100 for operator g in expression g
-- an error. To debug this try: debugmode(true);
(%o1)

substitute (a, b, c) [Function]
Substitutes a for b in c. b must be an atom or a complete subexpression of c.
For example, x+y+z is a complete subexpression of 2*(x+y+z)/w while x+y is not.
When b does not have these characteristics, one may sometimes use substpart or ratsubst (see below). Alternatively, if b is of the form e/f then one could use subst (a*f, e, c) while if b is of the form e^(-1/f) then one could use subst (a^-f, e, c). The subst command also discerns the x^-y in x^-y so that subst (a, sqrt(x),
1/sqrt(x)) yields 1/a. a and b may also be operators of an expression enclosed in double-quotes " or they may be function names. If one wishes to substitute for the independent variable in derivative forms then the at function (see below) should be used.

**subst** is an alias for substitute.

The commands subst (eq_1, expr) or subst ([eq_1, ..., eq_k], expr) are other permissible forms. The eq._i are equations indicating substitutions to be made. For each equation, the right side will be substituted for the left in the expression expr. The equations are substituted in serial from left to right in expr. See the functions sublis and psubst for making parallel substitutions.

**exptsubst** if true permits substitutions like y for %e^-x in %e^(-a*x) to take place.

When opsubst is false, subst will not attempt to substitute into the operator of an expression. E.g. (opsubst: false, subst (x^2, r, r+r[0])) will work.

See also at, ev and psubst.

**Examples:**

```
(%i1) subst (a, x+y, x + (x+y)^2 + y);
  2
(%o1) y + x + a

(%i2) subst (-%i, %i, a + b*%i);
    a - %i b

(%i3) subst(["+"]="-"],a+b-c);
  c-b+a
```

The substitution is done in serial for a list of equations. Compare this with a parallel substitution:

```
(%i1) subst([a=b, b=c], a+b);
  2
(%o1) c

(%i2) sublis([a=b, b=c], a+b);
    c + b
```

Single-character Operators like + and - have to be quoted in order to be replaced by subst. It is to note, though, that a+b-c might be expressed as a+b+(-1*c) internally.

```
(%i3) subst(["+"="-"],a+b-c);
  c-b+a
```

The difference between subst and at can be seen in the following example:

```
(%i1) g1:y(t)=a*x(t)+b*diff(x(t),t);
   d
(%o1) y(t) = b (-- (x(t))) + a x(t)
   dt

(%i2) subst('diff(x(t),t)=1,g1);
(%o2) y(t) = a x(t) + b

(%i3) at(g1,'diff(x(t),t)=1);
                           d
    !
  !
  !d
(%o3) y(t) = b (-- (x(t))! + a x(t)
     !d
      !-- (x(t)) = 1
```

For further examples, do `example subst`.

`substinpart (x, expr, n_1, ..., n_k)`

Similar to `substpart`, but `substinpart` works on the internal representation of `expr`.

Examples:

```lisp
(\%i1) x . 'diff (f(x), x, 2);

   2
2
x . (--- (f(x)))
   2
dx

(\%o1)

(\%i2) substinpart (d^2, \%, 2);

   2
2
x . d

(\%o2)

(\%i3) substinpart (f1, f[1](x + 1), 0);

f1(x + 1)

(\%o3)

If the last argument to a `part` function is a list of indices then several subexpressions
are picked out, each one corresponding to an index of the list. Thus

(\%i1) part (x + y + z, [1, 3]);

   z + x

(\%o1)

`piece` holds the value of the last expression selected when using the `part` functions. It
is set during the execution of the function and thus may be referred to in the function
itself as shown below. If `partswitch` is set to `true` then `end` is returned when a
selected part of an expression doesn’t exist, otherwise an error message is given.

(\%i1) expr: 27*y^3 + 54*x*y^2 + 36*x^2*y + y + 8*x^3 + x + 1;

                   3  2
3 2 2 3
(\%o1) 27 y + 54 x y + 36 x y + y + 8 x + x + 1

(\%i2) part (expr, 2, [1, 3]);

   2
2
54 y

(\%o2)

(\%i3) sqrt (piece/54);

   abs(y)

(\%o3)

(\%i4) substpart (factor (piece), expr, [1, 2, 3, 5]);

                   3
(\%o4) (3 y + 2 x) + y + x + 1

(\%i5) expr: 1/x + y/x - 1/z;

                   1 y 1
(\%o5) (-) + - + -
       z x x

(\%i6) substpart (xthru (piece), expr, [2, 3]);

                      y + 1 1
(\%o6) ----- - -
                      x z

Also, setting the option `inflag` to `true` and calling `part` or `substpart` is the same
as calling `inpart` or `substinpart`.
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**substpart** \((x, \text{expr}, n_1, \ldots, n_k)\)  
[Function]  
Substitutes \(x\) for the subexpression picked out by the rest of the arguments as in \(\text{part}\). It returns the new value of \(\text{expr}\). \(x\) may be some operator to be substituted for an operator of \(\text{expr}\). In some cases \(x\) needs to be enclosed in double-quotes " (e.g. \(\text{substpart} ("+", a*b, 0)\) yields \(b + a\)).

Example:

\[
\begin{align*}
(\%i1) & \quad 1/(x^2 + 2); \\
(\%o1) & \quad \frac{1}{2}x + 2 \\
(\%i2) & \quad \text{substpart} (3/2, \%, 1, 2); \\
(\%o2) & \quad \frac{3/2}{2}x + 2 \\
(\%i3) & \quad a*x + f(b, y); \\
(\%o3) & \quad a x + f(b, y) \\
(\%i4) & \quad \text{substpart} ("+", \%, 1, 0); \\
(\%o4) & \quad x + f(b, y) + a
\end{align*}
\]

Also, setting the option \(\text{inflag}\) to \(\text{true}\) and calling \(\text{part}\) or \(\text{substpart}\) is the same as calling \(\text{inpart}\) or \(\text{substinpart}\).

**symbolp** \((\text{expr})\)  
[Function]  
Returns \(\text{true}\) if \(\text{expr}\) is a symbol, else \(\text{false}\).

See also Section 6.3 [Identifiers], page 82.

**unorder** ()  
[Function]  
Disables the aliasing created by the last use of the ordering commands \(\text{ordergreat}\) and \(\text{orderless}\). \(\text{ordergreat}\) and \(\text{orderless}\) may not be used more than one time each without calling \(\text{unorder}\). \(\text{unorder}\) does not substitute back in expressions the original symbols for the aliases introduced by \(\text{ordergreat}\) and \(\text{orderless}\). Therefore, after execution of \(\text{unorder}\) the aliases appear in previous expressions.

See also \(\text{ordergreat}\) and \(\text{orderless}\).

Examples:

\(\text{ordergreat}(a)\) introduces an alias for the symbol \(a\). Therefore, the difference of \(\%o2\) and \(\%o4\) does not vanish. \(\text{unorder}\) does not substitute back the symbol \(a\) and the alias appears in the output \(\%o7\).

\[
\begin{align*}
(\%i1) & \quad \text{unorder}(); \\
(\%o1) & \quad [] \\
(\%i2) & \quad b*x + a^2; \\
(\%o2) & \quad b x + a \\
(\%i3) & \quad \text{ordergreat} (a); \\
(\%o3) & \quad \text{done}
\end{align*}
\]
(%i4) b*x + a^2;
   %th(1) - %th(3);
(%o4) a + b x
(%i5) unordered();
(%o5) 2 2
   a - a
(%i6) %th(2);
(%o6) [a]

verbify (f)

 Returns the verb form of the function name f. See also verb, noun, and nounify.

Examples:

(%i1) verbify ('foo);
   foo
(%i2) :lisp $%FOO
   $FOO
(%i2) nounify (foo);
   foo
(%i3) :lisp $%FOO
   %FOO
7 Operators

7.1 Introduction to operators

It is possible to define new operators with specified precedence, to undefine existing operators, or to redefine the precedence of existing operators. An operator may be unary prefix or unary postfix, binary infix, n-ary infix, matchfix, or nofix. "Matchfix" means a pair of symbols which enclose their argument or arguments, and "nofix" means an operator which takes no arguments. As examples of the different types of operators, there are the following.

unary prefix
   negation \(-\ a\)

unary postfix
   factorial \(a!\)

binary infix
   exponentiation \(a^b\)

n-ary infix
   addition \(a + b\)

matchfix
   list construction \([a, b]\)

(There are no built-in nofix operators; for an example of such an operator, see nofix.)

The mechanism to define a new operator is straightforward. It is only necessary to declare a function as an operator; the operator function might or might not be defined.

An example of user-defined operators is the following. Note that the explicit function call "dd" \((a)\) is equivalent to \(dd\ a\), likewise "<-" \((a, b)\) is equivalent to \(a <- b\). Note also that the functions "dd" and "<-" are undefined in this example.

```
(%i1) prefix ("dd");
(%o1) dd
(%i2) dd a;
(%o2) dd a
(%i3) "dd" (a);
(%o3) dd a
(%i4) infix ("<-");
(%o4) <-
(%i5) a <- dd b;
(%o5) a <- dd b
(%i6) "<-" (a, "dd" (b));
(%o6) a <- dd b
```

The Maxima functions which define new operators are summarized in this table, stating the default left and right binding powers \((lbp\ and\ rbp,\ respectively)\). (Binding power determines operator precedence. However, since left and right binding powers can differ, binding power is somewhat more complicated than precedence.) Some of the operation definition functions take additional arguments; see the function descriptions for details.

```
prefix   rbp=180
postfix  lbp=180
```
infix  lbp=180, rbp=180
nary   lbp=180, rbp=180
matchfix  (binding power not applicable)
nofix  (binding power not applicable)

For comparison, here are some built-in operators and their left and right binding powers.

<table>
<thead>
<tr>
<th>Operator</th>
<th>lbp</th>
<th>rbp</th>
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<tbody>
<tr>
<td>:</td>
<td>180</td>
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<td>::</td>
<td>180</td>
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<tr>
<td>:=</td>
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<td>::=</td>
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<td>-</td>
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<td>or</td>
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<td>,</td>
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<td>$</td>
<td>-1</td>
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<tr>
<td>;</td>
<td>-1</td>
<td></td>
</tr>
</tbody>
</table>

\textit{remove} and \textit{kill} remove operator properties from an atom. \texttt{remove ("a", op)} removes only the operator properties of \texttt{a}. \texttt{kill ("a")} removes all properties of \texttt{a}, including the operator properties. Note that the name of the operator must be enclosed in quotation marks.

\begin{verbatim}
(%i1) infix ("##");  
        ##
(%o1)      ##

(%i2) "##" (a, b) := a^b

(%o2)        a ## b := a^b

(%i3) 5 ## 3;
(%o3) 125

(%i4) remove ("##", op);

(%o4) done
\end{verbatim}
The symbols $+$, $\times$, $/$ and $^\wedge$ represent addition, multiplication, division, and exponentiation, respectively. The names of these operators are "+" "\times" "/" and "^\wedge", which may appear where the name of a function or operator is required.

The symbols $+$ and $-$ represent unary addition and negation, respectively, and the names of these operators are "+" and "-", respectively.

Subtraction $a - b$ is represented within Maxima as addition, $a + (- b)$. Expressions such as $a + (- b)$ are displayed as subtraction. Maxima recognizes "-" only as the name of the unary negation operator, and not as the name of the binary subtraction operator.

Division $a / b$ is represented within Maxima as multiplication, $a * b^(- 1)$. Expressions such as $a * b^(- 1)$ are displayed as division. Maxima recognizes "/" as the name of the division operator.

Addition and multiplication are n-ary, commutative operators. Division and exponentiation are binary, noncommutative operators.

Maxima sorts the operands of commutative operators to construct a canonical representation. For internal storage, the ordering is determined by orderlessp. For display, the ordering for addition is determined by ordergreatp, and for multiplication, it is the same as the internal ordering.

Arithmetic computations are carried out on literal numbers (integers, rationals, ordinary floats, and bigfloats). Except for exponentiation, all arithmetic operations on
numbers are simplified to numbers. Exponentiation is simplified to a number if either operand is an ordinary float or bigfloat or if the result is an exact integer or rational; otherwise an exponentiation may be simplified to \texttt{sqrt} or another exponentiation or left unchanged.

Floating-point contagion applies to arithmetic computations: if any operand is a bigfloat, the result is a bigfloat; otherwise, if any operand is an ordinary float, the result is an ordinary float; otherwise, the operands are rationals or integers and the result is a rational or integer.

Arithmetic computations are a simplification, not an evaluation. Thus arithmetic is carried out in quoted (but simplified) expressions.

Arithmetic operations are applied element-by-element to lists when the global flag \texttt{listarith} is \texttt{true}, and always applied element-by-element to matrices. When one operand is a list or matrix and another is an operand of some other type, the other operand is combined with each of the elements of the list or matrix.

Examples:

Addition and multiplication are n-ary, commutative operators. Maxima sorts the operands to construct a canonical representation. The names of these operators are "+" and "*".

\begin{verbatim}
(%i1) c + g + d + a + b + e + f;
  (%o1) g + f + e + d + c + b + a
(%i2) [op (%), args (%)];
  (%o2) [+ , [g, f, e, d, c, b, a]]
(%i3) c * g * d * a * b * e * f;
  (%o3) a b c d e f g
(%i4) [op (%), args (%)];
  (%o4) [* , [a, b, c, d, e, f, g]]
(%i5) apply ("+", [a, 8, x, 2, 9, x, x, a]);
  (%o5) 3 x + 2 a + 19
(%i6) apply ("*", [a, 8, x, 2, 9, x, x, a]);
  (%o6) 144 a x
\end{verbatim}

Division and exponentiation are binary, noncommutative operators. The names of these operators are "/" and "-".

\begin{verbatim}
(%i1) [a / b, a ^ b];
  (%o1) [- , a ]
  b
(%i2) [map (op, %), map (args, %)];
  (%o2) [[/ , []], [[a, b], [a, b]]]
(%i3) [apply ("/", [a, b]), apply ("-", [a, b])];
  (%o3) [- , a ]
  b
\end{verbatim}

Subtraction and division are represented internally in terms of addition and multiplication, respectively.
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(%i1) [inpart (a - b, 0), inpart (a - b, 1), inpart (a - b, 2)];
(%o1) [+ , a , - b]
     0 1

(%i2) [inpart (a / b, 0), inpart (a / b, 1), inpart (a / b, 2)];
(%o2) [*, a, -]
     b

Computations are carried out on literal numbers. Floating-point contagion applies.

(%i1) 17 + b - (1/2)*29 + 11^(2/4); 5
(%o1) b + sqrt(11) + -
     2

(%i2) [17 + 29, 17 + 29.0, 17 + 29b0];
(%o2) [46, 46.0, 4.6b1]

Arithmetic computations are a simplification, not an evaluation.

(%i1) simp : false;
(%o1) false

(%i2) '(17 + 29*11/7 - 5^3);
29 11 3
(%o2) 17 + ----- - 5
     7

(%i3) simp : true;
(%o3) true

(%i4) '(17 + 29*11/7 - 5^3);
437
(%o4) - ---
     7

Arithmetic is carried out element-by-element for lists (depending on listarith) and
matrices.

(%i1) matrix ([a, x], [h, u]) - matrix ([1, 2], [3, 4]);
      [ a - 1 x - 2 ]
(%o1) [ ]
      [ h - 3 u - 4 ]

(%i2) 5 * matrix ([a, x], [h, u]);
      [ 5 a 5 x ]
(%o2) [ ]
      [ 5 h 5 u ]

(%i3) listarith : false;
(%o3) false

(%i4) [a, c, m, t] / [1, 7, 2, 9];
      [a, c, m, t]
(%o4) [-----------------]
      [1, 7, 2, 9]

(%i5) [a, c, m, t] ^ x;
      [a, c, m, t]
(%o5) [ ]
(%i6) listarith : true;
(%o6)   true
(%i7) [a, c, m, t] / [1, 7, 2, 9];
     c m t
(%o7) [a, -, -, -]
     7 2 9
(%i8) [a, c, m, t] ^ x;
         x x x x
(%o8) [a , c , m , t ]

**

Exponentiation operator. Maxima recognizes ** as the same operator as ^ in input, and it is displayed as ^ in 1-dimensional output, or by placing the exponent as a superscript in 2-dimensional output.

The fortran function displays the exponentiation operator as **, whether it was input as ** or ^.

Examples:

(%i1) is (a**b = a^b);
(%o1)   true
(%i2) x**y + x^z;
     z y
(%o2) [x , x ]
     + +
(%i3) string (x**y + x^z);
(%o3) [x^z+x^y]
(%i4) fortran (x**y + x^z);
(x**z+x**y)
(%o4) done

^^

Noncommutative exponentiation operator. ^^ is the exponentiation operator corresponding to noncommutative multiplication ., just as the ordinary exponentiation operator ^ corresponds to commutative multiplication *.

Noncommutative exponentiation is displayed by ^^ in 1-dimensional output, and by placing the exponent as a superscript within angle brackets <> in 2-dimensional output.

Examples:

(%i1) a . a . b . b . b + a * a * a * b * b;
(%o1)     a b + a b
(%i2) string (a . a . b . b . b + a * a * a * b * b);
(%o2)     a b + a b

.

The dot operator, for matrix (non-commutative) multiplication. When "." is used in this way, spaces should be left on both sides of it, e.g. A . B This distinguishes it plainly from a decimal point in a floating point number.
See also Dot, dot0nscsimp, dot0simp, dotisimp, dotassoc, dotconstrules, dotdistrib, dotexptsimp, dotident, and dotscrules.

7.3 Relational operators

<  [Operator]
<=  [Operator]
>=  [Operator]
>   [Operator]

The symbols < <= >= and > represent less than, less than or equal, greater than or equal, and greater than, respectively. The names of these operators are "<" "<=" ">=" and ">", which may appear where the name of a function or operator is required.

These relational operators are all binary operators; constructs such as a < b < c are not recognized by Maxima.

Relational expressions are evaluated to Boolean values by the functions is and maybe, and the programming constructs if, while, and unless. Relational expressions are not otherwise evaluated or simplified to Boolean values, although the arguments of relational expressions are evaluated (when evaluation is not otherwise prevented by quotation).

When a relational expression cannot be evaluated to true or false, the behavior of is and if are governed by the global flag prederror. When prederror is true, is and if trigger an error. When prederror is false, is returns unknown, and if returns a partially-evaluated conditional expression.

maybe always behaves as if prederror were false, and while and unless always behave as if prederror were true.

Relational operators do not distribute over lists or other aggregates.

See also =, #, equal, and notequal.

Examples:

Relational expressions are evaluated to Boolean values by some functions and programming constructs.

(%i1) [x, y, z] : [123, 456, 789];
(%o1) [123, 456, 789]
(%i2) is (x < y);
(%o2) true
(%i3) maybe (y > z);
(%o3) false
(%i4) if x >= z then 1 else 0;
(%o4) 0
(%i5) block ([S], S : 0, for i:1 while i <= 100 do S : S + i, return (S));
(%o5) 5050

Relational expressions are not otherwise evaluated or simplified to Boolean values, although the arguments of relational expressions are evaluated.

(%o1) [123, 456, 789]
(%i2) [x < y, y <= z, z >= y, y > z];
(%o2) [123 < 456, 456 <= 789, 789 >= 456, 456 > 789]
(%i3) map (is, %);
(%o3) [true, true, true, false]

7.4 Logical operators

and
The logical conjunction operator. and is an n-ary infix operator; its operands are Boolean expressions, and its result is a Boolean value.
and forces evaluation (like is) of one or more operands, and may force evaluation of all operands.
Operands are evaluated in the order in which they appear. and evaluates only as many of its operands as necessary to determine the result. If any operand is false, the result is false and no further operands are evaluated.
The global flag prederror governs the behavior of and when an evaluated operand cannot be determined to be true or false. and prints an error message when prederror is true. Otherwise, operands which do not evaluate to true or false are accepted, and the result is a Boolean expression.
and is not commutative: a and b might not be equal to b and a due to the treatment of indeterminate operands.

not
The logical negation operator. not is a prefix operator; its operand is a Boolean expression, and its result is a Boolean value.
not forces evaluation (like is) of its operand.
The global flag prederror governs the behavior of not when its operand cannot be determined to be true or false. not prints an error message when prederror is true. Otherwise, operands which do not evaluate to true or false are accepted, and the result is a Boolean expression.

or
The logical disjunction operator. or is an n-ary infix operator; its operands are Boolean expressions, and its result is a Boolean value.
or forces evaluation (like is) of one or more operands, and may force evaluation of all operands.
Operands are evaluated in the order in which they appear. or evaluates only as many of its operands as necessary to determine the result. If any operand is true, the result is true and no further operands are evaluated.
The global flag prederror governs the behavior of or when an evaluated operand cannot be determined to be true or false. or prints an error message when prederror is true. Otherwise, operands which do not evaluate to true or false are accepted, and the result is a Boolean expression.
or is not commutative: a or b might not be equal to b or a due to the treatment of indeterminate operands.
7.5 Operators for Equations

# 
[Operator]

Represents the negation of syntactic equality =.

Note that because of the rules for evaluation of predicate expressions (in particular because \texttt{not expr} causes evaluation of \texttt{expr}), \texttt{not a = b} is equivalent to \texttt{is(a \# b)}, instead of \texttt{a \# b}.

Examples:

\begin{verbatim}
(%i1) a = b;
   (%o1) a = b
(%i2) is(a = b);
   (%o2) false
(%i3) a \# b;
   (%o3) a \# b
(%i4) not a = b;
   (%o4) true
(%i5) is(a \# b);
   (%o5) true
(%i6) is(not a = b);
   (%o6) true
\end{verbatim}

= 
[Operator]

The equation operator.

An expression \texttt{a = b}, by itself, represents an unevaluated equation, which might or might not hold. Unevaluated equations may appear as arguments to \texttt{solve} and \texttt{algsys} or some other functions.

The function \texttt{is} evaluates = to a Boolean value. \texttt{is(a = b)} evaluates \texttt{a = b} to \texttt{true} when \texttt{a} and \texttt{b} are identical. That is, \texttt{a} and \texttt{b} are atoms which are identical, or they are not atoms and their operators are identical and their arguments are identical. Otherwise, \texttt{is(a = b)} evaluates to \texttt{false}; it never evaluates to \texttt{unknown}. When \texttt{is(a = b)} is \texttt{true}, \texttt{a} and \texttt{b} are said to be syntactically equal, in contrast to equivalent expressions, for which \texttt{is(equal(a, b))} is \texttt{true}. Expressions can be equivalent and not syntactically equal.

The negation of = is represented by \#. As with =, an expression \texttt{a \# b}, by itself, is not evaluated. \texttt{is(a \# b)} evaluates \texttt{a \# b} to \texttt{true} or \texttt{false}.

In addition to \texttt{is}, some other operators evaluate = and \# to \texttt{true} or \texttt{false}, namely \texttt{if}, \texttt{and}, \texttt{or}, and \texttt{not}.

Note that because of the rules for evaluation of predicate expressions (in particular because \texttt{not expr} causes evaluation of \texttt{expr}), \texttt{not a = b} is equivalent to \texttt{is(a \# b)}, instead of \texttt{a \# b}.

\texttt{rhs} and \texttt{lhs} return the right-hand and left-hand sides, respectively, of an equation or inequation.

See also \texttt{equal} and \texttt{notequal}.

Examples:
An expression \( a = b \), by itself, represents an unevaluated equation, which might or might not hold.

\[
\text{(%i1)} \quad \text{eq}_1 : a \times x - 5 \times y = 17;
\]
\[
\text{(%o1)} \quad a x - 5 y = 17
\]
\[
\text{(%i2)} \quad \text{eq}_2 : b \times x + 3 \times y = 29;
\]
\[
\text{(%o2)} \quad 3 y + b x = 29
\]
\[
\text{(%i3)} \quad \text{solve} ([\text{eq}_1, \text{eq}_2], [x, y]);
\]
\[
\text{(%o3)} \quad \begin{array}{l}
\left[ x = \frac{196}{5 b + 3 a} , \\
\frac{29 a - 17 b}{5 b + 3 a} , \\
\right]
\end{array}
\]
\[
\text{(%i4)} \quad \text{subst} (\% , [\text{eq}_1, \text{eq}_2]);
\]
\[
\text{(%o4)} \quad \begin{array}{l}
\left[ \frac{a}{5 b + 3 a} - \frac{5 (29 a - 17 b)}{5 b + 3 a} = 17, \\
\frac{b}{5 b + 3 a} + \frac{3 (29 a - 17 b)}{5 b + 3 a} = 29 \right]
\end{array}
\]
\[
\text{(%i5)} \quad \text{ratsimp} (\%);
\]
\[
\text{(%o5)} \quad [17 = 17, 29 = 29]
\]

\( \text{is}(a = b) \) evaluates \( a = b \) to \text{true} when \( a \) and \( b \) are syntactically equal (that is, identical). Expressions can be equivalent and not syntactically equal.

\[
\text{(%i1)} \quad a : (x + 1) \times (x - 1);
\]
\[
\text{(%o1)} \quad (x - 1) \times (x + 1)
\]
\[
\text{(%i2)} \quad b : \times x^2 - 1;
\]
\[
\text{(%o2)} \quad \times x^2 - 1
\]
\[
\text{(%i3)} \quad \text{[is} (a = b), \text{is} (a \# b)];
\]
\[
\text{(%o3)} \quad \text{[false, true]}
\]
\[
\text{(%i4)} \quad \text{[is} (\text{equal} (a, b)), \text{is} (\text{notequal} (a, b))];
\]
\[
\text{(%o4)} \quad \text{[true, false]}
\]

Some operators evaluate \( = \) and \# to \text{true} or \text{false}.

\[
\text{(%i1)} \quad \text{if expand} ((x + y) - 2) = \times x^2 + 2 \times x \times y + y^2 \text{ then FOO else BAR};
\]
\[
\text{(%o1)} \quad \text{FOO}
\]
\[
\text{(%i2)} \quad \text{eq}_3 : 2 \times x = 3 \times x;
\]
\[
\text{(%o2)} \quad 2 x = 3 x
\]
\[
\text{(%i3)} \quad \text{eq}_4 : \text{exp} (2) = \%e^{-2};
\]
\[
\text{(%o3)} \quad \text{2} \quad \text{2}
\]
\[
\text{(%i4)} \quad \text{[eq}_3 \text{ and eq}_4, \text{eq}_3 \text{ or eq}_4, \text{not eq}_3];
\]
\[
\text{(%o4)} \quad \text{[false, true, true]}
\]

Because \text{not expr} causes evaluation of \text{expr}, \text{not} \( a = b \) is equivalent to \( \text{is}(a \# b) \).

\[
\text{(%i1)} \quad [2 \times x \# 3 \times x, \text{not} (2 \times x = 3 \times x)];
\]
\[
\text{(%o1)} \quad [2 x \# 3 x, \text{true}]
\]
\[
\text{(%i2)} \quad \text{is} (2 \times x \# 3 \times x);
\]
7.6 Assignment operators

Assignment operator.

When the left-hand side is a simple variable (not subscripted), \( : \) evaluates its right-hand side and associates that value with the left-hand side.

When the left-hand side is a subscripted element of a list, matrix, declared Maxima array, or Lisp array, the right-hand side is assigned to that element. The subscript must name an existing element; such objects cannot be extended by naming nonexistent elements.

When the left-hand side is a subscripted element of an undeclared Maxima array, the right-hand side is assigned to that element, if it already exists, or a new element is allocated, if it does not already exist.

When the left-hand side is a list of simple and/or subscripted variables, the right-hand side must evaluate to a list, and the elements of the right-hand side are assigned to the elements of the left-hand side, in parallel.

See also \texttt{kill} and \texttt{remvalue}, which undo the association between the left-hand side and its value.

Examples:

Assignment to a simple variable.

\begin{verbatim}
(%i1) a;
(%o1) a
(%i2) a : 123;
(%o2) 123
(%i3) a;
(%o3) 123
\end{verbatim}

Assignment to an element of a list.

\begin{verbatim}
(%i1) b : [1, 2, 3];
(%o1) [1, 2, 3]
(%i2) b[3] : 456;
(%o2) 456
(%i3) b;
(%o3) [1, 2, 456]
\end{verbatim}

Assignment creates an undeclared array.

\begin{verbatim}
(%i1) c[99] : 789;
(%o1) 789
(%i2) c[99];
(%o2) 789
(%i3) c;
(%o3) c
(%i4) arrayinfo (c);
(%o4) [hashed, 1, [99]]
\end{verbatim}
Multiple assignment.

Multiple assignment is carried out in parallel. The values of \texttt{a} and \texttt{b} are exchanged in this example.

$$
\begin{align*}
\%i1) \ [a, \ b, \ c] & : [45, \ 67, \ 89]; \\
\%o1) & \ [45, \ 67, \ 89] \\
\%i2) \ a; \\
\%o2) & \ 45 \\
\%i3) \ b; \\
\%o3) & \ 67 \\
\%i4) \ c; \\
\%o4) & \ 89 \\
\end{align*}
$$

\textbf{::}

Assignment operator.

\textbf{::} is the same as \textbf{:} (which see) except that \textbf{::} evaluates its left-hand side as well as its right-hand side.

Examples:

$$
\begin{align*}
\%i1) \ x : 'foo; \\
\%o1) & \ foo \\
\%i2) \ x :: 123; \\
\%o2) & \ 123 \\
\%i3) \ foo; \\
\%o3) & \ 123 \\
\%i4) \ x : '[a, \ b, \ c]; \\
\%o4) & \ [a, \ b, \ c] \\
\%i5) \ x :: [11, \ 22, \ 33]; \\
\%o5) & \ [11, \ 22, \ 33] \\
\%i6) \ a; \\
\%o6) & \ 11 \\
\%i7) \ b; \\
\%o7) & \ 22 \\
\%i8) \ c; \\
\%o8) & \ 33 \\
\end{align*}
$$
Macro function definition operator. `::=` defines a function (called a "macro" for historical reasons) which quotes its arguments, and the expression which it returns (called the "macro expansion") is evaluated in the context from which the macro was called. A macro function is otherwise the same as an ordinary function.

`macroexpand` returns a macro expansion (without evaluating it). `macroexpand (foo (x))` followed by `''%` is equivalent to `foo (x)` when `foo` is a macro function.

`::=` puts the name of the new macro function onto the global list `macros`. `kill`, `remove`, and `remfunction` unbind macro function definitions and remove names from `macros`.

`fundef` or `dispfun` return a macro function definition or assign it to a label, respectively.

Macro functions commonly contain `buildq` and `splice` expressions to construct an expression, which is then evaluated.

Examples

A macro function quotes its arguments, so message (1) shows `y - z`, not the value of `y - z`. The macro expansion (the quoted expression `(print ("(2) x is equal to", x))`) is evaluated in the context from which the macro was called, printing message (2).

```
(%i1) x: %pi$
(%i2) y: 1234$
(%i3) z: 1729 * w$
(%i4) printq1 (x) ::= block (print ("(1) x is equal to", x),
      '(print ("(2) x is equal to", x)))$
(%i5) printq1 (y - z);
(1) x is equal to y - z
(2) x is equal to %pi
(%o5) %pi
```

An ordinary function evaluates its arguments, so message (1) shows the value of `y - z`. The return value is not evaluated, so message (2) is not printed until the explicit evaluation `''%`.

```
(%i1) x: %pi$
(%i2) y: 1234$
(%i3) z: 1729 * w$
(%i4) printe1 (x) := block (print ("(1) x is equal to", x),
      '(print ("(2) x is equal to", x)))$
(%i5) printe1 (y - z);
(1) x is equal to 1234 - 1729 w
(2) x is equal to %pi
(%o5) print((2) x is equal to, x)
(%i6) ''%;
(2) x is equal to %pi
(%o6) %pi
```

`macroexpand` returns a macro expansion. `macroexpand (foo (x))` followed by `''%` is equivalent to `foo (x)` when `foo` is a macro function.

```
(%i11) x: %pi$
```
(\%i2) y: 1234$
(\%i3) z: 1729 * w$
(\%i4) g (x) ::= buildq ([x], print ("x is equal to", x))$
(\%i5) macroexpand (g (y - z));
(\%o5) \quad \text{print(x is equal to, y - z)}$
(\%i6) ' %$;
\quad \quad \quad \text{x is equal to 1234 - 1729 w}$
(\%o6) 1234 - 1729 w$
(\%i7) g (y - z);$
\quad \quad \quad \text{x is equal to 1234 - 1729 w}$
(\%o7) 1234 - 1729 w$

[Operator]

The function definition operator.

\text{$f(x_1, \ldots, x_n) := expr$ defines a function named $f$ with arguments $x_1, \ldots, x_n$ and function body $expr$. $:= \,$ never evaluates the function body (unless explicitly evaluated by quote-quote \\"\)). The function body is evaluated every time the function is called.}$

\text{$f[x_1, \ldots, x_n] := expr$ defines a so-called array function. Its function body is evaluated just once for each distinct value of its arguments, and that value is returned, without evaluating the function body, whenever the arguments have those values again. (A function of this kind is commonly known as a \text{\text{“memoizing function\text{“}}.}$

\text{$f[x_1, \ldots, x_n](y_1, \ldots, y_m) := \text{expr}$ is a special case of an array function.}$

\text{$f[x_1, \ldots, x_n]$ is an array function which returns a lambda expression with arguments $y_1, \ldots, y_m$. The function body is evaluated once for each distinct value of $x_1, \ldots, x_n$, and the body of the lambda expression is that value.}$

When the last or only function argument $x_n$ is a list of one element, the function defined by $:= \,$ accepts a variable number of arguments. Actual arguments are assigned one-to-one to formal arguments $x_1, \ldots, x_{(n - 1)}$, and any further actual arguments, if present, are assigned to $x_n$ as a list.

All function definitions appear in the same namespace; defining a function $f$ within another function $g$ does not automatically limit the scope of $f$ to $g$. However, \text{local($f$)} makes the definition of function $f$ effective only within the block or other compound expression in which \text{local} appears.

If some formal argument $x_k$ is a quoted symbol, the function defined by $:= \,$ does not evaluate the corresponding actual argument. Otherwise all actual arguments are evaluated.

See also \text{define} and $::=\,$.

Examples:

$:= \,$ never evaluates the function body (unless explicitly evaluated by quote-quote).

(\%i1) \text{expr : cos(y) - sin(x);}$
(\%o1)$\quad \quad \quad \quad \text{cos(y) - sin(x)}$
(\%i2) \text{F1 (x, y) := expr;}$
(\%o2)$\quad \quad \quad \quad \text{F1(x, y) := expr}$
(\%i3) \text{F1 (a, b);}$
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(\%o3) \cos(y) - \sin(x)
(\%i4) F2(x, y) := 'expr;
(\%o4) \quad F2(x, y) := \cos(y) - \sin(x)
(\%i5) F2(a, b);
(\%o5) \quad \cos(b) - \sin(a)

f(x_1, \ldots, x_n) := \ldots \text{ defines an ordinary function.}

(\%i1) G1(x, y) := (print("Evaluating G1 for x=", x, " and y=", y), x.y - y.x);
(\%o1) G1(x, y) := (print("Evaluating G1 for x=", x, " and y=", y), x . y - y . x)
(\%i2) G1([1, a], [2, b]);
Evaluating G1 for x=[1, a] and y=[2, b]
(\%o2) 0
(\%i3) G1([1, a], [2, b]);
Evaluating G1 for x=[1, a] and y=[2, b]
(\%o3) 0

f[x_1, \ldots, x_n] := \ldots \text{ defines an array function.}

(\%i1) G2[a] := (print("Evaluating G2 for a=", a), a^2);
(\%o1) \quad G2 := (print("Evaluating G2 for a=", a), a)
(\%i2) G2[1234];
Evaluating G2 for a= 1234
(\%o2) 1522756
(\%i3) G2[1234];
(\%o3) 1522756
(\%i4) G2[2345];
Evaluating G2 for a= 2345
(\%o4) 5499025
(\%i5) arrayinfo(G2);
(\%o5) \quad \text{[hashed, 1, [1234], [2345]]}
(\%i6) listarray(G2);
(\%o6) \quad \text{[1522756, 5499025]}

f[x_1, \ldots, x_n](y_1, \ldots, y_m) := \text{expr} \text{ is a special case of an array function.}

(\%i1) G3[n](x) := (print("Evaluating G3 for n=", n), diff(sin(x)^2, x, n));
(\%o1) \quad G3(x) := (print("Evaluating G3 for n=", n),
\quad \quad 2 \text{ diff(sin(x), x, n))}
(\%i2) G3[2];
Evaluating G3 for n= 2
(\%o2) \quad \lambda([x], 2 \cos(x) - 2 \sin(x))
(\%i3) G3[2];
(\%o3) \quad \lambda([x], 2 \cos(x) - 2 \sin(x))
(%i4) G3[2](1);

2 2
(%o4) 2 cos (1) - 2 sin (1)

(%i5) arrayinfo (G3);

(%o5) [hashed, 1, [2]]

(%i6) listarray (G3);

2 2
(%o6) [lambda([x], 2 cos (x) - 2 sin (x))]

When the last or only function argument \( x_n \) is a list of one element, the function
defined by \( := \) accepts a variable number of arguments.

(%i1) H ([L]) := apply ("+", L);

(%o1) H([L]) := apply("+", L)

(%i2) H (a, b, c);

c + b + a

local makes a local function definition.

(%i1) foo (x) := 1 - x;

(%o1) foo(x) := 1 - x

(%i2) foo (100);

(%o2) - 99

(%i3) block (local (foo), foo (x) := 2 * x, foo (100));

(%o3) 200

(%i4) foo (100);

(%o4) - 99

7.7 User defined operators

infix

infix (op)
infix (op, lbp, rbp)
infix (op, lbp, rbp, lpos, rpos, pos)

Declares \( op \) to be an infix operator. An infix operator is a function of two arguments,
with the name of the function written between the arguments. For example, the
subtraction operator \(-\) is an infix operator.

\( \text{infix (op)} \) declares \( op \) to be an infix operator with default binding powers (left and
right both equal to 180) and parts of speech (left and right both equal to \textit{any}).

\( \text{infix (op, lbp, rbp)} \) declares \( op \) to be an infix operator with stated left and right
binding powers and default parts of speech (left and right both equal to \textit{any}).

\( \text{infix (op, lbp, rbp, lpos, rpos, pos)} \) declares \( op \) to be an infix operator with
stated left and right binding powers and parts of speech \( lpos, rpos, \) and \( pos \) for the
left operand, the right operand, and the operator result, respectively.

"Part of speech", in reference to operator declarations, means expression type. Three
types are recognized: \textit{expr}, \textit{clause}, and \textit{any}, indicating an algebraic expression, a
Boolean expression, or any kind of expression, respectively. Maxima can detect some
syntax errors by comparing the declared part of speech to an actual expression.
The precedence of \textit{op} with respect to other operators derives from the left and right binding powers of the operators in question. If the left and right binding powers of \textit{op} are both greater the left and right binding powers of some other operator, then \textit{op} takes precedence over the other operator. If the binding powers are not both greater or less, some more complicated relation holds.

The associativity of \textit{op} depends on its binding powers. Greater left binding power (\textit{lbp}) implies an instance of \textit{op} is evaluated before other operators to its left in an expression, while greater right binding power (\textit{rbp}) implies an instance of \textit{op} is evaluated before other operators to its right in an expression. Thus greater \textit{lbp} makes \textit{op} right-associative, while greater \textit{rbp} makes \textit{op} left-associative. If \textit{lbp} is equal to \textit{rbp}, \textit{op} is left-associative.

See also Section 7.1 [Introduction to operators], page 107.

Examples:

If the left and right binding powers of \textit{op} are both greater the left and right binding powers of some other operator, then \textit{op} takes precedence over the other operator.

\begin{verbatim}
(%i1) :lisp (get '$+ 'lbp)
 100
(%i1) :lisp (get '$+ 'rbp)
 100
(%i1) infix ("##", 101, 101);
  #
(%i2) "##"(a, b) := sconcat("", a, ",", b, ")");
(%o2) (a ## b) := sconcat("", a, ",", b, ")
(%i3) 1 + a ## b + 2;
  (a,b) + 3
(%i4) infix ("##", 99, 99);
  #
(%i5) 1 + a ## b + 2;
  (a+1,b+2)
\end{verbatim}

Greater \textit{lbp} makes \textit{op} right-associative, while greater \textit{rbp} makes \textit{op} left-associative.

\begin{verbatim}
(%i1) infix ("##", 100, 99);
  #
(%i2) "##"(a, b) := sconcat("", a, ",", b, ")$%
(%o2) (foo,(bar,baz))
(%i3) foo ## bar ## baz;
  (foo,(bar,baz))
(%i4) infix ("##", 100, 101);
  #
(%i5) foo ## bar ## baz;
  ((foo,bar),baz)
\end{verbatim}

Maxima can detect some syntax errors by comparing the declared part of speech to an actual expression.

\begin{verbatim}
(%i1) infix ("##", 100, 99, expr, expr, expr);
  #
(%i1) if x ## y then 1 else 0;
Incorrect syntax: Found algebraic expression where logical
\end{verbatim}
expression expected

if x ## y then

(%i2) infix ("##", 100, 99, expr, expr, clause);
(%o2) ##

(%i3) if x ## y then 1 else 0;
(%o3) if x ## y then 1 else 0

matchfix

matchfix (ldelimiter, rdelimiter)
matchfix (ldelimiter, rdelimiter, arg_pos, pos)
Declares a matchfix operator with left and right delimiters \texttt{ldelimiter} and \texttt{rdelimiter}. The delimiters are specified as strings.

A "matchfix" operator is a function of any number of arguments, such that the arguments occur between matching left and right delimiters. The delimiters may be any strings, so long as the parser can distinguish the delimiters from the operands and other expressions and operators. In practice this rules out unparsable delimiters such as %, ,, $ and ;, and may require isolating the delimiters with white space. The right delimiter can be the same or different from the left delimiter.

A left delimiter can be associated with only one right delimiter; two different matchfix operators cannot have the same left delimiter.

An existing operator may be redeclared as a matchfix operator without changing its other properties. In particular, built-in operators such as addition + can be declared matchfix, but operator functions cannot be defined for built-in operators.

The command \texttt{matchfix (ldelimiter, rdelimiter, arg\_pos, pos)} declares the argument part-of-speech \texttt{arg\_pos} and result part-of-speech \texttt{pos}, and the delimiters \texttt{ldelimiter} and \texttt{rdelimiter}.

"Part of speech", in reference to operator declarations, means expression type. Three types are recognized: \texttt{expr}, \texttt{clause}, and \texttt{any}, indicating an algebraic expression, a Boolean expression, or any kind of expression, respectively. Maxima can detect some syntax errors by comparing the declared part of speech to an actual expression.

The function to carry out a matchfix operation is an ordinary user-defined function. The operator function is defined in the usual way with the function definition operator := or define. The arguments may be written between the delimiters, or with the left delimiter as a quoted string and the arguments following in parentheses. \texttt{dispfun (ldelimiter)} displays the function definition.

The only built-in matchfix operator is the list constructor [ ]. Parentheses ( ) and double-quotes " " act like matchfix operators, but are not treated as such by the Maxima parser.

\texttt{matchfix} evaluates its arguments. \texttt{matchfix} returns its first argument, \texttt{ldelimiter}.

Examples:

Delimiters may be almost any strings.

(%i1) matchfix ("@@", "-");
(%o1) @@
(%i2) @@ a, b, c -;
Matchfix operators are ordinary user-defined functions.

(%i1) matchfix ("!-", "-!");
(%o1) "!-"
(%i2) !- x, y :- x/y - y/x;
    x y
(%o2) !-x, y :- - -
    y x
(%i3) define (!-x, y, x/y - y/x);
    x y
(%o3) !-x, y :- - -
    y x
(%i4) define ("!-" (x, y), x/y - y/x);
    x y
(%o4) !-x, y :- - -
    y x
(%i5) dispfun ("!-"); 
    x y
(%t5) !-x, y :- - -
    y x
(%i6) done
(%i7) !-3, 5-!
(%o7) 16
(%i8) !-" (3, 5);
(%o8) 16
(%i9) - --
(%o9) 15
nary

\[ \text{nary}(\text{op}) \]
\[ \text{nary}(\text{op}, \text{bp}, \text{arg-pos}, \text{pos}) \]

An \text{nary} operator is used to denote a function of any number of arguments, each of which is separated by an occurrence of the operator, e.g. A+B or A+B+C. The \text{nary}("x") function is a syntax extension function to declare \text{x} to be an \text{nary} operator. Functions may be declared to be \text{nary}. If \text{declare(j,nary);} is done, this tells the simplifier to simplify, e.g. \text{j(j(a,b),j(c,d))} to \text{j(a, b, c, d)}.

See also Section 7.1 [Introduction to operators], page 107.

nofix

\[ \text{nofix}(\text{op}) \]
\[ \text{nofix}(\text{op}, \text{pos}) \]

\text{nofix} operators are used to denote functions of no arguments. The mere presence of such an operator in a command will cause the corresponding function to be evaluated. For example, when one types "exit;" to exit from a Maxima break, "exit" is behaving similar to a \text{nofix} operator. The function \text{nofix("x")} is a syntax extension function which declares \text{x} to be a \text{nofix} operator.

See also Section 7.1 [Introduction to operators], page 107.

postfix

\[ \text{postfix}(\text{op}) \]
\[ \text{postfix}(\text{op}, \text{lbp}, \text{lpos}, \text{pos}) \]

\text{postfix} operators like the \text{prefix} variety denote functions of a single argument, but in this case the argument immediately precedes an occurrence of the operator in the input string, e.g. 3!. The \text{postfix("x")} function is a syntax extension function to declare \text{x} to be a \text{postfix} operator.

See also Section 7.1 [Introduction to operators], page 107.

prefix

\[ \text{prefix}(\text{op}) \]
\[ \text{prefix}(\text{op}, \text{rbp}, \text{rpos}, \text{pos}) \]

A \text{prefix} operator is one which signifies a function of one argument, which argument immediately follows an occurrence of the operator. \text{prefix("x")} is a syntax extension function to declare \text{x} to be a \text{prefix} operator.

See also Section 7.1 [Introduction to operators], page 107.
8 Evaluation

8.1 Functions and Variables for Evaluation

The single quote operator ' prevents evaluation.

Applied to a symbol, the single quote prevents evaluation of the symbol.

Applied to a function call, the single quote prevents evaluation of the function call, although the arguments of the function are still evaluated (if evaluation is not otherwise prevented). The result is the noun form of the function call.

Applied to a parenthesized expression, the single quote prevents evaluation of all symbols and function calls in the expression. E.g., '(f(x)) means do not evaluate the expression f(x). 'f(x) (with the single quote applied to f instead of f(x)) means return the noun form of f applied to [x].

The single quote does not prevent simplification.

When the global flag noundisp is true, nouns display with a single quote. This switch is always true when displaying function definitions.

See also the quote-quote operator [quote-quote], page 129, and nouns.

Examples:

Applied to a symbol, the single quote prevents evaluation of the symbol.

\[
\begin{align*}
(\%i1) & \text{ aa: 1024;} \\
(\%o1) & \text{ 1024} \\
(\%i2) & \text{ aa^2;} \\
(\%o2) & \text{ 1048576} \\
(\%i3) & \text{ 'aa^2;} \\
(\%o3) & \text{ 2} \\
(\%i4) & \text{ 'aa;} \\
(\%o4) & \text{ aa} \\
& \text{ 1048576}
\end{align*}
\]

Applied to a function call, the single quote prevents evaluation of the function call. The result is the noun form of the function call.

\[
\begin{align*}
(\%i1) & \text{ x: 5;} \\
(\%o1) & \text{ 5} \\
(\%i2) & \text{ x1: 7;} \\
(\%o2) & \text{ 7} \\
(\%i3) & \text{ integrate (x^2, x, x0, x1);} \\
& \text{ 218} \\
(\%o3) & \text{ ---} \\
(\%i4) & \text{ 'integrate (x^2, x, x0, x1);} \\
& \text{ 3}
\end{align*}
\]
Applied to a parenthesized expression, the single quote prevents evaluation of all symbols and function calls in the expression.

\begin{verbatim}
(%i1) aa: 1024;
(%o1) 1024
(%i2) bb: 19;
(%o2) 19
(%i3) sqrt(aa) + bb;
(%o3) 51
(%i4) '(sqrt(aa) + bb);
(%o4) bb + sqrt(aa)
(%i5) ''%;
(%o5) 51
\end{verbatim}

The single quote does not prevent simplification.

\begin{verbatim}
(%i1) sin (17 * %pi) + cos (17 * %pi);
(%o1) - 1
(%i2) '(sin (17 * %pi) + cos (17 * %pi));
(%o2) - 1
\end{verbatim}

Maxima considers floating point operations by its in-built mathematical functions to be a simplification.

\begin{verbatim}
(%i1) sin(1.0);
(%o1) .8414709848078965
(%i2) '(sin(1.0));
(%o2) .8414709848078965
\end{verbatim}

When the global flag \texttt{noundisp} is \texttt{true}, nouns display with a single quote.

\begin{verbatim}
(%i1) x:%pi;
(%o1) %pi
(%i2) bfloat(x);
(%o2) 3.141592653589793b0
(%i3) sin(x);
(%o3) 0
(%i4) noundisp;
(%o4) false
(%i5) 'bfloat(x);
(%o5) bfloat(3.141592653589793b0)
\end{verbatim}
The quote-quote operator "" (two single quote marks) modifies evaluation in input expressions.

Applied to a general expression expr, quote-quote causes the value of expr to be substituted for expr in the input expression.

Applied to the operator of an expression, quote-quote changes the operator from a noun to a verb (if it is not already a verb).

The quote-quote operator is applied by the input parser; it is not stored as part of a parsed input expression. The quote-quote operator is always applied as soon as it is parsed, and cannot be quoted. Thus quote-quote causes evaluation when evaluation is otherwise suppressed, such as in function definitions, lambda expressions, and expressions quoted by single quote '.

Quote-quote is recognized by batch and load.

See also ev, the single-quote operator [quote], page 127, and nouns.

Examples:

Applied to a general expression expr, quote-quote causes the value of expr to be substituted for expr in the input expression.

(\%i1) expand ((a + b)^3);
(\%o1) \(\frac{1}{3} a + \frac{2}{3} b + \frac{3}{3} a b + a\)
(\%i2) [...,""...];
(\%o2) [expand((b + a) ), b + 3 a b + 3 a b + a ]
(\%o3) [\%i1, ""\%i1];
(\%o3) [expand((b + a) ), b + 3 a b + 3 a b + a ]
(\%i4) [aa : cc, bb : dd, cc : 17, dd : 29];
(%o4) [cc, dd, 17, 29]
(%i5) foo_1 (x) := aa - bb * x;
(%o5) foo_1(x) := aa - bb x
(%i6) foo_1 (10);
(%o6) cc - 10 dd
(%i7) '%;
(%o7) - 273
(%i8) '(foo_1 (10));
(%o8) - 273
(%i9) foo_2 (x) := ''aa - ''bb * x;
(%o9) foo_2(x) := cc - dd x
(%i10) foo_2 (10);
(%o10) - 273
(%i11) [x0 : x1, x1 : x2, x2 : x3];
(%o11) [x1, x2, x3]
(%i12) x0;
(%o12) x1
(%i13) ''x0;
(%o13) x2
(%i14) ''''x0;
(%o14) x3

Applied to the operator of an expression, quote-quote changes the operator from a noun to a verb (if it is not already a verb).

(%i1) declare (foo, noun);  
(%o1) done
(%i2) foo (x) := x - 1729;
(%o2) foo(x) := x - 1729
(%i3) foo (100);
(%o3) foo(100)
(%o4) - 1629

The quote-quote operator is applied by the input parser; it is not stored as part of a parsed input expression.

(%i1) [aa : bb, cc : dd, bb : 1234, dd : 5678];
(%o1) [bb, dd, 1234, 5678]
(%i2) aa + cc;
(%o2) dd + bb
(%i3) display (_, op (_), args (_));
_ = cc + aa
op(cc + aa) = +
args(cc + aa) = [cc, aa]
(%o3) done
(%i4) ''(aa + cc);
Quote-quote causes evaluation when evaluation is otherwise suppressed, such as in function definitions, lambda expressions, and expressions quoted by single quote '.

(%i11) foo_1a (x) := ''(integrate (log (x), x));
(%o11) foo_1a(x) := x log(x) - x
(%i12) foo_1b (x) := integrate (log (x), x);
(%o12) foo_1b(x) := integrate(log(x), x)
(%i13) dispfun (foo_1a, foo_1b);
(%o13) [foo_1a(x) := x log(x) - x, foo_1b(x) := integrate(log(x), x)]
(%i14) integrate (log (x), x);
(%o14) x log(x) - x
(%i15) foo_2a (x) := ''%;
(%o15) foo_2a(x) := %
(%i16) foo_2b (x) := %;
(%o16) foo_2b(x) := %
(%i17) dispfun (foo_2a, foo_2b);
(%o17) [foo_2a(x) := %, foo_2b(x) := %]
(%i18) F : lambda ([u], diff (sin (u), u));
(%o18) lambda([u], diff(sin(u), u))
(%i19) G : lambda ([u], ''(diff (sin (u), u)));
(%o19) lambda([u], cos(u))
(%i20) '('(sum (a[k], k, 1, 3) + sum (b[k], k, 1, 3));
(%o20) b[3] + a + b + a + b + a
(%i21) '(''(sum (a[k], k, 1, 3)) + '''(sum (b[k], k, 1, 3)));
(%o21) b + a + b + a + b + a
     3 3 2 2 1 1
ev (expr, arg_1, ..., arg_n)  [Function]
Evaluates the expression expr in the environment specified by the arguments arg_1, ...
arg_n. The arguments are switches (Boolean flags), assignments, equations, and functions.
ev returns the result (another expression) of the evaluation.

The evaluation is carried out in steps, as follows.
1. First the environment is set up by scanning the arguments which may be any or
all of the following.
   - simp causes expr to be simplified regardless of the setting of the switch simp
     which inhibits simplification if false.
   - noeval suppresses the evaluation phase of ev (see step (4) below). This
     is useful in conjunction with the other switches and in causing expr to be
     resimplified without being reevaluated.
   - nouns causes the evaluation of noun forms (typically unevaluated functions
     such as 'integrate or 'diff) in expr.
   - expand causes expansion.
   - expand (m, n) causes expansion, setting the values of maxposex and
     maxnegex to m and n respectively.
   - detout causes any matrix inverses computed in expr to have their determi-
     nant kept outside of the inverse rather than dividing through each element.
   - diff causes all differentiations indicated in expr to be performed.
   - derivlist (x, y, z, ...) causes only differentiations with respect to the
     indicated variables. See also derivlist.
   - risch causes integrals in expr to be evaluated using the Risch algorithm. See
     risch. The standard integration routine is invoked when using the special
     symbol nouns.
   - float causes non-integral rational numbers to be converted to floating point.
   - numer causes some mathematical functions (including exponentiation) with
     numerical arguments to be evaluated in floating point. It causes variables
     in expr which have been given numervals to be replaced by their values. It
     also sets the float switch on.
   - pred causes predicates (expressions which evaluate to true or false) to be
     evaluated.
   - eval causes an extra post-evaluation of expr to occur. (See step (5) below.)
     eval may occur multiple times. For each instance of eval, the expression is
     evaluated again.
   - A where A is an atom declared to be an evaluation flag evflag causes A to
     be bound to true during the evaluation of expr.
   - V: expression (or alternately V=expression) causes V to be bound to the
     value of expression during the evaluation of expr. Note that if V is a
     Maxima option, then expression is used for its value during the evaluation
     of expr. If more than one argument to ev is of this type then the binding is
done in parallel. If V is a non-atomic expression then a substitution rather
than a binding is performed.
• $F$ where $F$, a function name, has been declared to be an evaluation function $\text{evfun}$ causes $F$ to be applied to $\text{expr}$.

• Any other function names, e.g. $\text{sum}$, cause evaluation of occurrences of those names in $\text{expr}$ as though they were verbs.

• In addition a function occurring in $\text{expr}$ (say $F(x)$) may be defined locally for the purpose of this evaluation of $\text{expr}$ by giving $F(x) := \text{expression}$ as an argument to $\text{ev}$.

• If an atom not mentioned above or a subscripted variable or subscripted expression was given as an argument, it is evaluated and if the result is an equation or assignment then the indicated binding or substitution is performed. If the result is a list then the members of the list are treated as if they were additional arguments given to $\text{ev}$. This permits a list of equations to be given (e.g. $[X=1, Y=A**2]$) or a list of names of equations (e.g., [%t1, %t2] where %t1 and %t2 are equations) such as that returned by $\text{solve}$.

The arguments of $\text{ev}$ may be given in any order with the exception of substitution equations which are handled in sequence, left to right, and evaluation functions which are composed, e.g., $\text{ev} (\text{expr}, \text{ratsimp}, \text{realpart})$ is handled as $\text{realpart} (\text{ratsimp} (\text{expr}))$.

The $\text{simp}$, $\text{numer}$, and $\text{float}$ switches may also be set locally in a block, or globally in Maxima so that they will remain in effect until being reset.

If $\text{expr}$ is a canonical rational expression (CRE), then the expression returned by $\text{ev}$ is also a CRE, provided the $\text{numer}$ and $\text{float}$ switches are not both true.

2. During step (1), a list is made of the non-subscripted variables appearing on the left side of equations in the arguments or in the value of some arguments if the value is an equation. The variables (subscripted variables which do not have associated array functions as well as non-subscripted variables) in the expression $\text{expr}$ are replaced by their global values, except for those appearing in this list. Usually, $\text{expr}$ is just a label or % (as in %i2 in the example below), so this step simply retrieves the expression named by the label, so that $\text{ev}$ may work on it.

3. If any substitutions are indicated by the arguments, they are carried out now.

4. The resulting expression is then re-evaluated (unless one of the arguments was $\text{noeval}$) and simplified according to the arguments. Note that any function calls in $\text{expr}$ will be carried out after the variables in it are evaluated and that $\text{ev}(F(x))$ thus may behave like $F(\text{ev}(x))$.

5. For each instance of $\text{eval}$ in the arguments, steps (3) and (4) are repeated.

See also $[\text{quote-quote}]$, page 129, $\text{at}$ and $\text{subst}$.

Examples:

((%i1) $\sin(x) + \cos(y) + (w+1)^2 + \text{'diff} (\sin(w), w);$  

$$\frac{d}{dw} 2$$

((%o1) $\cos(y) + \sin(x) + \frac{-- (\sin(w)) + (w + 1)}{dw};$  

((%i2) $\text{ev} (\%, \text{numer}, \text{expand}, \text{diff}, x=2, y=1);$  

((%o2) $\cos(w) + w + 2 w + 2.449599732693821$
An alternate top level syntax has been provided for \texttt{ev}, whereby one may just type in its arguments, without the \texttt{ev()}. That is, one may write simply
\begin{verbatim}
  expr, arg_1, ..., arg_n
\end{verbatim}
This is not permitted as part of another expression, e.g., in functions, blocks, etc. Notice the parallel binding process in the following example.

\begin{verbatim}
(%i3) programmode: false;
(%o3) false
(%i4) x+y, x: a+y, y: 2;
(%o4) y + a + 2
(%i5) 2*x - 3*y = 3$
(%i6) -3*x + 2*y = -4$
(%i7) solve ([%o5, %o6]);
Solution

1
(%t7) y = - -
      5

(%t8) x = -
      5

(%o8) [[%t7, %t8]]
(%i8) %o6, %o8;
(%o8) - 4 = - 4
(%i9) x + 1/x > gamma (1/2);

1
(%o9) x + - > sqrt(%pi)
    x
(%i10) %, numer, x=1/2;
(%o10) 2.5 > 1.772453850905516
(%i11) %, pred;
(%o11) true
\end{verbatim}

\texttt{eval}

As an argument in a call to \texttt{ev} (\texttt{expr}), \texttt{eval} causes an extra evaluation of \texttt{expr}. See \texttt{ev}.

**Example:**

\begin{verbatim}
(%i1) [a:b,b:c,c:d,d:e];
(%o1) [b, c, d, e]
(%i2) a;
(%o2) b
(%i3) ev(a);
(%o3) c
(%i4) ev(a),eval;
(%o4) e
(%i5) a,eval,eval;
\end{verbatim}
evflag [Property]

When a symbol \( x \) has the \texttt{evflag} property, the expressions \texttt{ev(expr, x)} and \texttt{expr}, \( x \) (at the interactive prompt) are equivalent to \texttt{ev(expr, x = true)}. That is, \( x \) is bound to \texttt{true} while \texttt{expr} is evaluated.

The expression \texttt{declare(x, evflag)} gives the \texttt{evflag} property to the variable \( x \).

The flags which have the \texttt{evflag} property by default are the following:

- \texttt{algebraic}
- \texttt{cauchysum}
- \texttt{demoivre}
- \texttt{dottstyles}
- \texttt{emode}
- \texttt{enumear}
- \texttt{exponentialize}
- \texttt{exptisolate}
- \texttt{factorflag}
- \texttt{float}
- \texttt{halfangles}
- \texttt{infeval}
- \texttt{isolate_wrt_times}
- \texttt{keefloat}
- \texttt{letrat}
- \texttt{listarith}
- \texttt{logabs}
- \texttt{logarc}
- \texttt{logexpand}
- \texttt{lognegint}
- \texttt{logarc}
- \texttt{logexpand}
- \texttt{lognegint}
- \texttt{m1pbranch}
- \texttt{numer_pbranch}
- \texttt{programmode}
- \texttt{radexpand}
- \texttt{ratalgdenom}
- \texttt{ratfac}
- \texttt{ratmx}
- \texttt{ratsimpexpons}
- \texttt{simp}
- \texttt{simpproduct}
- \texttt{simpsum}
- \texttt{sumexpand}
- \texttt{trigexpand}

Examples:

\begin{verbatim}
(%i1) sin (1/2);
    1
(%o1) sin(-)
    2

(%i2) sin (1/2), float;
(%o2) 0.479425538604203

(%i3) sin (1/2), float=true;
(%o3) 0.479425538604203

(%i4) simp : false;
(%o4) false

(%i5) 1 + 1;
(%o5) 1 + 1

(%i6) 1 + 1, simp;
(%o6) 2

(%i7) simp : true;
(%o7) true

(%i8) sum (1/k^2, k, 1, inf);
inf
====
\ |
/ 1

(%o8) > --

/ 2

==== k
k = 1

(%i9) sum (1/k^2, k, 1, inf), simppsum;
\end{verbatim}
\[
\frac{\pi}{6}
\]

(%i10) declare (aa, evflag);
(%o10) done

(%i11) if aa = true then YES else NO;
(%o11) NO

(%i12) if aa = true then YES else NO, aa;
(%o12) YES

**evfun** [Property]

When a function \( F \) has the **evfun** property, the expressions \( \text{ev}(\text{expr}, F) \) and \( \text{expr} \), \( F \) (at the interactive prompt) are equivalent to \( F(\text{ev}(\text{expr})) \).

If two or more **evfun** functions \( F, G, \) etc., are specified, the functions are applied in the order that they are specified.

The expression \( \text{declare}(F, \text{evfun}) \) gives the **evfun** property to the function \( F \). The functions which have the **evfun** property by default are the following:

- bfloat
- factor
- fullratsimp
- logcontract
- polarform
- radcan
- ratexpand
- ratsimp
- rectform
- rootscontract
- trigexpand
- trigreduce

Examples:

(%i1) \( x^3 - 1 \);
(%o1) \( x - 1 \)

(%i2) \( x^3 - 1 \), factor;
(%o2) \( (x - 1) (x + x + 1) \)

(%i3) factor (\( x^3 - 1 \));
(%o3) \( (x - 1) (x + x + 1) \)

(%i4) \( \frac{\cos(4 \times x)}{\sin(x)^4} \);
(%o4) \( \frac{\cos(4 \times x)}{4\sin(x)} \)

(%i5) \( \frac{\cos(4 \times x)}{\sin(x)^4}, \text{trigexpand} \);
(%o5) \( \frac{\sin(x) - 6\cos(x)\sin(x) + \cos(x)}{\sin(x) - 6\cos(x)\sin(x) + \cos(x)} \)

(%i6) \( \frac{\cos(4 \times x)}{\sin(x)^4}, \text{trigexpand, ratexpand} \);
(%o6) \( 2\sin(x) - 6\cos(x)\cos(x) \)
(\%o6) \quad \frac{- \frac{1}{2} \sin(x) + \frac{1}{4} \sin(x)}{\sin(x)} + 1

(\%i7) \text{ratexpand(\text{trigexpand(} \frac{\cos(4x)}{\sin(x)^4})\text{)}};

\quad \frac{6 \cos(x) \cos(x)}{\sin(x) \sin(x)}

(\%o7) \quad \frac{- \frac{1}{2} \cos(x) + \frac{1}{4} \cos(x)}{\sin(x)} + 1

(\%i8) \text{declare([F, G], evfun)};

(\%o8) \text{done}

(\%i9) (aa : bb, bb : cc, cc : dd);

(\%o9) dd

(\%i10) aa;

(\%o10) bb

(\%i11) aa, F;

(\%o11) F(cc)

(\%i12) F(aa);

(\%o12) F(bb)

(\%i13) F(ev(aa));

(\%o13) F(cc)

(\%i14) aa, F, G;

(\%o14) G(F(cc))

(\%i15) G(F(ev(aa)));

(\%o15) G(F(cc))

\textbf{infeval} \quad [\text{Option variable}]

Enables "infinite evaluation" mode. \textbf{ev} repeatedly evaluates an expression until it stops changing. To prevent a variable, say \textit{X}, from being evaluated away in this mode, simply include \textit{X}'X as an argument to \textbf{ev}. Of course expressions such as \textit{ev}(X, X=X+1, \text{infeval}) will generate an infinite loop.

\textbf{noeval} \quad [\text{Special symbol}]

\textbf{noeval} suppresses the evaluation phase of \textbf{ev}. This is useful in conjunction with other switches and in causing expressions to be resimplified without being reevaluated.

\textbf{nouns} \quad [\text{Special symbol}]

\textbf{nouns} is an \textbf{evflag}. When used as an option to the \textbf{ev} command, \textbf{nouns} converts all "noun" forms occurring in the expression being \textit{ev’d} to "verbs", i.e., evaluates them. See also \textit{noun}, \textit{nounify}, \textit{verb}, and \textit{verbify}.

\textbf{pred} \quad [\text{Special symbol}]

As an argument in a call to \textit{ev (expr)}, \textbf{pred} causes predicates (expressions which evaluate to \textit{true} or \textit{false}) to be evaluated. See \textit{ev}.

Example:

(\%i1) 1<2;
(\%o1) 1 < 2
(\%i2) 1<2,pred;
(%o2) true
9 Simplification

9.1 Introduction to Simplification

Maxima performs a cycle of actions in response to each new user-typed command. This consists of four steps: reading or "parsing" the input, evaluation, simplification and output. Parsing converts a syntactically valid sequence of typed characters into a data structure to be used for the rest of the operations. Evaluation includes replacing names with their assigned values. Simplification means rewriting an expression to be easier for the user or other programs to understand. Output includes displaying computational results in a variety of different formats and notations.

Evaluation and simplification sometimes appear to have similar functionality as they both have the goal of removing "complexity" and system designers have sometimes divided a task so that it is performed partly in each. For example, \texttt{integrate(x,x)} evaluates the answer as \(\frac{x^2}{2}\), which is then simplified to \(\frac{x^2}{2}\).

Evaluation is always present: it is the consequence of having a programming system with functions, subroutines, variables, values, loops, assignments and so on. In the evaluation step, built-in or user-defined function names are replaced by their definitions, variables are replaced by their values. This is largely the same as activities of a conventional programming language, but extended to work with symbolic mathematical data. Because of the generality of the mathematics at hand, there are different possible models of evaluation and so the systems has optional "flags" that can steer the process of evaluation. See Section 8.1 [Functions and Variables for Evaluation], page 127.

By contrast, the intent of simplification is to maintain the value of an expression while re-formulating its representation to be smaller, simpler to understand, or to conform to particular specifications (like factored, expanded). For example, \(\sin(0)\) to 0 or \(x+x\) to \(2x\). There are several powerful tools to alter the results of simplification, since it is largely in this part of the system that a user can incorporate knowledge of newly introduced functions or symbolic notation into Maxima.

Simplification is generally done at four different levels:

- The internal, built-in automated simplifier,
- Built-in simplification routines that can be explicitly called by the user at selected places in a program or command sequence,
- User-written simplification routines, linked to the simplifier by using "tellsimp" or "tellsimpafter" and called automatically,
- User-written routines that can be explicitly called by the user at selected places in a program or command sequence.

The internal simplifier belongs to the heart of Maxima. It is a large and complicated collection of programs, and it has been refined over many years and by thousands of users. Nevertheless, especially if you are trying out novel ideas or unconventional notation, you may find it helpful to make small (or large) changes to the program yourself. For details see for example the paper at the end of \url{https://people.eecs.berkeley.edu/~fateman/papers/intro5.txt}. 
Maxima internally represents expressions as "trees" with operators or "roots" like +, *, = and operands ("leaves") which are variables like \(x, y, z\), functions or sub-trees, like \(x*y\). Each operator has a simplification program associated with it. + (which also covers binary - since \(a-b = a+(-1)*b\)) and * (which also covers / since \(a/b = a*b^(-1)\)) have rather elaborate simplification programs. These simplification programs (simpus, simptimes, simpexpt, etc.) are called whenever the simpler encounters the respective arithmetic operators in an expression tree to be analyzed.

The structure of the simpler dates back to 1965, and many hands have worked on it through the years. The structure turns out to be, in modern jargon, data-directed, or object-oriented. The program dispatches to the appropriate routine depending on the root of some sub-tree of the expression, recursively. This general notion means you can make modifications to the simplification process by very local changes to the program. In many cases it is conceptually straightforward to add an operator and add its simplification routine without disturbing existing code.

We note that in addition to this general simpler operating on algebraic expression trees, there are several other representations of expressions in Maxima which have separate methods and simplifiers. For example, the \texttt{rat()} function converts polynomials to vectors of coefficients to assist in rapid manipulation of such forms. Other representations include Taylor series and the (rarely used) Poisson series.

All operators introduced by the user initially have no simplification programs associated with them. Maxima does not know anything about function "f" and so typing \texttt{f(a,b)} will result in simplifying \(a*b\), but not \(f\). Even some built-in operators have no simplifications. For example, = does not "simplify" – it is a place-holder with no simplification semantics other than to simplify its two arguments, in this case referred to as the left and right sides. Other parts of Maxima such as the solve program take special note of equations, that is, trees with = as the root. (Note – in Maxima, the assignment operation is \\texttt{:}. That is, \texttt{q: 4} sets the value of the symbol \texttt{q} to \texttt{4}. Function definition is done with \\texttt{:=}.)

The general simpler returns results with an internal flag indicating the expression and each sub-expression has been simplified. This does not guarantee that it is unique over all possible equivalent expressions. That’s too hard (theoretically, not possible given the generality of what can be expressed in Maxima). However, some aspects of the expression, such as the ordering of terms in a sum or product, are made uniform. This is important for the other programs to work properly.

You can set a number of option variables which direct Maxima’s processing to favor particular kinds of patterns as being goals. You can even use the most extreme option which is to turn the simpler off by \texttt{simp:false}. We do not recommend this since many internal routines expect their arguments to be simplified. (About the only time it seems plausible to turn off the simpler is in the rare case that you want to over-ride a built-in simplification. In that case you might temporarily disable the simpler, put in the new transformation via \texttt{tellsimp}, and then re-enable the simpler by \texttt{simp:true}.)

It is more plausible for you to associate user-defined symbolic function names or operators with properties (\texttt{additive}, \texttt{lassociative}, \texttt{oddfun}, \texttt{antisymmetric}, \texttt{linear}, \texttt{outative}, \texttt{commutative}, \texttt{multiplicative}, \texttt{rassociative}, \texttt{evenfun}, \texttt{nary} and \texttt{symmetric}). These options steer the simpler processing in systematic directions.
For example, `declare(f, oddfun)` specifies that \( f \) is an odd function. Maxima will simplify \( f(-x) \) to \(-f(x)\). In the case of an even function, that is `declare(g, evenfun)`, Maxima will simplify \( g(-x) \) to \( g(x) \). You can also associate a programming function with a name such as \( h(x) := x^2 + 1 \). In that case the evaluator will immediately replace \( h(3) \) by 10, and \( h(a+1) \) by \((a+1)^2 + 1\), so any properties of \( h \) will be ignored.

In addition to these directly related properties set up by the user, facts and properties from the actual context may have an impact on the simplifier’s behavior, too. See Section 11.1 [Introduction to Maximas Database], page 183.

Example: \( \sin(n \cdot \%pi) \) is simplified to zero, if \( n \) is an integer.

```lisp
(%i1) sin(n * %pi);
(%o1) sin(%pi n)
(%i2) declare(n, integer);
(%o2) done
(%i3) sin(n * %pi);
(%o3) 0
```

If automated simplification is not sufficient, you can consider a variety of built-in, but explicitly called simplification functions (`ratsimp`, `expand`, `factor`, `radcan` and others). There are also flags that will push simplification into one or another direction. Given `demoivre:true` the simplifier rewrites complex exponentials as trigonometric forms. Given `exponentialize:true` the simplifier tries to do the reverse: rewrite trigonometric forms as complex exponentials.

As everywhere in Maxima, by writing your own functions (be it in the Maxima user language or in the implementation language Lisp) and explicitly calling them at selected places in the program, you can respond to your individual simplification needs. Lisp gives you a handle on all the internal mechanisms, but you rarely need this full generality. "Tellsimp" is designed to generate much of the Lisp internal interface into the simplifier automatically. See Chapter 34 [Rules and Patterns], page 549.

Over the years (Maxima/Macsyma’s origins date back to about 1966!) users have contributed numerous application packages and tools to extend or alter its functional behavior. Various non-standard and "share" packages exist to modify or extend simplification as well. You are invited to look into this more experimental material where work is still in progress. See Chapter 80 [simplification-pkg], page 1039.

The following appended material is optional on a first reading, and reading it is not necessary for productive use of Maxima. It is for the curious user who wants to understand what is going on, or the ambitious programmer who might wish to change the (open-source) code. Experimentation with redefining Maxima Lisp code is easily possible: to change the definition of a Lisp program (say the one that simplifies \( \cos() \), named `simp%cos`), you simply load into Maxima a text file that will overwrite the `simp%cos` function from the `maxima` package.

### 9.2 Functions and Variables for Simplification

**Property**

- **additive**

  If `declare(f, additive)` has been executed, then:
(1) If \( f \) is univariate, whenever the simplifier encounters \( f \) applied to a sum, \( f \) will be distributed over that sum. I.e. \( f(y+x) \) will simplify to \( f(y)+f(x) \).

(2) If \( f \) is a function of 2 or more arguments, additivity is defined as additivity in the first argument to \( f \), as in the case of \texttt{sum} or \texttt{integrate}, i.e. \( f(h(x)+g(x),x) \) will simplify to \( f(h(x),x)+f(g(x),x) \). This simplification does not occur when \( f \) is applied to expressions of the form \texttt{sum(x[i],i,lower-limit,upper-limit)}.

Example:

\[
\begin{align*}
(\%i1) & \quad \text{F3 (a + b + c);} \\
(\%o1) & \quad \text{F3(c + b + a)} \\
(\%i2) & \quad \text{declare (F3, additive);} \\
(\%o2) & \quad \text{done} \\
(\%i3) & \quad \text{F3 (a + b + c);} \\
(\%o3) & \quad \text{F3(c) + F3(b) + F3(a)}
\end{align*}
\]

\textbf{Property antisymmetric}

If \texttt{declare(h,antisymmetric)} is done, this tells the simplifier that \( h \) is antisymmetric. E.g. \( h(x,z,y) \) will simplify to \(- h(x, y, z) \). That is, it will give \((-1)^n\) times the result given by \texttt{symmetric} or \texttt{commutative}, where \( n \) is the number of interchanges of two arguments necessary to convert it to that form.

Examples:

\[
\begin{align*}
(\%i1) & \quad \text{S (b, a);} \\
(\%o1) & \quad \text{S(b, a)} \\
(\%i2) & \quad \text{declare (S, symmetric);} \\
(\%o2) & \quad \text{done} \\
(\%i3) & \quad \text{S (b, a);} \\
(\%o3) & \quad \text{S(a, b)} \\
(\%i4) & \quad \text{S (a, c, e, d, b);} \\
(\%o4) & \quad \text{S(a, b, c, d, e)} \\
(\%i5) & \quad \text{T (b, a);} \\
(\%o5) & \quad \text{T(b, a)} \\
(\%i6) & \quad \text{declare (T, antisymmetric);} \\
(\%o6) & \quad \text{done} \\
(\%i7) & \quad \text{T (b, a);} \\
(\%o7) & \quad \text{- T(a, b)} \\
(\%i8) & \quad \text{T (a, c, e, d, b);} \\
(\%o8) & \quad \text{T(a, b, c, d, e)}
\end{align*}
\]

\textbf{Function combine (expr)}

Simplifies the sum \( expr \) by combining terms with the same denominator into a single term.

Example:

\[
\begin{align*}
(\%i1) & \quad 1*f/2*b + 2*c/3*a + 3*f/4*b +c/5*b*a; \\
(\%o1) & \quad ----- + ----- + ----- \\
& \quad 4 \quad 5 \quad 3
\end{align*}
\]
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(%i2) combine (%);
75 b f + 4 (3 a b c + 10 a c)
(%o2) -------------------------------------
60

[Property] commutative

If declare(h, commutative) is done, this tells the simplifier that h is a commutative function. E.g. h(x, z, y) will simplify to h(x, y, z). This is the same as symmetric.

Exemplo:

(%i1) S (b, a);
(%o1) S(b, a)
(%i2) S (a, b) + S (b, a);
(%o2) S(b, a) + S(a, b)
(%i3) declare (S, commutative);
(%o3) done
(%i4) S (b, a);
(%o4) S(a, b)
(%i5) S (a, b) + S (b, a);
(%o5) 2 S(a, b)
(%i6) S (a, c, e, d, b);
(%o6) S(a, b, c, d, e)

demoivre (expr)

demoivre

The function demoivre (expr) converts one expression without setting the global variable demoivre.

When the variable demoivre is true, complex exponentials are converted into equivalent expressions in terms of circular functions: \( \exp (a + b*%i) \) simplifies to \( \%e^a * (\cos(b) + %i*\sin(b)) \) if \( b \) is free of \( %i \). \( a \) and \( b \) are not expanded.

The default value of demoivre is false.

demoivre and exponentialize cannot both be true at the same time.

distrib (expr)

distrib

Distributes sums over products. It differs from expand in that it works at only the top level of an expression, i.e., it doesn’t recurse and it is faster than expand. It differs from multthru in that it expands all sums at that level.

Examples:

(%i1) distrib ((a+b) * (c+d));
(%o1) b d + a d + b c + a c
(%i2) multthru ((a+b) * (c+d));
(%o2) (b + a) d + (b + a) c
(%i3) distrib (1/((a+b) * (c+d)));
(%o3) ------------------------
(b + a) (d + c)
\begin{verbatim}
(%i4) expand (1/((a+b) * (c+d)), 1, 0);
  1
(%o4) ---------------------
     b d + a d + b c + a c
\end{verbatim}

distribute_over

[Option variable]

Default value: true

distribute_over controls the mapping of functions over bags like lists, matrices, and equations. At this time not all Maxima functions have this property. It is possible to look up this property with the command properties.

The mapping of functions is switched off, when setting distribute_over to the value false.

Examples:

The \texttt{sin} function maps over a list:

\begin{verbatim}
(%i11) sin([x,1,1.0]);
(%o11) [sin(x), sin(1), 0.8414709848078965]
\end{verbatim}

\texttt{mod} is a function with two arguments which maps over lists. Mapping over nested lists is possible too:

\begin{verbatim}
(%i11) mod([x,11,2*a],10);
(%o11) [mod(x, 10), 1, 2 mod(a, 5)]
(%i12) mod([[x,y,z],11,2*a],10);
(%o12) [[mod(x, 10), mod(y, 10), mod(z, 10)], 1, 2 mod(a, 5)]
\end{verbatim}

Mapping of the \texttt{floor} function over a matrix and an equation:

\begin{verbatim}
(%i11) floor(matrix([a,b],[c,d]));
     [ floor(a)  floor(b) ]
(%o11) [                      ]
     [ floor(c)  floor(d) ]
(%i12) floor(a=b);
(%o12) floor(a) = floor(b)
\end{verbatim}

Functions with more than one argument map over any of the arguments or all arguments:

\begin{verbatim}
(%i11) expintegral_e([1,2],[x,y]);
(%o11) [[expintegral_e(1, x), expintegral_e(1, y)],
     [expintegral_e(2, x), expintegral_e(2, y)]]
\end{verbatim}

Check if a function has the property distribute_over:

\begin{verbatim}
(%i11) properties(abs);
(%o11) [integral, rule, distributes over bags, noun, gradef, system function]
\end{verbatim}

The mapping of functions is switched off, when setting distribute_over to the value false.

\begin{verbatim}
(%i11) distribute_over;
(%o11) true
(%i12) sin([x,1,1.0]);
(%o12) [sin(x), sin(1), 0.8414709848078965]
\end{verbatim}
(\%i3) distribute_over : not distribute_over;
(\%o3) false
(\%i4) sin([x,1,1.0]);
(\%o4) sin([x, 1, 1.0])

**domain**

[Option variable]

**Default value:** real

When **domain** is set to complex, sqrt (x^2) will remain sqrt (x^2) instead of returning abs(x).

**evenfun**

[Property]

**oddfun**

[Property]

```
declarate(f, evenfun) or declare(f, oddfun) tells Maxima to recognize the function f as an even or odd function.
```

**Examples:**

(\%i1) o (- x) + o (x);
(\%o1) o(x) + o(- x)
(\%i2) declare (o, oddfun);
(\%o2) done
(\%i3) o (- x) + o (x);
(\%o3) 0
(\%i4) e (- x) - e (x);
(\%o4) e(- x) - e(x)
(\%i5) declare (e, evenfun);
(\%o5) done
(\%i6) e (- x) - e (x);
(\%o6) 0

**expand**

[Function]

```
expand (expr)
expand (expr, p, n)
```

Expand expression expr. Products of sums and exponentiated sums are multiplied out, numerators of rational expressions which are sums are split into their respective terms, and multiplication (commutative and non-commutative) are distributed over addition at all levels of expr.

For polynomials one should usually use ratexpand which uses a more efficient algorithm.

**maxnegex** and **maxposex** control the maximum negative and positive exponents, respectively, which will expand.

**expand (expr, p, n)** expands expr, using p for maxposex and n for maxnegex. This is useful in order to expand part but not all of an expression.

**expon** - the exponent of the largest negative power which is automatically expanded (independent of calls to expand). For example if **expon** is 4 then (x+1)^(-5) will not be automatically expanded.

**expop** - the highest positive exponent which is automatically expanded. Thus (x+1)^(-3), when typed, will be automatically expanded only if **expop** is greater than
or equal to 3. If it is desired to have \((x+1)^n\) expanded where \(n\) is greater than \(\text{expop}\) then executing \(\text{expand} ((x+1)^n)\) will work only if \(\text{maxposex}\) is not less than \(n\).

\(\text{expand}(\text{expr}, 0, 0)\) causes a resimplification of \(\text{expr}\). \(\text{expr}\) is not reevaluated. In distinction from \(\text{ev}(\text{expr}, \text{noeval})\) a special representation (e. g. a CRE form) is removed. See also \(\text{ev}\).

The \(\text{expand}\) flag used with \(\text{ev}\) causes expansion.

The file \texttt{share/simplification/facexp.mac} contains several related functions (in particular \texttt{facsum, factorfacsum} and \texttt{collectterms}, which are autoloaded) and variables (\texttt{nextlayerfactor} and \texttt{facsum_combine}) that provide the user with the ability to structure expressions by controlled expansion. Brief function descriptions are available in \texttt{simplification/facexp.usg}. A demo is available by doing \texttt{demo("facexp")}.

Examples:

\begin{verbatim}
(%i1) expr: (x+1)^2*(y+1)^3;
2 3
(%o1) (x + 1) (y + 1)
(%i2) expand(expr);
2 3 3 3 2 2 2 2 2
x y + 2 x y + y + 3 x y + 6 x y + 3 y + 3 x y
2
+ 6 x y + 3 y + x + 2 x + 1
(%i3) expand(expr,2);
2 3 3
x (y + 1) + 2 x (y + 1) + (y + 1)
(%i4) expr: (x+1)^-2*(y+1)^3;
3
(%o4) ----------- + ----------- + ----------- + -----------
2
(%i5) expand(expr);
3 2
y + 3 y + 3 y + 1
2
x + 2 x + 1 x + 2 x + 1 x + 2 x + 1 x + 2 x + 1
(%i6) expand(expr,2,2);
3
(%o6) -----------
2
(%i1) expr: (1+x)^2*sin(x);
2
(%o1) (x + 1) sin(x)
\end{verbatim}
(\%i2) exponentialize:true;
(\%o2) true
(\%i3) expand(expr,0,0);
  2  %i x - %i x
  %i (x + 1) (%e - %e )
(\%o3) - -------------------------------
          2

expandwrt (expr, x_1, . . ., x_n)  [Function]
    Expands expression expr with respect to the variables x_1, . . ., x_n. All products
    involving the variables appear explicitly. The form returned will be free of products of
    sums of expressions that are not free of the variables. x_1, . . ., x_n may be variables,
    operators, or expressions.
    By default, denominators are not expanded, but this can be controlled by means of
    the switch expandwrt_denom.
    This function is autoloaded from simplification/stopex.mac.

expandwrt_denom  [Option variable]
    Default value: false
    expandwrt_denom controls the treatment of rational expressions by expandwrt. If
    true, then both the numerator and denominator of the expression will be expanded
    according to the arguments of expandwrt, but if expandwrt_denom is false, then
    only the numerator will be expanded in that way.

expandwrt_factored (expr, x_1, . . ., x_n)  [Function]
    is similar to expandwrt, but treats expressions that are products somewhat differently.
    expandwrt_factored expands only on those factors of expr that contain the variables
    x_1, . . ., x_n.
    This function is autoloaded from simplification/stopex.mac.

expon  [Option variable]
    Default value: 0
    expon is the exponent of the largest negative power which is automatically expanded
    (independent of calls to expand). For example, if expon is 4 then (x+1)^(-5) will
    not be automatically expanded.

exponentialize (expr)  [Function]
    The function exponentialize (expr) converts circular and hyperbolic functions in
    expr to exponentials, without setting the global variable exponentialize.
    When the variable exponentialize is true, all circular and hyperbolic functions are
    converted to exponential form. The default value is false.
    demoivre converts complex exponentials into circular functions. exponentialize
    and demoivre cannot both be true at the same time.

expop  [Option variable]
    Default value: 0
expop is the highest positive exponent which is automatically expanded. Thus \((x + 1)^3\), when typed, will be automatically expanded only if \texttt{expop}\ is greater than or equal to 3. If it is desired to have \((x + 1)^n\) expanded where \(n\) is greater than \texttt{expop}\ then executing \texttt{expand ((x + 1)^n)}\ will work only if \texttt{maxposex}\ is not less than \(n\).

lassociative
\texttt{declare (g, lassociative)}\ tells the Maxima simplifier that \(g\) is left-associative. E.g., \(g\ (g\ (a, b), g\ (c, d))\) will simplify to \(g\ (g\ (g\ (a, b), c), d)\).

linear
\texttt{declare (g, linear)}\ tells the Maxima simplifier that \(g\) is linear.

Example:
\begin{verbatim}
(\%i1) declare (f, linear);
(\%o1) done
(\%i2) f(x+y);
(\%o2) f(y) + f(x)
(\%i3) declare (a, constant);
(\%o3) done
(\%i4) f(a*x);
(\%o4) a f(x)
\end{verbatim}

Example:
\begin{verbatim}
(\%i1) 'sum (F(k) + G(k), k, 1, inf);
inf
====
\(\%o1\) > \((G(k) + F(k))\)
/====
\(\%i2\) declare (nounify (sum), linear);
(\%o2) done
(\%i3) 'sum (F(k) + G(k), k, 1, inf);
inf inf
==== ====
\(\%o3\) > \(G(k) + > \ F(k)\)
/ /====
\(\%o3\) > \(G(k) + \ F(k)\)
/ /====
\(\%o3\) k = 1 k = 1
\end{verbatim}

maxnegex
\texttt{default value: 1000}
maxnegex is the largest negative exponent which will be expanded by the `expand` command, see also `maxposex`.

**maxposex**

[Option variable]
Default value: 1000

maxposex is the largest exponent which will be expanded with the `expand` command, see also `maxnegex`.

**multiplicative**

[Property]

`declare(f, multiplicative)` tells the Maxima simplifier that `f` is multiplicative.

1. If `f` is univariate, whenever the simplifier encounters `f` applied to a product, `f` distributes over that product. E.g., `f(x*y)` simplifies to `f(x)*f(y)`. This simplification is not applied to expressions of the form `f('product(...))`.
2. If `f` is a function of 2 or more arguments, multiplicativity is defined as multiplicativity in the first argument to `f`, e.g., `f(g(x) * h(x), x)` simplifies to `f(g(x), x) * f(h(x), x)`.

`declare(nounify(product), multiplicative)` tells Maxima to simplify symbolic products.

Example:

```
(%i1) F2 (a * b * c);
            F2(a b c)
(%o1)           
(%i2) declare (F2, multiplicative);
            done
(%i3) F2 (a * b * c);
            F2(a) F2(b) F2(c)
```

`declare(nounify(product), multiplicative)` tells Maxima to simplify symbolic products.

```
(%i1) product (a[i] * b[i], i, 1, n);
           n
       /===\     
      ! ! a b
    ! ! i i
         i = 1
(%i2) declare (nounify (product), multiplicative);
            done
(%i3) product (a[i] * b[i], i, 1, n);
           n     n
       /===\ /===\     
      ! ! a ! ! b
    ! ! i ! ! i
         i = 1     i = 1
```
multthru

\[ \text{multthru}(\text{expr}) \]
\[ \text{multthru}(\text{expr}_1, \text{expr}_2) \]

Multiplies a factor (which should be a sum) of \text{expr} by the other factors of \text{expr}. That is, \text{expr} is \( f_1 f_2 \ldots f_n \) where at least one factor, say \( f_i \), is a sum of terms. Each term in that sum is multiplied by the other factors in the product. (Namely all the factors except \( f_i \)). \text{multthru} does not expand exponentiated sums. This function is the fastest way to distribute products (commutative or noncommutative) over sums. Since quotients are represented as products \text{multthru} can be used to divide sums by products as well.

\text{multthru}(\text{expr}_1, \text{expr}_2) \) multiplies each term in \text{expr}_2 (which should be a sum or an equation) by \text{expr}_1. If \text{expr}_1 is not itself a sum then this form is equivalent to \text{multthru}(\text{expr}_1*\text{expr}_2).

\begin{verbatim}
(%i1) x/(x-y)^2 - 1/(x-y) - f(x)/(x-y)^3;
\end{verbatim}

\begin{verbatim}
(%o1) - ------- + --------- - --------
     x - y           2            3

(%i2) multthru ((x-y)^3, %);
\end{verbatim}

\begin{verbatim}
(%o2) - (x - y) + x (x - y) - f(x)

(%i3) ratexpand (%);
\end{verbatim}

\begin{verbatim}
2
(%o3) - y + x y - f(x)

(%i4) ((a+b)^10*s^2 + 2*a*b*s + (a*b)^2)/(a*b*s^2);
\end{verbatim}

\begin{verbatim}
10 2 2 2
(b + a) s + 2 a b s + a b
\end{verbatim}

\begin{verbatim}
2
-----------
2 a b s
\end{verbatim}

\begin{verbatim}
(%i5) multthru (%); /* note that this does not expand (b+a)^10 */
\end{verbatim}

\begin{verbatim}
2  a b  (b + a)
(%o5) - + ----
     s  2

(%i6) multthru (a.(b+c.(d+e)+f));
\end{verbatim}

\begin{verbatim}
a . f + a . c . (e + d) + a . b
\end{verbatim}

\begin{verbatim}
(%i7) expand (a.(b+c.(d+e)+f));
\end{verbatim}

\begin{verbatim}
a . f + a . c . e + a . c . d + a . b
\end{verbatim}

nary

\[ \text{declare}(f, \text{nary}) \]

\text{declare}(f, \text{nary}) tells Maxima to recognize the function \( f \) as an n-ary function. The \text{nary} declaration is not the same as calling the \text{[function_nary]}, page 126, function. The sole effect of \text{declare}(f, nary) is to instruct the Maxima simplifier to flatten nested expressions, for example, to simplify \( \text{foo}(x, \text{foo}(y, z)) \) to \( \text{foo}(x, y, z) \). See also \text{declare}.
Example:

\begin{verbatim}
(%i1) H (H (a, b), H (c, H (d, e)));
(%o1) H(H(a, b), H(c, H(d, e)))
(%i2) declare (H, nary);
(%o2) done
(%i3) H (H (a, b), H (c, H (d, e)));
(%o3) H(a, b, c, d, e)
\end{verbatim}

negdistrib

[Option variable]

Default value: true

When negdistrib is true, -1 distributes over an expression. E.g., -(x + y) becomes - y - x. Setting it to false will allow - (x + y) to be displayed like that. This is sometimes useful but be very careful: like the simp flag, this is one flag you do not want to set to false as a matter of course or necessarily for other than local use in your Maxima.

Example:

\begin{verbatim}
(%i1) negdistrib;
(%o1) true
(%i2) -(x+y);
(%o2) (- y) - x
(%i3) negdistrib : not negdistrib ;
(%o3) false
(%i4) -(x+y);
(%o4) - (y + x)
\end{verbatim}

opproperties

[System variable]

opproperties is the list of the special operator properties recognized by the Maxima simplifier:

Example:

\begin{verbatim}
(%i1) opproperties;
(%o1) [linear, additive, multiplicative, outative, evenfun,
    oddfun, commutative, symmetric, antisymmetric, nary,
    lassociative, rassociative]
\end{verbatim}

outative

[Property]

\texttt{declare(f, outative)} tells the Maxima simplifier that constant factors in the argument of \texttt{f} can be pulled out.

1. If \texttt{f} is univariate, whenever the simplifier encounters \texttt{f} applied to a product, that product will be partitioned into factors that are constant and factors that are not and the constant factors will be pulled out. E.g., \texttt{f(a*x)} will simplify to \texttt{a*f(x)} where \texttt{a} is a constant. Non-atomic constant factors will not be pulled out.

2. If \texttt{f} is a function of 2 or more arguments, outativity is defined as in the case of \texttt{sum} or \texttt{integrate}, i.e., \texttt{f (a*g(x), x)} will simplify to \texttt{a * f(g(x), x)} for \texttt{a} free of \texttt{x}.

\texttt{sum}, \texttt{integrate}, and \texttt{limit} are all outative.
Example:

(%i1) F1 (100 * x);
(%o1) F1(100 x)
(%i2) declare (F1, outative);
(%o2) done
(%i3) F1 (100 * x);
(%o3) 100 F1(x)
(%i4) declare (zz, constant);
(%o4) done
(%i5) F1 (zz * y);
(%o5) zz F1(y)

Function radcan

Simplifies expr, which can contain logs, exponentials, and radicals, by converting it into a form which is canonical over a large class of expressions and a given ordering of variables; that is, all functionally equivalent forms are mapped into a unique form. For a somewhat larger class of expressions, radcan produces a regular form. Two equivalent expressions in this class do not necessarily have the same appearance, but their difference can be simplified by radcan to zero.

For some expressions radcan is quite time consuming. This is the cost of exploring certain relationships among the components of the expression for simplifications based on factoring and partial-fraction expansions of exponents.

Examples:

(%i1) radcan((log(x+x^2)-log(x))^a/log(1+x)^(a/2));
(%o1) log(x + 1)
(%i2) radcan((log(1+2*a^x+a^(2*x))/log(1+a^x)))
(%o2) 2
(%i3) radcan((%e^x-1)/(1+%e^(x/2)));
(%o3) %e - 1

Option variable radexpand

Default value: true

radexpand controls some simplifications of radicals.

When radexpand is all, causes nth roots of factors of a product which are powers of n to be pulled outside of the radical. E.g. if radexpand is all, sqrt (16*x^2) simplifies to 4*x.

More particularly, consider sqrt (x^2).

- If radexpand is all or assume (x > 0) has been executed, sqrt(x^2) simplifies to x.
- If radexpand is true and domain is real (its default), sqrt(x^2) simplifies to abs(x).
- If radexpand is false, or radexpand is true and domain is complex, sqrt(x^2) is not simplified.

Note that domain only matters when radexpand is true.
rassociative

declare (g, rassociative) tells the Maxima simplifier that g is right-associative. E.g., \(g(g(a, b), g(c, d))\) simplifies to \(g(a, g(b, g(c, d)))\).

scsimp (expr, rule_1, ..., rule_n)

Sequential Comparative Simplification (method due to Stoute). scsimp attempts to simplify expr according to the rules rule_1, ..., rule_n. If a smaller expression is obtained, the process repeats. Otherwise after all simplifications are tried, it returns the original answer.

element (scsimp) displays some examples.

simp

Default value: true

simp enables simplification. This is the default. simp is also an evflag, which is recognized by the function ev. See ev.

When simp is used as an evflag with a value false, the simplification is suppressed only during the evaluation phase of an expression. The flag does not suppress the simplification which follows the evaluation phase.

Many Maxima functions and operations require simplification to be enabled to work normally. When simplification is disabled, many results will be incomplete, and in addition there may be incorrect results or program errors.

Examples:

The simplification is switched off globally. The expression \(\sin(1.0)\) is not simplified to its numerical value. The simp-flag switches the simplification on.

```
(%i1) simp:false;
(%o1) false
(%i2) sin(1.0);
(%o2) sin(1.0)
(%i3) sin(1.0),simp;
(%o3) 0.8414709848078965
```

The simplification is switched on again. The simp-flag cannot suppress the simplification completely. The output shows a simplified expression, but the variable \(x\) has an unsimplified expression as a value, because the assignment has occurred during the evaluation phase of the expression.

```
(%i1) simp:true;
(%o1) true
(%i2) x:sin(1.0),simp:false;
(%o2) 0.8414709848078965
(%i3) :lisp $x
((%SIN) 1.0)
```

symmetric

declare (h, symmetric) tells the Maxima simplifier that h is a symmetric function. E.g., \(h(x, z, y)\) simplifies to \(h(x, y, z)\).

commutative is synonymous with symmetric.
\textbf{xthru (expr)}  

Combines all terms of \texttt{expr} (which should be a sum) over a common denominator without expanding products and exponentiated sums as \texttt{ratsimp} does. \texttt{xthru} cancels common factors in the numerator and denominator of rational expressions but only if the factors are explicit.

Sometimes it is better to use \texttt{xthru} before \texttt{ratsimping} an expression in order to cause explicit factors of the \texttt{gcd} of the numerator and denominator to be canceled thus simplifying the expression to be \texttt{ratsimped}.

Examples:

\begin{verbatim}
(%i1) ((x+2)^20 - 2*y)/(x+y)^20 + (x+y)^(-19) - x/(x+y)^20;

20
1 (x + 2) - 2 y x
(%o1) --------- + ------------- - ---------
19 20 20
(y + x) (y + x) (y + x)

(%i2) xthru (%);

20
(x + 2) - y
(%o2) -------------
20
(y + x)
\end{verbatim}
10 Mathematical Functions

10.1 Functions for Numbers

\texttt{abs (z)}

The \texttt{abs} function represents the mathematical absolute value function and works for both numerical and symbolic values. If the argument, \( z \), is a real or complex number, \texttt{abs} returns the absolute value of \( z \). If possible, symbolic expressions using the absolute value function are also simplified.

Maxima can differentiate, integrate and calculate limits for expressions containing \texttt{abs}. The \texttt{abs_integrate} package further extends Maxima’s ability to calculate integrals involving the abs function. See (%i12) in the examples below.

When applied to a list or matrix, \texttt{abs} automatically distributes over the terms. Similarly, it distributes over both sides of an equation. To alter this behaviour, see the variable \texttt{distribute_over}.

See also \texttt{cabs}.

Examples:

Calculation of \texttt{abs} for real and complex numbers, including numerical constants and various infinities. The first example shows how \texttt{abs} distributes over the elements of a list.

\begin{verbatim}
(%i1) abs([-4, 0, 1, 1+%i]);
(%o1) [4, 0, 1, sqrt(2)]

(%i2) abs((1+%i)*(1-%i));
(%o2) 2

(%i3) abs(%e+%i);
(%o3) sqrt(%e + 1)

(%i4) abs([inf, infinity, minf]);
(%o4) [inf, inf, inf]
\end{verbatim}

Simplification of expressions containing \texttt{abs}:

\begin{verbatim}
(%i5) abs(x^2);
(%o5) x sqrt(x^2)

(%i6) abs(x^3);
(%o6) x abs(x)^2

(%i7) abs(abs(x));
(%o7) abs(x)

(%i8) abs(conjugate(x));
(%o8) abs(x)
\end{verbatim}
Integrating and differentiating with the \texttt{abs} function. Note that more integrals involving the \texttt{abs} function can be performed, if the \texttt{abs_integrate} package is loaded. The last example shows the Laplace transform of \texttt{abs}: see \texttt{laplace}.

\begin{verbatim}
(%i9) diff(x*abs(x),x),expand;
          2 abs(x)
(%o9)    2

(%i10) integrate(abs(x),x);
        2 x abs(x)
(%o10) --------
      2

(%i11) integrate(x*abs(x),x);

I x abs(x) dx
(%o11) [-----------------]
    2

(%i12) load(abs_integrate)$

(%i13) integrate(x*abs(x),x);
        2 3
x abs(x)  x signum(x)
(%o13) --------- - ------------
      2 6

(%i14) integrate(abs(x),x,-2,%pi);

2
(%o14) ---- + 2
    2

(%i15) laplace(abs(x),x,s);

1 --
(%o15)    2
    s
\end{verbatim}

\section*{ceiling ($x$)}

When $x$ is a real number, return the least integer that is greater than or equal to $x$. If $x$ is a constant expression (10 * \%pi, for example), \texttt{ceiling} evaluates $x$ using big floating point numbers, and applies \texttt{ceiling} to the resulting big float. Because \texttt{ceiling} uses floating point evaluation, it’s possible, although unlikely, that \texttt{ceiling} could return an erroneous value for constant inputs. To guard against errors, the floating point evaluation is done using three values for \texttt{fpprec}.

For non-constant inputs, \texttt{ceiling} tries to return a simplified value. Here are examples of the simplifications that \texttt{ceiling} knows about:
Chapter 10: Mathematical Functions

The ceiling function distributes over lists, matrices and equations. See `distribute_over`.

Finally, for all inputs that are manifestly complex, `ceiling` returns a noun form.

If the range of a function is a subset of the integers, it can be declared to be `integervalued`. Both the ceiling and floor functions can use this information; for example:

```
(%i1) declare (f, integervalued)$
(%i2) floor (f(x));
   1
(%o2) f(x)
(%i3) ceiling (f(x) - 1);
   f(x) - 1

Example use:
```

Example use:

```
(%i1) unitfrac(r) := block([uf : [], q],
    if not(ratnump(r)) then
        error("unitfrac: argument must be a rational number"),
    while r # 0 do (uf : cons(q : 1/ceiling(1/r), uf),
        r : r - q),
    reverse(uf));
(%i2) unitfrac (9/10);
   1 1 1
(%o2) \([-, -, --\]
   2 3 15`
```
entier (x) [Function]

Returns the largest integer less than or equal to x where x is numeric. fix (as in fixnum) is a synonym for this, so fix(x) is precisely the same.

floor (x) [Function]

When x is a real number, return the largest integer that is less than or equal to x. If x is a constant expression (10 * %pi, for example), floor evaluates x using big floating point numbers, and applies floor to the resulting big float. Because floor uses floating point evaluation, it’s possible, although unlikely, that floor could return an erroneous value for constant inputs. To guard against errors, the floating point evaluation is done using three values for fpprec.

For non-constant inputs, floor tries to return a simplified value. Here are examples of the simplifications that floor knows about:

```
(%i1) floor (ceiling (x));
(%o1) ceiling(x)
(%i2) floor (floor (x));
(%o2) floor(x)
(%i3) declare (n, integer)$
(%i4) [floor (n), floor (abs (n)), floor (min (n, 6))];
(%o4) [n, abs(n), min(6, n)]
(%i5) assume (x > 0, x < 1)$
(%i6) floor (x);
(%o6) 0
(%i7) tex (floor (a));
$$\left \lfloor a \right \rfloor$$
(%o7) false
```
The **floor** function distributes over lists, matrices and equations. See `distribute_over`.

Finally, for all inputs that are manifestly complex, **floor** returns a noun form.

If the range of a function is a subset of the integers, it can be declared to be **integervalued**. Both the **ceiling** and **floor** functions can use this information; for example:

```
(%i1) declare (f, integervalued)$
(%i2) floor (f(x));
  (%o2) f(x)
(%i3) ceiling (f(x) - 1);
  (%o3) f(x) - 1
```

**fix**

A synonym for **entier** (*x*).

**lmax** (*L*)

When *L* is a list or a set, return `apply ('max, args (L))`. When *L* is not a list or a set, signal an error. See also **lmin** and **max**.

**lmin** (*L*)

When *L* is a list or a set, return `apply ('min, args (L))`. When *L* is not a list or a set, signal an error. See also **lmax** and **min**.

**max** (*x_1*, ..., *x_n*)

Return a simplified value for the maximum of the expressions *x_1* through *x_n*. When `get (trylevel, maxmin)`, is 2 or greater, **max** uses the simplification `max (e, -e) --> |e|`. When `get (trylevel, maxmin)` is 3 or greater, **max** tries to eliminate expressions that are between two other arguments; for example, `max (x, 2*x, 3*x) --> max (x, 3*x)`. To set the value of **trylevel** to 2, use `put (trylevel, 2, maxmin)`. See also **min** and **lmax**.

**min** (*x_1*, ..., *x_n*)

Return a simplified value for the minimum of the expressions *x_1* through *x_n*. When `get (trylevel, maxmin)`, is 2 or greater, **min** uses the simplification `min (e, -e) --> -|e|`. When `get (trylevel, maxmin)` is 3 or greater, **min** tries to eliminate expressions that are between two other arguments; for example, `min (x, 2*x, 3*x) --> min (x, 3*x)`. To set the value of **trylevel** to 2, use `put (trylevel, 2, maxmin)`. See also **max** and **lmin**.

**round** (*x*)

When *x* is a real number, returns the closest integer to *x*. Multiples of 1/2 are rounded to the nearest even integer. Evaluation of *x* is similar to **floor** and **ceiling**.

The **round** function distributes over lists, matrices and equations. See `distribute_over`.

**signum** (*x*)

For either real or complex numbers *x*, the signum function returns 0 if *x* is zero; for a nonzero numeric input *x*, the signum function returns *x/abs(x)*.
For non-numeric inputs, Maxima attempts to determine the sign of the input. When the sign is negative, zero, or positive, \texttt{signum} returns -1, 0, 1, respectively. For all other values for the sign, \texttt{signum} a simplified but equivalent form. The simplifications include reflection (\texttt{signum(-x)} gives \texttt{-signum(x)}) and multiplicative identity (\texttt{signum(x*y)} gives \texttt{signum(x) * signum(y)}).

The \texttt{signum} function distributes over a list, a matrix, or an equation. See \texttt{distribute_over}.

\texttt{truncate} (x)

When \( x \) is a real number, return the closest integer to \( x \) not greater in absolute value than \( x \). Evaluation of \( x \) is similar to \texttt{floor} and \texttt{ceiling}.

The \texttt{truncate} function distributes over lists, matrices and equations. See \texttt{distribute_over}.

\section*{10.2 Functions for Complex Numbers}

\texttt{cabs} (expr)

Calculates the absolute value of an expression representing a complex number. Unlike the function \texttt{abs}, the \texttt{cabs} function always decomposes its argument into a real and an imaginary part. If \( x \) and \( y \) represent real variables or expressions, the \texttt{cabs} function calculates the absolute value of \( x + \%i*y \) as

\begin{verbatim}
(\%i1) cabs (1); 1
(\%i2) cabs (1 + \%i); sqrt(2)
(\%i3) cabs (exp (\%i)); 1
(\%i4) cabs (exp (\%pi * \%i)); 1
(\%i5) cabs (exp (3/2 * \%pi * \%i)); 1
(\%i6) cabs (17 * exp (2 * \%i)); 17
\end{verbatim}

If \texttt{cabs} returns a noun form this most commonly is caused by some properties of the variables involved not being known:

\begin{verbatim}
(\%i1) cabs (a+\%i*b); 2
(\%o1) sqrt(b^2 + a^2)
(\%i2) declare(a,real,b,real); done
(\%i3) cabs (a+\%i*b); 2
(\%o3) sqrt(b^2 + a^2)
(\%i4) assume(a>0,b>0); [a > 0, b > 0]
\end{verbatim}
(95) \text{cabs} (a+\text{i}\cdot b);

\text{sqrt}(b^2 + a^2)

The \text{cabs} function can use known properties like symmetry properties of complex functions to help it calculate the absolute value of an expression. If such identities exist, they can be advertised to \text{cabs} using function properties. The symmetries that \text{cabs} understands are: mirror symmetry, conjugate function and complex characteristic.

\text{cabs} is a verb function and is not suitable for symbolic calculations. For such calculations (including integration, differentiation and taking limits of expressions containing absolute values), use \text{abs}.

The result of \text{cabs} can include the absolute value function, \text{abs}, and the arc tangent, \text{atan2}.

When applied to a list or matrix, \text{cabs} automatically distributes over the terms. Similarly, it distributes over both sides of an equation.

For further ways to compute with complex numbers, see the functions \text{rectform}, \text{realpart}, \text{imagpart}, \text{carg}, \text{conjugate} and \text{polarform}.

Examples:

Examples with \text{sqrt} and \text{sin}.

(91) \text{cabs}(\text{sqrt}(1+\text{i}\cdot x));

\text{sqrt}((x+1)^{1/4})

(92) \text{cabs}(\text{sin}(x+\text{i}\cdot y));

\text{sqrt}((\text{cos}(x)\text{sinh}(y)+\text{sin}(x)\text{cosh}(y))^{2})

The error function, \text{erf}, has mirror symmetry, which is used here in the calculation of the absolute value with a complex argument:

(93) \text{cabs}(\text{erf}(x+\text{i}\cdot y));

\text{sqrt}\left(\frac{\text{erf}(\text{i}\cdot y+x)-\text{erf}(\text{i}\cdot y-x)}{4}\right)

\text{sqrt}\left(\frac{\text{erf}(\text{i}\cdot y+x)+\text{erf}(\text{i}\cdot y-x)}{4}\right)

Maxima knows complex identities for the Bessel functions, which allow it to compute the absolute value for complex arguments. Here is an example for \text{bessel}_j.

(94) \text{cabs}(\text{bessel}_j(1,\text{i})));

\text{abs}(\text{bessel}_j(1,\text{i}))

\text{carg} (z) 

Returns the complex argument of \(z\). The complex argument is an angle \(\theta\) in \((-\pi, \pi]\) such that \(r \cdot \text{exp}(\theta \cdot \text{i}) = z\) where \(r\) is the magnitude of \(z\).
**carg** is a computational function, not a simplifying function.

See also **abs** (complex magnitude), **polarform**, **rectform**, **realpart**, and **imagpart**.

Examples:

```lisp
(%i1) carg (1);
(%o1) 0
(%i2) carg (1 + %i);
   %pi
(%o2) ----
   4
(%i3) carg (exp (%i));
   sin(1)
(%o3) atan(------)
   cos(1)
(%i4) carg (exp (%pi * %i));
   %pi
(%o4) ----
   2
(%i5) carg (exp (3/2 * %pi * %i));
   %pi
(%o5) --
   2
(%i6) carg (17 * exp (2 * %i));
   sin(2)
(%o6) atan(------) + %pi
   cos(2)

If **carg** returns a noun form this most communly is caused by some properties of the variables involved not being known:

```lisp
(%i1) carg (a+%i*b);
   atan2(b, a)
(%o1) atan2(b, a)
(%i2) declare(a,real,b,real);
   done
(%i3) carg (a+%i*b);
   atan2(b, a)
(%i4) assume(a>0,b>0);
   [a > 0, b > 0]
(%i5) carg (a+%i*b);
   b
(%o5) atan(-)
   a
```

**conjugate** *(x)*  
[Function]

Returns the complex conjugate of x.

```lisp
(%i1) declare ([aa, bb], real, cc, complex, ii, imaginary);
   done
(%i2) conjugate (aa + bb*%i);
   aa - %i bb
(%i3) conjugate (cc);
   conjugate(cc)
```
(%i4) conjugate (ii);  
(%o4) - ii  
(%i5) conjugate (xx + yy);  
(%o5) yy + xx  

imagpart (expr)  
Returns the imaginary part of the expression expr.  
imagpart is a computational function, not a simplifying function.  
See also abs, carg, polarform, rectform, and realpart.  
Example:  
  (%i1) imagpart (a+b*%i);  
  (%o1) b  
  (%i2) imagpart (1+sqrt(2)*%i);  
  (%o2) sqrt(2)  
  (%i3) imagpart (1);  
  (%o3) 0  
  (%i4) imagpart (sqrt(2)*%i);  
  (%o4) sqrt(2)  

polarform (expr)  
Returns an expression r %e^(%i theta) equivalent to expr, such that r and theta are purely real.  
Example:  
  (%i1) polarform(a+b*%i);  
  (%o1) sqrt(b + a^2) %e^(%i atan2(b, a))  
  (%i2) polarform(1+%i);  
  (%o2) sqrt(2) %e^(%i pi/4)  
  (%i3) polarform(1+2*%i);  
  (%o3) sqrt(5) %e^(%i atan(2))  

realpart (expr)  
Returns the real part of expr. realpart and imagpart will work on expressions involving trigonometric and hyperbolic functions, as well as square root, logarithm, and exponentiation.  
Example:  
  (%i1) realpart (a+b*%i);  
  (%o1) a  
  (%i2) realpart (1+sqrt(2)*%i);  
  (%o2) 1  
  (%i3) realpart (sqrt(2)*%i);  
  (%o3) 0
rectform (expr)  [Function]
Returns an expression a + b %i equivalent to expr, such that a and b are purely real.

Example:

(%i1) rectform(sqrt(2)*%e^(%i*%pi/4));
(%o1) %i + 1
(%i2) rectform(sqrt(b^2+a^2)*%e^(%i*atan2(b, a)));  
(%o2) %i b + a
(%i3) rectform(sqrt(5)*%e^(%i*atan(2)));  
(%o3) 2 %i + 1

10.3 Combinatorial Functions

!!  [Operator]
The double factorial operator.
For an integer, float, or rational number n, n!! evaluates to the product n (n-2) (n-4) ... (n - 2 (k-1)) where k is equal to entier (n/2), that is, the largest integer less than or equal to n/2. Note that this definition does not coincide with other published definitions for arguments which are not integers.

For an even (or odd) integer n, n!! evaluates to the product of all the consecutive even (or odd) integers from 2 (or 1) through n inclusive.

For an argument n which is not an integer, float, or rational, n!! yields a noun form genfact (n, n/2, 2).

binomial (x, y)  [Function]
The binomial coefficient x!/((y! (x - y)!)). If x and y are integers, then the numerical value of the binomial coefficient is computed. If y, or x - y, is an integer, the binomial coefficient is expressed as a polynomial.

Examples:

(%i1) binomial (11, 7);
(%o1) 330
(%i2) 11! / 7! / (11 - 7)!
(%o2) 330
(%i3) binomial (x, 7);
(x - 6) (x - 5) (x - 4) (x - 3) (x - 2) (x - 1) x
(%o3) --------------------------------------
5040
(%i4) binomial (x + 7, x);
(x + 1) (x + 2) (x + 3) (x + 4) (x + 5) (x + 6) (x + 7)
(%o4) --------------------------------------
5040
(%i5) binomial (11, y);
(%o5) binomial(11, y)
factcomb (expr)  
Tries to combine the coefficients of factorials in expr with the factorials themselves by converting, for example, \((n + 1)\times n!\) into \((n + 1)!\).  
sumsplitfact if set to false will cause minfactorial to be applied after a factcomb.

Example:

\[
\begin{align*}
\text{(i1)} & \quad \text{sumsplitfact;} \\
\text{(i2)} & \quad (n + 1) \times (n + 1) \times n!; \\
\text{(i3)} & \quad \text{factcomb (i2);} \\
\text{(i4)} & \quad \text{sumsplitfact: not sumsplitfact;} \\
\text{(i5)} & \quad (n + 1) \times (n + 1) \times n!; \\
\text{(i6)} & \quad \text{factcomb (i5);} \\
\end{align*}
\]

factorial ![ ]   
Represents the factorial function. Maxima treats factorial \((x)\) the same as \(x!\).

For any complex number \(x\), except for negative integers, \(x!\) is defined as \(\gamma(x+1)\).

For an integer \(x\), \(x!\) simplifies to the product of the integers from 1 to \(x\) inclusive. 0! simplifies to 1. For a real or complex number in float or bigfloat precision \(x, x!\), simplifies to the value of \(\gamma(x+1)\). For \(x\) equal to \(n/2\) where \(n\) is an odd integer, \(x!\) simplifies to a rational factor times \(\sqrt{\pi}\) (since \(\gamma(1/2)\) is equal to \(\sqrt{\pi}\)).

The option variables factlim and gammalim control the numerical evaluation of factorials for integer and rational arguments. The functions minfactorial and factcomb simplifies expressions containing factorials.

The functions gamma, bffac, and cbffac are varieties of the gamma function. bffac and cbffac are called internally by gamma to evaluate the gamma function for real and complex numbers in bigfloat precision.

makegamma substitutes \(\gamma\) for factorials and related functions.

Maxima knows the derivative of the factorial function and the limits for specific values like negative integers.

The option variable factorial_expand controls the simplification of expressions like \((n+x)!\), where \(n\) is an integer.

See also binomial.

The factorial of an integer is simplified to an exact number unless the operand is greater than factlim. The factorial for real and complex numbers is evaluated in float or bigfloat precision.
\( \text{(%i1)} \) \text{factlim} : 10;
\( \text{(%o1)} \) 10
\( \text{(%i2)} \) \{0!, (7/2)!, 8!, 20!\];
\( \text{(%o2)} \) \{1, \frac{\sqrt{\pi}}{105}, 40320, 20!\]
\( \text{(%i3)} \) \{4, 77!, (1.0+i)\};
\( \text{(%o3)} \) \{4, 77!, 0.3430658398165453 \pi + 0.6529654964201667\]
\( \text{(%i4)} \) \{2.86b0!, 1.0b0+i\};
\( \text{(%o4)} \) \text{incorrect syntax: Missing }]
\[2.86b0!, 1.0b0+i\]

The factorial of a known constant, or general expression is not simplified. Even so it may be possible to simplify the factorial after evaluating the operand.

\( \text{(%i1)} \) \{\(\text{(%i + 1)!}, \pi!, \pi!, (\cos(1) + \sin(1))!\]\;
\( \text{(%o1)} \) \{\(\text{(%i + 1)!}, \pi!, \pi!, (\sin(1) + \cos(1))!\]\;
\( \text{(%i2)} \) \text{ev (%, numer, %enumer)};  
\( \text{(%o2)} \) \{0.3430658398165453 \pi + 0.6529654964201667, 7.18802728976031, 4.260820476357003, 1.227580202486819\]

Factorials are simplified, not evaluated. Thus \(x!\) may be replaced even in a quoted expression.

\( \text{(%i1)} \) \text{'}\{\{0!, (7/2)!, 4.77!, 8!, 20!\}\};
\( \text{(%o1)} \) \{1, \frac{\sqrt{\pi}}{105}, 81.44668037931197, 40320, 2432902008176640000\]

Maxima knows the derivative of the factorial function.

\( \text{(%i1)} \) \text{diff(x!,x)};
\( \text{(%o1)} \) \(x!\ psi (x + 1)\)
\( \quad \) 0

The option variable \text{factorial\_expand} controls expansion and simplification of expressions with the factorial function.

\( \text{(%i1)} \) \text{(n+1)!/n!,factorial\_expand:true;}
\( \text{(%o1)} \) \text{n + 1}

\text{factlim} \quad \text{[Option variable]}

Default value: 100000

\text{factlim} \text{ specifies the highest factorial which is automatically expanded. If it is -1 then all integers are expanded.}

\text{factorial\_expand} \quad \text{[Option variable]}

Default value: false

The option variable \text{factorial\_expand} \text{ controls the simplification of expressions like (n+1)!, where n is an integer. See factorial for an example.}
genfact \( (x, y, z) \) \[\text{[Function]}\]

Returns the generalized factorial, defined as \( x \ (x-z) \ (x-2 \ z) \ldots \ (x-(y-1) \ z) \). Thus, when \( x \) is an integer, \( \text{genfact} \ (x, x, 1) = x! \) and \( \text{genfact} \ (x, x/2, 2) = x!! \).

minfactorial \( (\text{expr}) \) \[\text{[Function]}\]

Examines \( \text{expr} \) for occurrences of two factorials which differ by an integer. minfactorial then turns one into a polynomial times the other.

\begin{verbatim}
(%i1) n!/(n+2)!
    n!          
(%o1)  ---------
   (n + 2)!

(%i2) minfactorial (%);
         1
(%o2)  ---------------
     (n + 1) (n + 2)
\end{verbatim}

sumsplitfact \[\text{[Option variable]}\]

Default value: true

When sumsplitfact is false, minfactorial is applied after a factcomb.

\begin{verbatim}
(%i1) sumsplitfact;
    true
(%o1)

(%i2) n!/(n+2)!
    n!          
(%o2)  ---------
   (n + 2)!

(%i3) factcomb(%);
         n!
(%o3)  ---------
     (n + 2)!

(%i4) sumsplitfact: not sumsplitfact ;
    false
(%o4)

(%i5) n!/(n+2)!
    n!          
(%o5)  ---------
   (n + 2)!

(%i6) factcomb(%);
         1
(%o6)  ---------------
     (n + 1) (n + 2)
\end{verbatim}

10.4 Root, Exponential and Logarithmic Functions

%e_to_numlog \[\text{[Option variable]}\]

Default value: false

When true, \( r \) some rational number, and \( x \) some expression, \( %e^{r \cdot \log(x)} \) will be simplified into \( x^r \). It should be noted that the radcan command also does
this transformation, and more complicated transformations of this ilk as well. The \texttt{logcontract} command "contracts" expressions containing log.

\%emode

\textbf{[Option variable]}

Default value: \texttt{true}

When \%emode is \texttt{true}, \(\%e^{(%\pi \%i \ x)}\) is simplified as follows. \(\%e^{(%\pi \%i \ x)}\) simplifies to \(\cos (\%pi \ x) + \%i \sin (\%pi \ x)\) if \(x\) is a floating point number, an integer, or a multiple of 1/2, 1/3, 1/4, or 1/6, and then further simplified. For other numerical \(x\), \(\%e^{(%\pi \%i \ x)}\) simplifies to \(\%e^{(%\pi \%i \ y)}\) where \(y\) is \(x - 2 \ k\) for some integer \(k\) such that \(\text{abs}(y) < 1\).

When \%emode is \texttt{false}, no special simplification of \(\%e^{(%\pi \%i \ x)}\) is carried out.

\begin{verbatim}
(%i1) %emode; (%o1) true
(%i2) %e^(%pi*%i*1); (%o2) - 1
(%i3) %e^(%pi*%i*216/144); (%o3) - %i
(%i4) %e^(%pi*%i*192/144); (%o4) (- ----------) - --
     2 2 (sqrt(3) %i 1
(%i5) %e^(%pi*%i*180/144); (%o5) (- --------) - ------
     sqrt(2) sqrt(2)
(%i6) %e^(%pi*%i*120/144); (%o6) ----- - -------
     2 2
(%i7) %e^(%pi*%i*121/144); (%o7) %e
\end{verbatim}

\%enumer

\textbf{[Option variable]}

Default value: \texttt{false}

When \%enumer is \texttt{true}, \%e is replaced by its numeric value 2.718... whenever \texttt{numer} is \texttt{true}.

When \%enumer is \texttt{false}, this substitution is carried out only if the exponent in \%e^x evaluates to a number.

See also \texttt{ev} and \texttt{numer}.

\begin{verbatim}
(%i1) %enumer; (%o1) false
(%i2) numer; (%o2) false
\end{verbatim}
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(%i3) 2*%e;
(%o3) 2 %e
(%i4) %enumer: not %enumer;
(%o4) true
(%i5) 2*%e;
(%o5) 2 %e
(%i6) numer: not numer;
(%o6) true
(%i7) 2*%e;
(%o7) 5.43656365691809
(%i8) 2*%e^1;
(%o8) 5.43656365691809
(%i9) 2*%e^x;

exp (x) [Function]
Represents the exponential function. Instances of exp (x) in input are simplified to %e^x; exp does not appear in simplified expressions.
demoivre if true causes %e^(a + b %i) to simplify to %e^(-(a (cos(b) + %i sin(b)))) if b is free of %i. See demoivre.
%emode, when true, causes %e^(%pi %i x) to be simplified. See %emode.
%enumer, when true causes %e to be replaced by 2.718... whenever numer is true. See %enumer.

(%i10) demoivre;
(%o10) false
(%i11) %e^(a + b*%i);
(%o11) %e (%i b + a)
(%i12) demoivre: not demoivre;
(%o12) true
(%i13) %e^(a + b*%i);
(%o13) %e^a (%i sin(b) + cos(b))

li [s] (z) [Function]
Represents the polylogarithm function of order s and argument z, defined by the infinite series

\[ \operatorname{Li}_s(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^s} \]

li [1] is \(- \log (1 - z)\). li [2] and li [3] are the dilogarithm and trilogarithm functions, respectively.

When the order is 1, the polylogarithm simplifies to \(- \log (1 - z)\), which in turn simplifies to a numerical value if z is a real or complex floating point number or the numer evaluation flag is present.
When the order is 2 or 3, the polylogarithm simplifies to a numerical value if \( z \) is a real floating point number or the \texttt{numer} evaluation flag is present.

Examples:

```plaintext
(%i1) assume (x > 0);
(%o1) \{x > 0\}
(%i2) integrate ((log (1 - t)) / t, t, 0, x);
\quad - li (x)
\quad 2
(%i3) li [2] (7);
\quad li (7)
\quad 2
(%i4) li [2] (7), numer;
\quad 1.248273182099423 - 6.113257028817991 \%i
(%i5) li [3] (7);
\quad li (7)
\quad 3
(%i6) li [2] (7), numer;
\quad 1.248273182099423 - 6.113257028817991 \%i
(%i7) L : makelist (i / 4.0, i, 0, 8);
\quad [0.0, 0.25, 0.5, 0.75, 1.0, 1.25, 1.5, 1.75, 2.0]
(%i8) map (lambda ([x], li [2] (x)), L);
\quad [0.0, 0.2676526390827326, 0.5822405264650125,
\quad 0.978469392930306, 1.644934066848226,
\quad 2.190177011441645 - 0.7010261415046585 \%i,
\quad 2.37439527027248 - 1.248273182099423 - 6.113257028817991 \%i,
\quad 2.448686765338205 - 1.758084848210787 \%i,
\quad 2.467401100272339 - 2.190177011441645 - 0.7010261415046585 \%i]
(%i9) map (lambda ([x], li [3] (x)), L);
\quad [0.0, 0.2584613953442624, 0.537213926780429,
\quad 0.8444258046482203, 1.2020569, 1.642866879503222
\quad - 0.07821473130035025 \%i, 2.060877505514697
\quad - 0.2582419849982037 \%i, 2.433418896388322
\quad - 0.491926018232965 \%i, 2.762071904015935
\quad - 0.7546938285978846 \%i]
```

\( \text{log} (x) \)  

Represents the natural (base \( e \)) logarithm of \( x \).

Maxima does not have a built-in function for the base 10 logarithm or other bases. \( \text{log10}(x) := \text{log}(x) / \text{log}(10) \) is a useful definition.

Simplification and evaluation of logarithms is governed by several global flags:

\( \text{logexpand} \)

causes \( \text{log}(a^b) \) to become \( b \times \text{log}(a) \). If it is set to \texttt{all}, \( \text{log}(a^b) \) will also simplify to \( \text{log}(a)+\text{log}(b) \). If it is set to \texttt{super}, then \( \text{log}(a/b) \) will also simplify to \( \text{log}(a) - \text{log}(b) \) for rational numbers \( a/b \), \( a\#1 \).

(\( \text{log}(1/b) \), for \( b \) integer, always simplifies.) If it is set to \texttt{false}, all of these simplifications will be turned off.
logsimp

if false then no simplification of %e to a power containing log's is done.

lognegint

if true implements the rule \( \log(-n) \to \log(n) + %i*%pi \) for \( n \) a positive integer.

%e_to_numlog

when true, \( r \) some rational number, and \( x \) some expression, the expression \( %e^{-r*(\log(x))} \) will be simplified into \( x^r \). It should be noted that the radcan command also does this transformation, and more complicated transformations of this as well. The logcontract command "contracts" expressions containing log.

logabs

[Option variable]
Default value: false
When doing indefinite integration where logs are generated, e.g. integrate(1/x,x), the answer is given in terms of \( \log(abs(...)) \) if logabs is true, but in terms of \( \log(...) \) if logabs is false. For definite integration, the logabs:true setting is used, because here "evaluation" of the indefinite integral at the endpoints is often needed.

logarc

[Option variable]

[Function]

logarc (expr)

When the global variable logarc is true, inverse circular and hyperbolic functions are replaced by equivalent logarithmic functions. The default value of logarc is false. The function logarc(expr) carries out that replacement for an expression expr without setting the global variable logarc.

logconcoeffp

[Option variable]
Default value: false
Controls which coefficients are contracted when using logcontract. It may be set to the name of a predicate function of one argument. E.g. if you like to generate SQRTs, you can do logconcoeffp:'logconfun$ logconfun(m):=featurep(m,integer) or ratnump(m)$ . Then logcontract(1/2*log(x)); will give \( \log(sqrt(x)) \).

logcontract

[Function]

Recursively scans the expression expr, transforming subexpressions of the form \( a1*log(b1) + a2*log(b2) + c \) into \( \log(ratsimp(b1^a1 * b2^a2)) + c \)

\[ \begin{align*}
(\%i1) & \quad 2*(a*\log(x) + 2*a*\log(y)) \\
(\%i2) & \quad \logcontract(%) \;
\end{align*} \]

The declaration declare(n,integer) causes logcontract(2*a*n*log(x)) to simplify to \( a*\log(x^(2*n)) \). The coefficients that "contract" in this manner are those such as the 2 and the \( n \) here which satisfy featurep(coef, integer). The user can control which coefficients are contracted by setting the option logconcoeffp to the name of a predicate function of one argument. E.g. if you like to generate SQRTs, you can do logconcoeffp:'logconfun$ logconfun(m):=featurep(m,integer) or ratnump(m)$ . Then logcontract(1/2*log(x)); will give \( \log(sqrt(x)) \).
**logexpand**

[Option variable]

Default value: `true`

If `true`, that is the default value, causes $\log(a^b)$ to become $b \log(a)$. If it is set to `all`, $\log(a*b)$ will also simplify to $\log(a)+\log(b)$. If it is set to `super`, then $\log(a/b)$ will also simplify to $\log(a)-\log(b)$ for rational numbers $a/b$, $a\neq 1$. ($\log(1/b)$, for integer $b$, always simplifies.) If it is set to `false`, all of these simplifications will be turned off.

When `logexpand` is set to `all` or `super`, the logarithm of a product expression simplifies to a summation of logarithms.

Examples:

When `logexpand` is `true`, $\log(a^b)$ simplifies to $b \log(a)$.

(%i1) log(n^2), logexpand=true;
(%o1) 2 log(n)

When `logexpand` is `all`, $\log(a*b)$ simplifies to $\log(a)+\log(b)$.

(%i1) log(10*x), logexpand=all;
(%o1) log(x) + log(10)

When `logexpand` is `super`, $\log(a/b)$ simplifies to $\log(a)-\log(b)$ for rational numbers $a/b$ with $a\neq 1$.

(%i1) log(a/(n + 1)), logexpand=super;
(%o1) log(a) - log(n + 1)

When `logexpand` is set to `all` or `super`, the logarithm of a product expression simplifies to a summation of logarithms.

(%i1) my_product : product (X(i), i, 1, n);
(%o1) \prod_{i=1}^{n} X(i)

(%i2) log(my_product), logexpand=all;
(%o2) \prod_{i=1}^{n} \log(X(i))

(%i3) log(my_product), logexpand=super;
(%o3) \prod_{i=1}^{n} \log(X(i))
When \texttt{logexpand} is \texttt{false}, these simplifications are disabled.

\begin{verbatim}
(%i1) logexpand : false $
(%i2) log(n^2);
2
(%o2) log(n )
(%i3) log(10*x);
(%o3) log(10 x)
(%i4) log(a/(n + 1));
  a
(%o4) log(-----)
  n + 1
(%i5) log ('product (X(i), i, 1, n));
  n
tax/===\n  ! !!!
(%o5) log( ! ! X(i))
     ! !!
     i = 1
\end{verbatim}

\texttt{lognegint} \hspace{1cm} \text{[Option variable]}
Default value: \texttt{false}
If \texttt{true} implements the rule \texttt{log(-n) \rightarrow log(n)+\%i*\pi} for \(n\) a positive integer.

\texttt{logsimp} \hspace{1cm} \text{[Option variable]}
Default value: \texttt{true}
If \texttt{false} then no simplification of \texttt{\%e} to a power containing \texttt{log}'s is done.

\texttt{plog (x)} \hspace{1cm} \text{[Function]}
Represents the principal branch of the complex-valued natural logarithm with \(-\pi < \text{carg}(x) \leq +\pi\).

\texttt{sqrt (x)} \hspace{1cm} \text{[Function]}
The square root of \(x\). It is represented internally by \(x^{(1/2)}\). See also \texttt{rootscontract} and \texttt{radexpand}. 
10.5 Trigonometric Functions

10.5.1 Introduction to Trigonometric
Maxima has many trigonometric functions defined. Not all trigonometric identities are programmed, but it is possible for the user to add many of them using the pattern matching capabilities of the system. The trigonometric functions defined in Maxima are: \( \text{acos}, \ \text{acosh}, \ \text{acot}, \ \text{acoth}, \ \text{acsc}, \ \text{acsch}, \ \text{asec}, \ \text{asech}, \ \text{asin}, \ \text{asinh}, \ \text{atan}, \ \text{atanh}, \ \text{cos}, \ \text{cosh}, \ \text{cot}, \ \text{coth}, \ \text{csc}, \ \text{csch}, \ \text{sec}, \ \text{sech}, \ \text{sin}, \ \text{sinh}, \ \text{tan}, \ \text{and} \ \tanh. \) There are a number of commands especially for handling trigonometric functions, see \textit{trigexpand}, \textit{trigreduce}, and the switch \textit{trigsign}. Two share packages extend the simplification rules built into Maxima, \textit{ntrig} and \textit{atrig1}. Do \textit{describe(command)} for details.

10.5.2 Functions and Variables for Trigonometric

\%piargs

Default value: \texttt{true}

When \%\texttt{piargs} is \texttt{true}, trigonometric functions are simplified to algebraic constants when the argument is an integer multiple of \( \pi, \pi/2, \pi/3, \pi/4, \) or \( \pi/6. \)

Maxima knows some identities which can be applied when \( \pi, \) etc., are multiplied by an integer variable (that is, a symbol declared to be integer).

Examples:

\begin{verbatim}
(%i1) %piargs : false$
(%i2) [sin (%pi), sin (%pi/2), sin (%pi/3)];
    %pi %pi
(%o2) [sin(%pi), sin(---), sin(---)]
        2   3
(%i3) [sin (%pi/4), sin (%pi/5), sin (%pi/6)];
   %pi   %pi   %pi
(%o3) [sin(---), sin(---), sin(---)]
      4   5   6
(%i4) %piargs : true$
(%i5) [sin (%pi), sin (%pi/2), sin (%pi/3)];
    sqrt(3)
(%o5) [0, 1, -------]
         2
(%i6) [sin (%pi/4), sin (%pi/5), sin (%pi/6)];

1  %pi  1
(%o6) [-------, sin(---), -]
      sqrt(2)  5  2
(%i7) [cos (%pi/3), cos (10*%pi/3), tan (10*%pi/3),
    cos (sqrt(2)*%pi/3)];

1  1  sqrt(2) %pi
(%o7) [-, -, sqrt(3), cos(----------)]
    2  2  3
\end{verbatim}

Some identities are applied when \( \pi \) and \( \pi/2 \) are multiplied by an integer variable.

\begin{verbatim}
(%i11) declare (n, integer, m, even)$
\end{verbatim}
(%i2) [sin (%pi * n), cos (%pi * m), sin (%pi/2 * m),
cos (%pi/2 * m)];

m/2
(%o2) [0, 1, 0, (- 1) ]

%iargs

[Option variable]

Default value: true

When %iargs is true, trigonometric functions are simplified to hyperbolic functions
when the argument is apparently a multiple of the imaginary unit \( i \).

Even when the argument is demonstrably real, the simplification is applied; Maxima
considers only whether the argument is a literal multiple of \( i \).

Examples:

(%i1) %iargs : false$
(%i2) [sin (%i * x), cos (%i * x), tan (%i * x)];

(%o2) [sin(%i x), cos(%i x), tan(%i x)]

(%i3) %iargs : true$
(%i4) [sin (%i * x), cos (%i * x), tan (%i * x)];

(%o4) [%i sinh(x), cosh(x), %i tanh(x)]

Even when the argument is demonstrably real, the simplification is applied.

(%i1) declare (x, imaginary)$
(%i2) [featurep (x, imaginary), featurep (x, real)];

(%o2) [true, false]

(%i3) sin (%i * x);

(%o3) %i sinh(x)

acos (x)

[Function]

– Arc Cosine.

acosh (x)

[Function]

– Hyperbolic Arc Cosine.

acot (x)

[Function]

– Arc Cotangent.

acoth (x)

[Function]

– Hyperbolic Arc Cotangent.

acsc (x)

[Function]

– Arc Cosecant.

acsch (x)

[Function]

– Hyperbolic Arc Cosecant.

asec (x)

[Function]

– Arc Secant.

asech (x)

[Function]

– Hyperbolic Arc Secant.
asin (x) 
  - Arc Sine.

asinh (x) 
  - Hyperbolic Arc Sine.

atan (x) 
  - Arc Tangent.

atan2 (y, x) 
  - yields the value of atan(y/x) in the interval -%pi to %pi.

atanh (x) 
  - Hyperbolic Arc Tangent.

atrig1 
  The atrig1 package contains several additional simplification rules for inverse trigonometric functions. Together with rules already known to Maxima, the following angles are fully implemented: 0, %pi/6, %pi/4, %pi/3, and %pi/2. Corresponding angles in the other three quadrants are also available. Do load(atrig1); to use them.

cos (x) 
  - Cosine.

cosh (x) 
  - Hyperbolic Cosine.

cot (x) 
  - Cotangent.

coth (x) 
  - Hyperbolic Cotangent.

csc (x) 
  - Cosecant.

csch (x) 
  - Hyperbolic Cosecant.

halfangles 
  Default value: false

When halfangles is true, trigonometric functions of arguments expr/2 are simplified to functions of expr.

For a real argument x in the interval 0 < x < 2*%pi the sine of the half-angle simplifies to a simple formula:

\[
\frac{\sqrt{1 - \cos(x)}}{\sqrt{2}}
\]

A complicated factor is needed to make this formula correct for all complex arguments z:

realpart(z)
floor(-----------)
    2 %pi
(- 1) (1 - unit_step(- imagpart(z))

realpart(z) realpart(z)
floor(-----------) - ceiling(-----------)
    2 %pi 2 %pi
((- 1) + 1))

Maxima knows this factor and similar factors for the functions \(\sin\), \(\cos\), \(\sinh\), and \(\cosh\). For special values of the argument \(z\) these factors simplify accordingly.

Examples:

(\%i1) halfangles : false$
(\%i2) \sin (x / 2);
          x
(\%o2) \sin(-)
          2
(\%i3) halfangles : true$
(\%i4) \sin (x / 2);
          x
floor(-----)
    2 %pi
(- 1) sqrt(1 - cos(x))
(\%o4) ----------------------------------
          sqrt(2)
(\%i5) assume(x>0, x<2*%pi)$
(\%i6) \sin(x / 2);
          sqrt(1 - cos(x))
          ----------------
          sqrt(2)

ntrig [Package]

The ntrig package contains a set of simplification rules that are used to simplify trigonometric function whose arguments are of the form \(f(n \times \frac{\pi}{10})\) where \(f\) is any of the functions \(\sin\), \(\cos\), \(\tan\), \(\csc\), \(\sec\) and \(\cot\).

sec (x) [Function]
  – Secant.

sech (x) [Function]
  – Hyperbolic Secant.

sin (x) [Function]
  – Sine.

sinh (x) [Function]
  – Hyperbolic Sine.

tan (x) [Function]
  – Tangent.
tanh (x)  
  
- Hyperbolic Tangent.

trigexpand (expr)  

Expands trigonometric and hyperbolic functions of sums of angles and of multiple angles occurring in expr. For best results, expr should be expanded. To enhance user control of simplification, this function expands only one level at a time, expanding sums of angles or multiple angles. To obtain full expansion into sines and cosines immediately, set the switch trigexpand: true.

trigexpand is governed by the following global flags:

trigexpand
  
If true causes expansion of all expressions containing sin's and cos's occurring subsequently.

halfangles
  
If true causes half-angles to be simplified away.

trigexpandplus
  
Controls the "sum" rule for trigexpand, expansion of sums (e.g. sin(x + y)) will take place only if trigexpandplus is true.

trigexpandtimes
  
Controls the "product" rule for trigexpand, expansion of products (e.g. sin(2 x)) will take place only if trigexpandtimes is true.

Examples:

(%i1) x+sin(3*x)/sin(x),trigexpand=true,expand;
     2  2
(%o1) (- sin(x)) + 3 cos(x) + x

(%i2) trigexpand(sin(10*x+y));

(%o2) cos(10 x) sin(y) + sin(10 x) cos(y)

trigexpandplus  

Default value: true

trigexpandplus controls the "sum" rule for trigexpand. Thus, when the trigexpand command is used or the trigexpand switch set to true, expansion of sums (e.g. sin(x+y)) will take place only if trigexpandplus is true.

trigexpandtimes  

Default value: true

trigexpandtimes controls the "product" rule for trigexpand. Thus, when the trigexpand command is used or the trigexpand switch set to true, expansion of products (e.g. sin(2*x)) will take place only if trigexpandtimes is true.

triginverses  

Default value: true

triginverses controls the simplification of the composition of trigonometric and hyperbolic functions with their inverse functions.

If all, both e.g. atan(tan(x)) and tan(atan(x)) simplify to x.
If \texttt{true}, the \texttt{arcfun(fun(x))} simplification is turned off.

If \texttt{false}, both the \texttt{arcfun(fun(x))} and \texttt{fun(arcfun(x))} simplifications are turned off.

\textbf{trigreduce} \hfill \textnormal{[Function]}

\begin{verbatim}
trigreduce (expr, x)
trigreduce (expr)
\end{verbatim}

Combines products and powers of trigonometric and hyperbolic sin's and cos's of \( x \) into those of multiples of \( x \). It also tries to eliminate these functions when they occur in denominators. If \( x \) is omitted then all variables in \( \texttt{expr} \) are used.

See also \texttt{poissimp}.

\begin{verbatim}
(%i1) trigreduce(-sin(x)^2+3*cos(x)^2+x);
\end{verbatim}

\begin{verbatim}
cos(2 x) cos(2 x) 1 1
-------- + 3 (-------- + -) + x - -
2 2 2 2
\end{verbatim}

\textbf{trigsimp (expr)} \hfill \textnormal{[Function]}

Employs the identities

\[
\sin(x)^2 + \cos(x)^2 = 1 \quad \text{and} \quad \cosh(x)^2 - \sinh(x)^2 = 1
\]

to simplify expressions containing \( \tan, \sec, \text{etc.} \), to \( \sin, \cos, \sinh, \cosh \).

\begin{verbatim}
trigreduce, ratsimp, and radcan may be able to further simplify the result.
\end{verbatim}

demo ("trgsmp.dem") displays some examples of \texttt{trigsimp}.

\textbf{trigrat (expr)} \hfill \textnormal{[Function]}

Gives a canonical simplified quasilinear form of a trigonometrical expression; \( \texttt{expr} \) is a rational fraction of several \( \sin, \cos \) or \( \tan \), the arguments of them are linear forms in some variables (or kernels) and \( \%\pi/n \) (\( n \) integer) with integer coefficients. The result is a simplified fraction with numerator and denominator linear in \( \sin \) and \( \cos \). Thus \texttt{trigrat} linearize always when it is possible.

\begin{verbatim}
(%i1) trigrat(sin(3*a)/sin(a+%pi/3));
\end{verbatim}

\begin{verbatim}
\sqrt(3) sin(2 a) + cos(2 a) - 1
\end{verbatim}

The following example is taken from Davenport, Siret, and Tournier, \textit{Calcul Formel}, Masson (or in English, Addison-Wesley), section 1.5.5, Morley theorem.

\begin{verbatim}
(%i1) c : %pi/3 - a - b$
\end{verbatim}

\begin{verbatim}
(%i2) bc : sin(a)*sin(3*c)/sin(a+b);
\end{verbatim}

\begin{verbatim}
\%pi
sin(a) sin(3 ((- b) - a + ---))
3
-----------------------------
sin(b + a)
\end{verbatim}
(%i3) ba : bc, c=a, a=c;

    %pi
  sin(3 a) sin(b + a - ---)
    3
(%o3) -----------------------------------------------

    %pi
  sin(a - ---)
    3
(%i4) ac2 : ba^2 + bc^2 - 2*bc*ba*cos(b);

    2    2  %pi
  sin (3 a) sin (b + a - ---)
    3
(%o4) ------------------------------

    %pi
  sin (a - ---)
    3

    %pi
  sin(b + a - ---))/(sin(a - ---) sin(b + a))
    3    3

    2    2  %pi
  sin (a) sin (3 ((- b) - a + ---))
    3

  %pi
  sin (b + a)
(%i5) trigrat (ac2);

(%o5) - (sqrt(3) sin(4 b + 4 a) - cos(4 b + 4 a)
- 2 sqrt(3) sin(4 b + 2 a) + 2 cos(4 b + 2 a)
- 2 sqrt(3) sin(2 b + 4 a) + 2 cos(2 b + 4 a)
+ 4 sqrt(3) sin(2 b + 2 a) - 8 cos(2 b + 2 a) - 4 cos(2 b - 2 a)
+ sqrt(3) sin(4 b) - cos(4 b) - 2 sqrt(3) sin(2 b) + 10 cos(2 b)
+ sqrt(3) sin(4 a) - cos(4 a) - 2 sqrt(3) sin(2 a) + 10 cos(2 a)
- 9)/4
10.6 Random Numbers

**make_random_state**

- `make_random_state (n)`
- `make_random_state (s)`
- `make_random_state (true)`
- `make_random_state (false)`

A random state object represents the state of the random number generator. The state comprises 627 32-bit words.

- `make_random_state (n)` returns a new random state object created from an integer seed value equal to \( n \mod 2^{32} \). \( n \) may be negative.
- `make_random_state (s)` returns a copy of the random state \( s \).
- `make_random_state (true)` returns a new random state object, using the current computer clock time as the seed.
- `make_random_state (false)` returns a copy of the current state of the random number generator.

**set_random_state**

Copies \( s \) to the random number generator state.

**random**

Returns a pseudorandom number. If \( x \) is an integer, `random (x)` returns an integer from 0 through \( x - 1 \) inclusive. If \( x \) is a floating point number, `random (x)` returns a nonnegative floating point number less than \( x \). `random` complains with an error if \( x \) is neither an integer nor a float, or if \( x \) is not positive.

The functions `make_random_state` and `set_random_state` maintain the state of the random number generator.

The Maxima random number generator is an implementation of the Mersenne twister MT 19937.

Examples:

```
(%i1) s1: make_random_state (654321)$
(%i2) set_random_state (s1); done
(%i3) random (1000); 768
(%i4) random (9573684); 7657880
(%i5) random (2^75); 11804491615036831636390
(%i6) s2: make_random_state (false)$
(%i7) random (1.0); 0.2310127244107132
(%i8) random (10.0); 4.394553645870825
```
(%i9) random (100.0);
(%o9) 32.28666704056853
(%i10) set_random_state (s2);
(%o10) done
(%i11) random (1.0);
(%o11) 0.2310127244107132
(%i12) random (10.0);
(%o12) 4.394553645870825
(%i13) random (100.0);
(%o13) 32.28666704056853
11 Maximas Database

11.1 Introduction to Maximas Database

11.2 Functions and Variables for Properties

**alphabetical**

*Property*

alphabetical is a property type recognized by `declare`. The expression `declare(s, alphabetical)` tells Maxima to recognize as alphabetic all of the characters in `s`, which must be a string.

See also Section 6.3 [Identifiers], page 82.

Example:

```
(%i1) xx"yy"@ : 1729;
(%o1) 1729
(%i2) declare ("~'@", alphabetical);
(%o2) done
(%i3) xx~yy'@ + @yy'xx + 'xx@yy~;
(%o3) 'xx@yy~ + @yy'xx + 1729
(%i4) listofvars (%);
(%o4) [@yy'xx, 'xx@yy~]
```

**bindtest**

*Property*

The command `declare(x, bindtest)` tells Maxima to trigger an error when the symbol `x` is evaluated unbound.

```
(%i1) aa + bb;
(%o1) bb + aa
(%i2) declare (aa, bindtest);
(%o2) done
(%i3) aa + bb;
  aa unbound variable
  -- an error. Quitting. To debug this try debugmode(true);
(%i4) aa : 1234;
(%o4) 1234
(%i5) aa + bb;
(%o5) bb + 1234
```

**constant**

*Property*

`declare(a, constant)` declares `a` to be a constant. The declaration of a symbol to be constant does not prevent the assignment of a nonconstant value to the symbol.

See `constantp` and `declare`.

Example:

```
(%i1) declare(c, constant);
(%o1) done
(%i2) constantp(c);
(%o2) true
```
\begin{verbatim}
(%i3) c : x;
(%o3) x
(%i4) constantp(c);
(%o4) false
\end{verbatim}

\textbf{constantp (expr)} \hfill [Function]

Returns \texttt{true} if \texttt{expr} is a constant expression, otherwise returns \texttt{false}.

An expression is considered a constant expression if its arguments are numbers (including rational numbers, as displayed with \texttt{/R/}), symbolic constants such as \%pi, \%e, and \%i, variables bound to a constant or declared constant by \texttt{declare}, or functions whose arguments are constant.

\texttt{constantp} evaluates its arguments.

See the property \texttt{constant} which declares a symbol to be constant.

Examples:

\begin{verbatim}
(%i1) constantp (7 * sin(2));
(%o1) true
(%i2) constantp (rat (17/29));
(%o2) true
(%i3) constantp (%pi * sin(%e));
(%o3) true
(%i4) constantp (exp (x));
(%o4) false
(%i5) declare (x, constant);
(%o5) done
(%i6) constantp (exp (x));
(%o6) true
(%i7) constantp (foo (x) + bar (%e) + baz (2));
(%o7) false
(%i8)
\end{verbatim}

\textbf{declare (a_1, p_1, a_2, p_2, \ldots)} \hfill [Function]

Assigns the atom or list of atoms \texttt{a_i} the property or list of properties \texttt{p_i}. When \texttt{a_i} and/or \texttt{p_i} are lists, each of the atoms gets all of the properties.

\texttt{declare} quotes its arguments. \texttt{declare} always returns \texttt{done}.

As noted in the description for each declaration flag, for some flags \texttt{featurep(object, feature)} returns \texttt{true} if \texttt{object} has been declared to have \texttt{feature}.

For more information about the features system, see \texttt{features}. To remove a property from an atom, use \texttt{remove}.

\texttt{declare} recognizes the following properties:

\textbf{additive} \hfill Tells Maxima to simplify \texttt{a_i} expressions by the substitution \texttt{a_i(x + y + z + \ldots) \rightarrow a_i(x) + a_i(y) + a_i(z) + \ldots.} The substitution is carried out on the first argument only.

\textbf{alphabetic} \hfill Tells Maxima to recognize all characters in \texttt{a_i} (which must be a string) as alphabetic characters.
antisymmetric, commutative, symmetric

Tells Maxima to recognize $a_i$ as a symmetric or antisymmetric function. **commutative** is the same as symmetric.

bindtest

Tells Maxima to trigger an error when $a_i$ is evaluated unbound.

commutative

Tells Maxima to consider $a_i$ a symmetric function.

commute

Tells Maxima to consider $a_i$ a commutative function.

commutative

is the same as symmetric.

even, odd

Tells Maxima to recognize $a_i$ as an even or odd integer variable.

evenfun, oddfun

Tells Maxima to recognize $a_i$ as an even or odd function.

bindtest

Tells Maxima to trigger an error when $a_i$ is evaluated unbound.

even, odd

Tells Maxima to recognize $a_i$ as an even or odd integer variable.

evenfun, oddfun

Tells Maxima to recognize $a_i$ as an even or odd function.

bindtest

Tells Maxima to trigger an error when $a_i$ is evaluated unbound.

even, odd

Tells Maxima to recognize $a_i$ as an even or odd integer variable.

evenfun, oddfun

Tells Maxima to recognize $a_i$ as an even or odd function.

bindtest

Tells Maxima to trigger an error when $a_i$ is evaluated unbound.

even, odd

Tells Maxima to recognize $a_i$ as an even or odd integer variable.

evenfun, oddfun

Tells Maxima to recognize $a_i$ as an even or odd function.

bindtest

Tells Maxima to trigger an error when $a_i$ is evaluated unbound.

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evenfun, oddfun

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evenfun, oddfun

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bindtest

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even, odd

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evenfun, oddfun

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bindtest

Tells Maxima to trigger an error when $a_i$ is evaluated unbound.

even, odd

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evenfun, oddfun

Tells Maxima to recognize $a_i$ as an even or odd function.

bindtest

Tells Maxima to trigger an error when $a_i$ is evaluated unbound.

even, odd

Tells Maxima to recognize $a_i$ as an even or odd integer variable.

evenfun, oddfun

Tells Maxima to recognize $a_i$ as an even or odd function.

bindtest

Tells Maxima to trigger an error when $a_i$ is evaluated unbound.

even, odd

Tells Maxima to recognize $a_i$ as an even or odd integer variable.

evenfun, oddfun

Tells Maxima to recognize $a_i$ as an even or odd function.

bindtest

Tells Maxima to trigger an error when $a_i$ is evaluated unbound.

even, odd

Tells Maxima to recognize $a_i$ as an even or odd integer variable.

evenfun, oddfun

Tells Maxima to recognize $a_i$ as an even or odd function.

bindtest

Tells Maxima to trigger an error when $a_i$ is evaluated unbound.

even, odd

Tells Maxima to recognize $a_i$ as an even or odd integer variable.

evenfun, oddfun

Tells Maxima to recognize $a_i$ as an even or odd function.

bindtest

Tells Maxima to trigger an error when $a_i$ is evaluated unbound.

even, odd

Tells Maxima to recognize $a_i$ as an even or odd integer variable.

evenfun, oddfun

Tells Maxima to recognize $a_i$ as an even or odd function.
nonscalar
Tell Maxima to consider \(a\_i\) a nonscalar variable. The usual application is to declare a variable as a symbolic vector or matrix.

noun
Tell Maxima to parse \(a\_i\) as a noun. The effect of this is to replace instances of \(a\_i\) with '\(a\_i\) or nounify(\(a\_i\)), depending on the context.

outative
Tell Maxima to simplify \(a\_i\) expressions by pulling constant factors out of the first argument.

When \(a\_i\) has one argument, a factor is considered constant if it is a literal or declared constant.
When \(a\_i\) has two or more arguments, a factor is considered constant if the second argument is a symbol and the factor is free of the second argument.

posfun
Tell Maxima to recognize \(a\_i\) as a positive function.

rational, irrational
Tell Maxima to recognize \(a\_i\) as a rational or irrational real variable.

real, imaginary, complex
Tell Maxima to recognize \(a\_i\) as a real, pure imaginary, or complex variable.

scalar
Tell Maxima to consider \(a\_i\) a scalar variable.

Examples of the usage of the properties are available in the documentation for each separate description of a property.

\textbf{decreasing} \hspace{1cm} \textbf{increasing}

The commands \texttt{declare(f, decreasing)} or \texttt{declare(f, increasing)} tell Maxima to recognize the function \(f\) as an decreasing or increasing function.

See also \texttt{declare} for more properties.

Example:
\begin{verbatim}
(%i1) assume(a > b);
(%o1) [a > b]
(%i2) is(f(a) > f(b));
(%o2) unknown
(%i3) declare(f, increasing);
(%o3) done
(%i4) is(f(a) > f(b));
(%o4) true
\end{verbatim}

\textbf{even} \hspace{1cm} \textbf{odd}

\texttt{declare(a, even)} or \texttt{declare(a, odd)} tells Maxima to recognize the symbol \(a\) as an even or odd integer variable. The properties \texttt{even} and \texttt{odd} are not recognized by the functions \texttt{evenp}, \texttt{oddp}, and \texttt{integerp}.

See also \texttt{declare} and \texttt{askinteger}.
Example:

```maxima
(%i1) declare(n, even);  
(%o1) done
(%i2) askinteger(n, even);  
(%o2) yes
(%i3) askinteger(n);  
(%o3) yes
(%i4) evenp(n);  
(%o4) false
```

**feature**

Maxima understands two distinct types of features, system features and features which apply to mathematical expressions. See also `status` for information about system features. See also `features` and `featurep` for information about mathematical features.

*feature* itself is not the name of a function or variable.

**featurep**

Attempts to determine whether the object *a* has the feature *f* on the basis of the facts in the current database. If so, it returns `true`, else `false`.

Note that `featurep` returns `false` when neither *f* nor the negation of *f* can be established.

`featurep` evaluates its argument.

See also `declare` and `features`.

```maxima
(%i1) declare (j, even)$
(%i2) featurep (j, integer);
(%o2) true
```

**features**

Maxima recognizes certain mathematical properties of functions and variables. These are called "features".

`declare (x, foo)` gives the property *foo* to the function or variable *x*.

`declare (foo, feature)` declares a new feature *foo*. For example, `declare ([red, green, blue], feature)` declares three new features, *red*, *green*, and *blue*.

The predicate `featurep (x, foo)` returns `true` if *x* has the *foo* property, and `false` otherwise.

The infolist `features` is a list of known features. These are

<table>
<thead>
<tr>
<th>integer</th>
<th>noninteger</th>
<th>even</th>
</tr>
</thead>
<tbody>
<tr>
<td>odd</td>
<td>rational</td>
<td>irrational</td>
</tr>
<tr>
<td>real</td>
<td>imaginary</td>
<td>complex</td>
</tr>
<tr>
<td>analytic</td>
<td>increasing</td>
<td>decreasing</td>
</tr>
<tr>
<td>oddfun</td>
<td>evenfun</td>
<td>posfun</td>
</tr>
<tr>
<td>constant</td>
<td>commutative</td>
<td>lassociative</td>
</tr>
<tr>
<td>rassociative</td>
<td>symmetric</td>
<td>antisymmetric</td>
</tr>
<tr>
<td>integervalued</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
plus any user-defined features.

`features` is a list of mathematical features. There is also a list of non-mathematical, system-dependent features. See `status`.

Example:

```maxima
(%i1) declare (FOO, feature);
(%o1) done
(%i2) declare (x, FOO);
(%o2) done
(%i3) featurep (x, FOO);
(%o3) true
```

`get (a, i)`

Retrieves the user property indicated by `i` associated with atom `a` or returns `false` if `a` doesn’t have property `i`.

`get` evaluates its arguments.

See also `put` and `qput`.

```maxima
(%i1) put (%e, 'transcendental, 'type);
(%o1) transcendental
(%i2) put (%pi, 'transcendental, 'type)$
(%i3) put (%i, 'algebraic, 'type)$
(%i4) typeof (expr) := block ([q],
   if numberp (expr)
   then return ('algebraic),
   if not atom (expr)
   then return (maplist ('typeof, expr)),
   q: get (expr, 'type),
   if q=false
   then errcatch (error(expr,"is not numeric.")) else q)$
(%i5) typeof (2*%e + x*%pi);
x is not numeric.
(%o5) [[transcendental, []], [algebraic, transcendental]]
(%i6) typeof (2*%e + %pi);
(%o6) [transcendental, [algebraic, transcendental]]
```

`integer`  

`noninteger`

`declare(a, integer)` or `declare(a, noninteger)` tells Maxima to recognize `a` as an integer or noninteger variable.

See also `declare`.

Example:

```maxima
(%i1) declare(n, integer, x, noninteger);
(%o1) done
(%i2) askinteger(n);
(%o2) yes
(%i3) askinteger(x);
(%o3) no
```
**integervalued**

[Property]

`declare(f, integervalued)` tells Maxima to recognize \( f \) as an integer-valued function.

See also `declare`.

Example:

```
(%i1) exp(%i)^f(x);
     %i f(x)
(%o1) (%e )
(%i2) declare(f, integervalued);
     done
(%i3) exp(%i)^f(x);
     %i f(x)
(%o3) %e
```

**nonarray**

[Property]

The command `declare(a, nonarray)` tells Maxima to consider \( a \) not an array. This declaration prevents multiple evaluation, if \( a \) is a subscripted variable.

See also `declare`.

Example:

```
(%i1) a:'b$ b:'c$ c:'d$
(%i4) a[x];
    d    
   x
(%i5) declare(a, nonarray);
     done
(%i6) a[x];
    a    
   x
```

**nonscalar**

[Property]

Makes atoms behave as does a list or matrix with respect to the dot operator.

See also `declare`.

**nonscalarp (expr)**

[Function]

Returns `true` if \( expr \) is a non-scalar, i.e., it contains atoms declared as non-scalars, lists, or matrices.

See also the predicate function `scalarp` and `declare`.

**posfun**

[Property]

`declare (f, posfun)` declares \( f \) to be a positive function. \( \text{is} \ (f(x) > 0) \) yields `true`.

See also `declare`.
printprops

printprops (a, i)
printprops ([a_1, ..., a_n], i)
printprops (all, i)

Displays the property with the indicator \( i \) associated with the atom \( a \). \( a \) may also be a list of atoms or the atom \( \text{all} \) in which case all of the atoms with the given property will be used. For example, \text{printprops} ([f, g], \text{atvalue})$. printprops is for properties that cannot otherwise be displayed, i.e. for \text{atvalue}, \text{atomgrad}, \text{gradef}, and \text{matchdeclare}.

properties (a)

Returns a list of the names of all the properties associated with the atom \( a \).

props

Default value: []

props are atoms which have any property other than those explicitly mentioned in infolists, such as specified by \text{atvalue}, \text{matchdeclare}, etc., as well as properties specified in the \text{declare} function.

propvars (prop)

Returns a list of those atoms on the props list which have the property indicated by \( prop \). Thus \text{propvars (atvalue)} returns a list of atoms which have atvalues.

put (atom, value, indicator)

Assigns \( value \) to the property (specified by \( indicator \)) of \( atom \). \( indicator \) may be the name of any property, not just a system-defined property. rem reverses the effect of put.

put evaluates its arguments. put returns value.

See also qput and get.

Examples:

```
(%i1) put (foo, (a+b)^5, expr);
      5
(%o1) (b + a)
(%i2) put (foo, "Hello", str);
(%o2) Hello
(%i3) properties (foo);
(%o3) [[user properties, str, expr]]
(%i4) get (foo, expr);
      5
(%o4) (b + a)
(%i5) get (foo, str);
(%o5) Hello
```

qput (atom, value, indicator)

Assigns \( value \) to the property (specified by \( indicator \)) of \( atom \). This is the same as put, except that the arguments are quoted.

See also get.
Example:

```maxima
(%i1) foo: aa$
(%i2) bar: bb$
(%i3) baz: cc$
(%i4) put (foo, bar, baz);
(%o4) bb
(%i5) properties (aa);
(%o5) [[user properties, cc]]
(%i6) get (aa, cc);
(%o6) bb
(%i7) qput (foo, bar, baz);
(%o7) bar
(%i8) properties (foo);
(%o8) [value, [user properties, baz]]
(%i9) get ('foo, 'baz);
(%o9) bar
```

**rational**  
**irrational**

```maxima
declare(a, rational) or declare(a, irrational) tells Maxima to recognize a as a rational or irrational real variable.
```

See also `declare`.

**real**  
**imaginary**  
**complex**

```maxima
declare(a, real), declare(a, imaginary), or declare(a, complex) tells Maxima to recognize a as a real, pure imaginary, or complex variable.
```

See also `declare`.

**rem (atom, indicator)**

Removes the property indicated by `indicator` from `atom`. `rem` reverses the effect of `put`.

`rem` returns `done` if `atom` had an `indicator` property when `rem` was called, or `false` if it had no such property.

**remove**

```maxima
remove (a_1, p_1, ..., a_n, p_n)
remove ([a_1, ..., a_m], [p_1, ..., p_n], ...)
remove ("a", operator)
remove (a, transfun)
remove (all, p)
```

Removes properties associated with atoms.

```maxima
remove (a_1, p_1, ..., a_n, p_n) removes property p_k from atom a_k.
remove ([a_1, ..., a_m], [p_1, ..., p_n], ...) removes properties p_1, ..., p_n from atoms a_1, ..., a_m. There may be more than one pair of lists.
remove (all, p) removes the property p from all atoms which have it.
```
The removed properties may be system-defined properties such as function, macro, or mode_declare. remove does not remove properties defined by put.

A property may be transfun to remove the translated Lisp version of a function. After executing this, the Maxima version of the function is executed rather than the translated version.

remove ("a", operator) or, equivalently, remove ("a", op) removes from a the operator properties declared by prefix, infix, [function_nary], page 126, postfix, matchfix, or nofix. Note that the name of the operator must be written as a quoted string.

remove always returns done whether or not an atom has a specified property. This behavior is unlike the more specific remove functions remvalue, remarray, remfunction, and remrule.

remove quotes its arguments.

\textbf{scalar}

\begin{verbatim}
declare(a, scalar)
\end{verbatim}

tells Maxima to consider a as a scalar variable.

See also declare.

\textbf{scalarp (expr)}

\begin{verbatim}
Returns true if expr is a number, constant, or variable declared scalar with declare, or composed entirely of numbers, constants, and such variables, but not containing matrices or lists.
\end{verbatim}

See also the predicate function nonscalarp.

\section{11.3 Functions and Variables for Facts}

\textbf{activate (context_1, \ldots, context_n)}

\begin{verbatim}
Activates the contexts context_1, \ldots, context_n. The facts in these contexts are then available to make deductions and retrieve information. The facts in these contexts are not listed by facts ()
\end{verbatim}

The variable \texttt{activecontexts} is the list of contexts which are active by way of the activate function.

\textbf{activecontexts}

\begin{verbatim}
Default value: []
\end{verbatim}

\texttt{activecontexts} is a list of the contexts which are active by way of the \texttt{activate} function, as opposed to being active because they are subcontexts of the current context.

\textbf{askinteger}

\begin{verbatim}
askinteger (expr, integer)
askinteger (expr)
askinteger (expr, even)
askinteger (expr, odd)
\end{verbatim}

\texttt{askinteger (expr, integer)} attempts to determine from the assume database whether expr is an integer. \texttt{askinteger} prompts the user if it cannot tell otherwise,
and attempt to install the information in the database if possible. \texttt{askinteger} (\texttt{expr}) is equivalent to \texttt{askinteger (expr, integer)}.

\texttt{askinteger (expr, even)} and \texttt{askinteger (expr, odd)} likewise attempt to determine if \texttt{expr} is an even integer or odd integer, respectively.

\texttt{asksign (expr)} \hspace{1cm} \textbf{[Function]}

First attempts to determine whether the specified expression is positive, negative, or zero. If it cannot, it asks the user the necessary questions to complete its deduction. The user’s answer is recorded in the database for the duration of the current computation. The return value of \texttt{asksign} is one of \texttt{pos}, \texttt{neg}, or \texttt{zero}.

\texttt{assume (pred_1, \ldots, pred_n)} \hspace{1cm} \textbf{[Function]}

Adds predicates \texttt{pred_1, \ldots, pred_n} to the current context. If a predicate is inconsistent or redundant with the predicates in the current context, it is not added to the context. The context accumulates predicates from each call to \texttt{assume}.

\texttt{assume} returns a list whose elements are the predicates added to the context or the atoms \texttt{redundant} or \texttt{inconsistent} where applicable.

The predicates \texttt{pred_1, \ldots, pred_n} can only be expressions with the relational operators \texttt{< <= equal notequal >= >}. Predicates cannot be literal equality = or literal inequality # expressions, nor can they be predicate functions such as \texttt{integerp}.

Compound predicates of the form \texttt{pred_1 and ... and pred_n} are recognized, but not \texttt{pred_1 or ... or pred_n}. \texttt{not pred_k} is recognized if \texttt{pred_k} is a relational predicate. Expressions of the form \texttt{not (pred_1 and pred_2)} and \texttt{not (pred_1 or pred_2)} are not recognized.

Maxima’s deduction mechanism is not very strong; there are many obvious consequences which cannot be determined by \texttt{is}. This is a known weakness.

\texttt{assume} does not handle predicates with complex numbers. If a predicate contains a complex number \texttt{assume} returns \texttt{inconsistent} or \texttt{redundant}.

\texttt{assume} evaluates its arguments.

See also \texttt{is, facts, forget, context}, and \texttt{declare}.

Examples:

```
(%i1) assume (xx > 0, yy < -1, zz >= 0);
(%o1) [xx > 0, yy < -1, zz >= 0]
(%i2) assume (aa < bb and bb < cc);
(%o2) [bb > aa, cc > bb]
(%i3) facts ();
(%o3) [xx > 0, - 1 > yy, zz >= 0, bb > aa, cc > bb]
(%i4) is (xx > yy);
(%o4) true
(%i5) is (yy < -yy);
(%o5) true
(%i6) is (sinh (bb - aa) > 0);
(%o6) true
(%i7) forget (bb > aa);
(%o7) [bb > aa]
```
(%i8) prederror : false;  
(%o8) false
(%i9) is (sinh (bb - aa) > 0);  
(%o9) unknown
(%i10) is (bb^2 < cc^2);  
(%o10) unknown

assumescalar  
[Option variable]
Default value: true
assumescalar helps govern whether expressions expr for which nonscalarp (expr) is false are assumed to behave like scalars for certain transformations.
Let expr represent any expression other than a list or a matrix, and let [1, 2, 3] represent any list or matrix. Then expr . [1, 2, 3] yields [expr, 2 expr, 3 expr] if assumescalar is true, or scalarp (expr) is true, or constantp (expr) is true.
If assumescalar is true, such expressions will behave like scalars only for commutative operators, but not for noncommutative multiplication.
When assumescalar is false, such expressions will behave like non-scalars.
When assumescalar is all, such expressions will behave like scalars for all the operators listed above.

assume_pos  
[Option variable]
Default value: false
When assume_pos is true and the sign of a parameter x cannot be determined from the current context or other considerations, sign and asksign (x) return true. This may forestall some automatically-generated asksign queries, such as may arise from integrate or other computations.
By default, a parameter is x such that symbolp (x) or subvarp (x). The class of expressions considered parameters can be modified to some extent via the variable assume_pos_pred.
sign and asksign attempt to deduce the sign of expressions from the sign of operands within the expression. For example, if a and b are both positive, then a + b is also positive.
However, there is no way to bypass all asksign queries. In particular, when the asksign argument is a difference x - y or a logarithm log(x), asksign always requests an input from the user, even when assume_pos is true and assume_pos_pred is a function which returns true for all arguments.

assume_pos_pred  
[Option variable]
Default value: false
When assume_pos_pred is assigned the name of a function or a lambda expression of one argument x, that function is called to determine whether x is considered a parameter for the purpose of assume_pos. assume_pos_pred is ignored when assume_pos is false.
The assume_pos_pred function is called by sign and asksign with an argument x which is either an atom, a subcripted variable, or a function call expression. If the
assume_pos_pred function returns true, x is considered a parameter for the purpose of assume_pos.

By default, a parameter is x such that symbolp (x) or subvarp (x).

See also assume and assume_pos.

Examples:

```maxima
(%i1) assume_pos: true$
(%i2) assume_pos_pred: symbolp$
(%i3) sign (a);
   pos
(%i4) sign (a[1]);
   pnz
(%i5) assume_pos_pred: lambda ([x], display (x), true)$
(%i6) asksign (a);
   x = a
   pos
(%o6)
(%i7) asksign (a[1]);
   x = a
   1
   pos
(%o7)
(%i8) asksign (foo (a));
   x = foo(a)
   pos
(%o8)
(%i9) asksign (foo (a) + bar (b));
   x = foo(a)
   x = bar(b)
   pos
(%o9)
(%i10) asksign (log (a));
   x = a
   pos
(%o10)
(%i11) asksign (a - b);
   x = a
   x = b
   x = a
   x = b
```
Is b - a positive, negative, or zero?

p;'  
(%o11) neg

context  [Option variable]

Default value: initial

decortext names the collection of facts maintained by assume and forget. assume
adds facts to the collection named by context, while forget removes facts.

Binding context to a name foo changes the current context to foo. If the specified
context foo does not yet exist, it is created automatically by a call to newcontext.
The specified context is activated automatically.

See contexts for a general description of the context mechanism.

contexts  [Option variable]

Default value: [initial, global]

ccontexts is a list of the contexts which currently exist, including the currently active
context.

The context mechanism makes it possible for a user to bind together and name a
collection of facts, called a context. Once this is done, the user can have Maxima
assume or forget large numbers of facts merely by activating or deactivating their
context.

Any symbolic atom can be a context, and the facts contained in that context will be
retained in storage until destroyed one by one by calling forget or destroyed as a
whole by calling kill to destroy the context to which they belong.

Contexts exist in a hierarchy, with the root always being the context global, which
contains information about Maxima that some functions need. When in a given
context, all the facts in that context are "active" (meaning that they are used in
deductions and retrievals) as are all the facts in any context which is a subcontext of
the active context.

When a fresh Maxima is started up, the user is in a context called initial, which
has global as a subcontext.

See also facts, newcontext, supcontext, killcontext, activate, deactivate,
assume, and forget.

decactive (context_1, ..., context_n)  [Function]

Deactivates the specified contexts context_1, ..., context_n.

facts  [Function]

dfacts (item)

dfacts ()

If item is the name of a context, facts (item) returns a list of the facts in the
specified context.

If item is not the name of a context, facts (item) returns a list of the facts known
about item in the current context. Facts that are active, but in a different context,
are not listed.
facts () (i.e., without an argument) lists the current context.

**forget**

\[ \text{function: forget (pred_1, \ldots, pred_n)} \]
\[ \text{function: forget (L)} \]

Removes predicates established by **assume**. The predicates may be expressions equivalent to (but not necessarily identical to) those previously assumed.

**forget (L),** where \( L \) is a list of predicates, forgets each item on the list.

**is (expr)**

**Function**

Attempts to determine whether the predicate \( expr \) is provable from the facts in the **assume** database.

If the predicate is provably **true** or **false**, **is** returns **true** or **false**, respectively. Otherwise, the return value is governed by the global flag **prederror**. When **prederror** is **true**, **is** complains with an error message. Otherwise, **is** returns **unknown**.

**ev(expr, pred)** (which can be written \( expr, pred \) at the interactive prompt) is equivalent to **is(expr)**.

See also **assume**, **facts**, and **maybe**.

Examples:

**is** causes evaluation of predicates.

\[
\begin{align*}
\text{(%i1) } & \%\pi > \%e; \\
\text{(%o1)} & \%\pi > \%e \\
\text{(%i2) } & \text{is (\%\pi > \%e);} \\
\text{(%o2)} & \text{true}
\end{align*}
\]

**is** attempts to derive predicates from the **assume** database.

\[
\begin{align*}
\text{(%i1) } & \text{assume (a > b);} \\
\text{(%o1)} & \text{[a > b] } \\
\text{(%i2) } & \text{assume (b > c);} \\
\text{(%o2)} & \text{[b > c] } \\
\text{(%i3) } & \text{is (a < b);} \\
\text{(%o3)} & \text{false } \\
\text{(%i4) } & \text{is (a > c);} \\
\text{(%o4)} & \text{true } \\
\text{(%i5) } & \text{is (equal (a, c));} \\
\text{(%o5)} & \text{false}
\end{align*}
\]

If **is** can neither prove nor disprove a predicate from the **assume** database, the global flag **prederror** governs the behavior of **is**.

\[
\begin{align*}
\text{(%i1) } & \text{assume (a > b);} \\
\text{(%o1)} & \text{[a > b] } \\
\text{(%i2) } & \text{prederror: true$} \\
\text{(%i3) } & \text{is (a > 0);} \\
\text{Maxima was unable to evaluate the predicate:}
\text{a > 0}
\text{-- an error. Quitting. To debug this try debugmode(true);}\n\end{align*}
\]
(\%i4) prederror: false$
(\%i5) is (a > 0);
(\%o5) unknown

\textbf{killcontext} (context\_1, \ldots, context\_n)
\textbf{[Function]}
Kills the contexts context\_1, \ldots, context\_n.

If one of the contexts is the current context, the new current context will become the first available subcontext of the current context which has not been killed. If the first available un kills context is \texttt{global} then \texttt{initial} is used instead. If the \texttt{initial} context is killed, a new, empty \texttt{initial} context is created.

\texttt{killcontext} refuses to kill a context which is currently active, either because it is a subcontext of the current context, or by use of the function \texttt{activate}.

\texttt{killcontext} evaluates its arguments. \texttt{killcontext} returns \texttt{done}.

\textbf{maybe} (expr)
\textbf{[Function]}
Attempts to determine whether the predicate expr is provable from the facts in the \texttt{assume} database.

If the predicate is provably \texttt{true} or \texttt{false}, \texttt{maybe} returns \texttt{true} or \texttt{false}, respectively. Otherwise, \texttt{maybe} returns \texttt{unknown}.

\texttt{maybe} is functionally equivalent to \texttt{is} with \texttt{prederror}: \texttt{false}, but the result is computed without actually assigning a value to \texttt{prederror}.

See also \texttt{assume}, \texttt{facts}, and \texttt{is}.

Examples:

(\%i1) maybe (x > 0);
(\%o1) unknown
(\%i2) assume (x > 1);
(\%o2) [x > 1]
(\%i3) maybe (x > 0);
(\%o3) true

\textbf{newcontext}
\textbf{[Function]}
\texttt{newcontext (name)}
\texttt{newcontext ()}
Creates a new, empty context, called \texttt{name}, which has \texttt{global} as its only subcontext. The newly-created context becomes the currently active context.

If \texttt{name} is not specified, a new name is created (via \texttt{gensym}) and returned.

\texttt{newcontext} evaluates its argument. \texttt{newcontext} returns \texttt{name} (if specified) or the new context name.

\textbf{sign} (expr)
\textbf{[Function]}
Attempts to determine the sign of expr on the basis of the facts in the current data base. It returns one of the following answers: \texttt{pos} (positive), \texttt{neg} (negative), \texttt{zero}, \texttt{pz} (positive or zero), \texttt{nz} (negative or zero), \texttt{pn} (positive or negative), or \texttt{pnz} (positive, negative, or zero, i.e. nothing known).
supcontext

supcontext (name, context)
supcontext (name)
supcontext ()

Creates a new context, called name, which has context as a subcontext. context must exist.

If context is not specified, the current context is assumed.

If name is not specified, a new name is created (via gensym) and returned.

supcontext evaluates its argument. supcontext returns name (if specified) or the new context name.

11.4 Functions and Variables for Predicates

charfun (p)

Return 0 when the predicate p evaluates to false; return 1 when the predicate evaluates to true. When the predicate evaluates to something other than true or false (unknown), return a noun form.

Examples:

(%i1) charfun (x < 1);
(%o1) charfun(x < 1)
(%i2) subst (x = -1, %);
(%o2) 1
(%i3) e : charfun ('"and" (-1 < x, x < 1))$
(%i4) [subst (x = -1, e), subst (x = 0, e), subst (x = 1, e)];
(%o4) [0, 1, 0]

compare (x, y)

Return a comparison operator op (<, <=, >, >=, =, or #) such that is (x op y) evaluates to true; when either x or y depends on %i and x # y, return notcomparable; when there is no such operator or Maxima isn’t able to determine the operator, return unknown.

Examples:

(%i1) compare (1, 2);
(%o1) <
(%i2) compare (1, x);
(%o2) unknown
(%i3) compare (%i, %i);
(%o3) =
(%i4) compare (%i, %i + 1);
(%o4) notcomparable
(%i5) compare (1/x, 0);
(%o5) #
(%i6) compare (x, abs(x));
(%o6) <=
The function \texttt{compare} doesn’t try to determine whether the real domains of its arguments are nonempty; thus

\begin{verbatim}
(%i1) compare (acos (x^2 + 1), acos (x^2 + 1) + 1);
(%o1) <
\end{verbatim}

The real domain of \texttt{acos (x^2 + 1)} is empty.

\textbf{equal (a, b)}

[Function]

Represents equivalence, that is, equal value.

By itself, \texttt{equal} does not evaluate or simplify. The function \texttt{is} attempts to evaluate \texttt{equal} to a Boolean value. \texttt{is(equal(a, b))} returns \texttt{true} (or \texttt{false}) if and only if \texttt{a} and \texttt{b} are equal (or not equal) for all possible values of their variables, as determined by evaluating \texttt{ratsimp(a - b)}; if \texttt{ratsimp} returns 0, the two expressions are considered equivalent. Two expressions may be equivalent even if they are not syntactically equal (i.e., identical).

When \texttt{is} fails to reduce \texttt{equal} to \texttt{true} or \texttt{false}, the result is governed by the global flag \texttt{prederror}. When \texttt{prederror} is \texttt{true}, \texttt{is} complains with an error message. Otherwise, \texttt{is} returns \texttt{unknown}.

In addition to \texttt{is}, some other operators evaluate \texttt{equal} and \texttt{notequal} to \texttt{true} or \texttt{false}, namely \texttt{if}, \texttt{and}, \texttt{or}, and \texttt{not}.

The negation of \texttt{equal} is \texttt{notequal}.

Examples:

By itself, \texttt{equal} does not evaluate or simplify.

\begin{verbatim}
(%i1) equal (x^2 - 1, (x + 1) * (x - 1));
2
(%o1) equal(x - 1, (x - 1) (x + 1))
(%i2) equal (x, x + 1);
(%o2) equal(x, x + 1)
(%i3) equal (x, y);
(%o3) equal(x, y)
\end{verbatim}

The function \texttt{is} attempts to evaluate \texttt{equal} to a Boolean value. \texttt{is(equal(a, b))} returns \texttt{true} when \texttt{ratsimp(a - b)} returns 0. Two expressions may be equivalent even if they are not syntactically equal (i.e., identical).

\begin{verbatim}
(%i1) ratsimp (x^2 - 1 - (x + 1) * (x - 1));
(%o1) 0
(%i2) is (equal (x^2 - 1, (x + 1) * (x - 1)));
(%o2) true
(%i3) is (x^2 - 1 = (x + 1) * (x - 1));
(%o3) false
(%i4) ratsimp (x - (x + 1));
(%o4) - 1
(%i5) is (equal (x, x + 1));
(%o5) false
(%i6) is (x = x + 1);
(%o6) false
(%i7) ratsimp (x - y);
\end{verbatim}
When \texttt{is} fails to reduce \texttt{equal} to \texttt{true} or \texttt{false}, the result is governed by the global flag \texttt{prederror}.

Some operators evaluate \texttt{equal} and \texttt{notequal} to \texttt{true} or \texttt{false}.

Because \texttt{not expr} causes evaluation of \texttt{expr}, \texttt{not equal(a, b)} is equivalent to \texttt{is(notequal(a, b))}.

\texttt{notequal(a, b)}

Represents the negation of \texttt{equal(a, b)}.

Examples:

```lisp
(%i1) equal (a, b);
(%o1) equal(a, b)
```
unknown (expr)  [Function]
Returns true if and only if expr contains an operator or function not recognized by the Maxima simplifier.

zeroequiv (expr, v)  [Function]
Tests whether the expression expr in the variable v is equivalent to zero, returning true, false, or dontknow.

zeroequiv has these restrictions:
1. Do not use functions that Maxima does not know how to differentiate and evaluate.
2. If the expression has poles on the real line, there may be errors in the result (but this is unlikely to occur).
3. If the expression contains functions which are not solutions to first order differential equations (e.g. Bessel functions) there may be incorrect results.
4. The algorithm uses evaluation at randomly chosen points for carefully selected subexpressions. This is always a somewhat hazardous business, although the algorithm tries to minimize the potential for error.

For example zeroequiv (sin(2 * x) - 2 * sin(x) * cos(x), x) returns true and zeroequiv (%e^x + x, x) returns false. On the other hand zeroequiv (log(a * b) - log(a) - log(b), a) returns dontknow because of the presence of an extra parameter b.
12 Plotting

12.1 Introduction to Plotting

Maxima uses an external plotting package to make the plots (see the section on Plotting Formats). The plotting functions calculate a set of points and pass them to the plotting package together with a set of commands. That information can be passed to the external program either through a pipe or by calling the program with the name of a file where the data has been saved. The data file is given the name maxout_xxx.format, where xxx is a number that is unique to every concurrently-running instance of Maxima and format is the name of the plotting format being used (gnuplot, xmaxima, mgnuplot, gnuplot_pipes or geomview).

There are to save the plot in a graphic format file. In those cases, the file maxout_xxx.format created by Maxima includes commands that will make the external plotting program save the result in a graphic file. The default name for that graphic file is maxplot.extension, where extension is the extension normally used for the kind of graphic file selected.

The maxout_xxx.format and maxplot.extension files are created in the directory specified by the system variable maxima_tempdir. That location can be changed by assigning to that variable (or to the environment variable MAXIMA_TEMPDIR) a string that represents a valid directory where Maxima can create new files. The output of the Maxima plotting command will be a list with the names of the file(s) created, including their complete path.

If the format used is either gnuplot or xmaxima, the external programs gnuplot or xmaxima can be run, giving it the file maxout_xxx.format as argument, in order to view again a plot previously created in Maxima. Thus, when a Maxima plotting command fails, the format can be set to gnuplot or xmaxima and the plain-text file maxout_xxx.gnuplot (or maxout_xxx.xmaxima) can be inspected to look for the source of the problem.

The additional package [draw], page 748, provides functions similar to the ones described in this section with some extra features. Note that some plotting options have the same name in both plotting packages, but their syntax and behavior is different. To view the documentation for a graphic option opt, type ?? opt in order to choose the information for either of those two packages.

12.2 Plotting Formats

Maxima can use either Gnuplot, Xmaxima or Geomview as graphics program. Gnuplot and Geomview are external programs which must be installed separately, while Xmaxima is distributed with Maxima. There are various different formats for those programs, which can be selected with the option plot_format (see also the Plotting Options section).

The plotting formats are the following:

- **gnuplot** (default on Windows)
  
  Used to launch the external program gnuplot, which must be installed in your system. All plotting commands and data are saved into the file maxout_xxx.gnuplot.

- **gnuplot_pipes** (default on non-Windows platforms)
  
  This format is not available in Windows platforms. It is similar to the gnuplot format except that the commands are sent to gnuplot through a pipe, while the data are saved
into the file **maxout_xxx.gnuplot_pipes**. A single gnuplot process is kept open and subsequent plot commands will be sent to the same process, replacing previous plots, unless the gnuplot pipe is closed with the function **gnuplot_close**. When this format is used, the function **gnuplot_replot** can be used to modify a plot that has already displayed on the screen.

This format is only used to plot to the screen; whenever graphic files are created, the format is silently switched to **gnuplot** and the commands needed to create the graphic file are saved with the data in file **maxout_xxx.gnuplot**.

- **mgnuplot**

  Mgnuplot is a Tk-based wrapper around gnuplot. It is included in the Maxima distribution. Mgnuplot offers a rudimentary GUI for gnuplot, but has fewer overall features than the plain gnuplot interface. Mgnuplot requires an external gnuplot installation and, in Unix systems, the Tcl/Tk system.

- **xmaxima**

  Xmaxima is a Tcl/Tk graphical interface for Maxima that can also be used to display plots created when Maxima is run from the console or from other graphical interfaces. To use this format, the xmaxima program, which is distributed together with Maxima, must be installed. If Maxima is being run from the Xmaxima console, the data and commands are passed to xmaxima through the same socket used for the communication between Maxima and the Xmaxima console. When used from a terminal or from graphical interfaces different from Xmaxima, the commands and data are saved in the file **maxout_xxx.xmaxima** and xmaxima is run with the name of that file as argument.

  In previous versions this format used to be called **openmath**; that old name still works as a synonym for **xmaxima**.

- **geomview**

  Geomview, a Motif based interactive 3D viewing program for Unix, can also be used to display plots created by Maxima. To use this format, the geomview program must be installed.

### 12.3 Functions and Variables for Plotting

**contour_plot (expr, x_range, y_range, options, ...)**  
([Function])

It plots the contours (curves of equal value) of **expr** over the region **x_range** by **y_range**. Any additional arguments are treated the same as in **plot3d**.

This function only works when the plot format is either **gnuplot** or **gnuplot_pipes**. The additional package **implicit_plot**, which works in any graphic format, can also be used to plot contours but a separate expression must be given for each contour.

Examples:
You can add any options accepted by plot3d; for instance, the option legend with a value of false, to remove the legend. By default, Gnuplot chooses and displays 3 contours. To increase the number of contours, it is necessary to use a custom gnuplot_preamble, as in the next example:

(%i1) contour_plot (u^3 + v^2, [u, -4, 4], [v, -4, 4],
[legend,false],
[gnuplot_preamble, "set cntrparam levels 12"])$

geomview_command [System variable]
This variable stores the name of the command used to run the geomview program when the plot format is geomview. Its default value is "geomview". If the geomview program is not found unless you give its complete path or if you want to try a different version of it, you may change the value of this variable. For instance,

(%i1) geomview_command: "/usr/local/bin/my_geomview"$

(%)
`get_plot_option (keyword, index)`  
Returns the current default value of the option named `keyword`, which is a list. The optional argument `index` must be a positive integer which can be used to extract only one element from the list (element 1 is the name of the option).

See also `set_plot_option`, `remove_plot_option` and the section on Plotting Options.

`gnuplot_command`  
This variable stores the name of the command used to run the gnuplot program when the plot format is `gnuplot`. Its default value is "gnuplot". If the gnuplot program is not found unless you give its complete path or if you want to try a different version of it, you may change the value of this variable. For instance,

```lisp
(%i1) gnuplot_command: "#/usr/local/bin/my_gnuplot"
```

`gnuplot_file_args`  
When a graphic file is going to be created using `gnuplot`, this variable is used to specify the way the file name should be passed to gnuplot. Its default value is "~-s", which means that the name of the file will be passed directly. The contents of this variable can be changed in order to add options for the gnuplot program, adding those options before the format directive "~-s".

`gnuplot_view_args`  
This variable is used to parse the argument that will be passed to the gnuplot program when the plot format is `gnuplot`. Its default value is "-persist ~s", where "s" will be replaced with the name of the file where the gnuplot commands have been written (usually "maxout.xxx.gnuplot"). The option `-persist` tells gnuplot to exit after the commands in the file have been executed, without closing the window that displays the plot.

Those familiar with gnuplot, might want to change the value of this variable. For example, by changing it to:

```lisp
(%i1) gnuplot_view_args: "~s -"$gnuplot will not be closed after the commands in the file have been executed; thus, the window with the plot will remain, as well as the gnuplot interactive shell where other commands can be issued in order to modify the plot.

In Windows versions of Gnuplot older than 4.6.3 the behavior of "~-s -" and "-persist ~s" were the opposite; namely, "-persist ~s" made the plot window and the gnuplot interactive shell remain, while "~-s -" closed the gnuplot shell keeping the plot window. Therefore, when older gnuplot versions are used in Windows, it might be necessary to adjust the value of `gnuplot_view_args`.

`implicit_plot`  
Displays a plot of a function on the real plane, defined implicitly by the expression `expr`. The domain in the plane is defined by `x_range` and `y_range`. Several functions can be represented on the same plot, giving a list `[expr_1, . . ., expr_n]` of expressions that define them. This function uses the global format options set up with the
**set_plot_option.** Additional options can also be given as extra arguments for the `implicit_plot` command.

The method used by `implicit_plot` consists of tracking sign changes on the domain given and it can fail for complicated expressions.

`load(implicit_plot)` loads this function.

Example:

```
(%i1) load(implicit_plot)
(%i2) implicit_plot (x^2 = y^3 - 3*y + 1, [x, -4, 4], [y, -4, 4])
```

---

**julia** *x, y, ...options...*  
[Function]  
Creates a graphic representation of the Julia set for the complex number *(x + i y).*  
The two mandatory parameters *x* and *y* must be real. This program is part of the additional package `dynamics`, but that package does not have to be loaded; the first time `julia` is used, it will be loaded automatically.

Each pixel in the grid is given a color corresponding to the number of iterations it takes the sequence that starts at that point to move out of the convergence circle of radius 2 centered at the origin. The number of pixels in the grid is controlled by the `grid` plot option (default 30 by 30). The maximum number of iterations is set with the option `iterations`. The program uses its own default palette: magenta, violet, blue, cyan, green, yellow, orange, red, brown and black, but it can be changed by adding an explicit `palette` option in the command.

The default domain used goes from -2 to 2 in both axes and can be changed with the *x* and *y* options. By default, the two axes are shown with the same scale, unless the option `yx_ratio` is used or the option `same_xy` is disabled. Other general plot options are also accepted.

The following example shows a region of the Julia set for the number -0.55 + 0.6. The option `color_bar_tics` is used to prevent Gnuplot from adjusting the color box up to 40, in which case the points corresponding the maximum 36 iterations would not be black.
(%i1) julia (-0.55, 0.6, [iterations, 36], [x, -0.3, 0.2],
               [y, 0.3, 0.9], [grid, 400, 400], [color_bar_tics, 0, 6, 36])$

make_transform ([var1, var2, var3], fx, fy, fz)

Returns a function suitable to be used in the option transform_xy of plot3d. The
three variables var1, var2, var3 are three dummy variable names, which represent
the 3 variables given by the plot3d command (first the two independent variables
and then the function that depends on those two variables). The three functions fx,
fy, fz must depend only on those 3 variables, and will give the corresponding x, y
and z coordinates that should be plotted. There are two transformations defined by
default: polar_to_xy and spherical_to_xyz. See the documentation for those two
transformations.

mandelbrot (options)

Creates a graphic representation of the Mandelbrot set. This program is part of the
additional package dynamics, but that package does not have to be loaded; the first
time mandelbrot is used, the package will be loaded automatically.

This program can be called without any arguments, in which case it will use a default
value of 9 iterations per point, a grid with dimensions set by the grid plot option
(default 30 by 30) and a region that extends from -2 to 2 in both axes. The options are
all the same that plot2d accepts, plus an option iterations to change the number
of iterations.

Each pixel in the grid is given a color corresponding to the number of iterations it
takes the sequence starting at zero to move out of the convergence circle of radius
2, centered at the origin. The maximum number of iterations is set by the option
iterations. The program uses its own default palette: magenta,violet, blue, cyan,
green, yellow, orange, red, brown and black, but it can be changed by adding an
explicit palette option in the command. By default, the two axes are shown with
the same scale, unless the option yx_ratio is used or the option same_xy is disabled.

Example:
[grid,400,400])$

(%i1) mandelbrot ([iterations, 30], [x, -2, 1], [y, -1.2, 1.2],
polar_to_xy

It can be given as value for the `transform_xy` option of plot3d. Its effect will be to interpret the two independent variables in plot3d as the distance from the z axis and the azimuthal angle (polar coordinates), and transform them into x and y coordinates.

plot2d

`plot2d(plot, x_range, ..., options, ...)`
`plot2d([plot_1, ..., plot_n], ..., options, ...)`
`plot2d([plot_1, ..., plot_n], x_range, ..., options, ...)`

Where `plot, plot_1, ..., plot_n` can be either expressions, function names or a list with the any of the forms: `[discrete, [x1, ..., xn], [y1, ..., yn]], [discrete, [[x1, y1], ..., [xn, ..., yn]]]` or `[parametric, x_expr, y_expr, t_range]`.

Displays a plot of one or more expressions as a function of one variable or parameter. `plot2d` displays one or several plots in two dimensions. When expressions or function name are used to define the plots, they should all depend on only one variable `var` and the use of `x_range` will be mandatory, to provide the name of the variable and its minimum and maximum values; the syntax for `x_range` is: `[variable, min, max].`

A plot can also be defined in the discrete or parametric forms. The discrete form is used to plot a set of points with given coordinates. A discrete plot is defined by a list starting with the keyword `discrete`, followed by one or two lists of values. If two lists are given, they must have the same length; the first list will be interpreted as the x coordinates of the points to be plotted and the second list as the y coordinates. If only one list is given after the `discrete` keyword, each element on the list could also be a list with two values that correspond to the x and y coordinates of a point, or it could be a sequence of numerical values which will be plotted at consecutive integer values (1,2,3,...) on the x axis.

A parametric plot is defined by a list starting with the keyword `parametric`, followed by two expressions or function names and a range for the parameter. The range for the parameter must be a list with the name of the parameter followed by its minimum and maximum values: `[param, min, max]`. The plot will show the path traced out
by the point with coordinates given by the two expressions or functions, as \textit{param}
increases from \textit{min} to \textit{max}.

A range for the vertical axis is an optional argument with the form: \([y, \textit{min}, \textit{max}]\)
(the keyword \textit{y} is always used for the vertical axis). If that option is used, the plot
will show that exact vertical range, independently of the values reached by the plot.
If the vertical range is not specified, it will be set up according to the minimum and
maximum values of the second coordinate of the plot points.

All other options should also be lists, starting with a keyword and followed by one or
more values. See \texttt{plot\_options}.

If there are several plots to be plotted, a legend will be written to identity each of
the expressions. The labels that should be used in that legend can be given with
the option \texttt{legend}. If that option is not used, Maxima will create labels from the
expressions or function names.

\textbf{Examples:}

Plot of a common function:

\begin{verbatim}
(\%i1) plot2d (sin(x), [x, -%pi, %pi])$
\end{verbatim}

If the function grows too fast, it might be necessary to limit the values in the vertical
axis using the \texttt{y} option:
When the plot box is disabled, no labels are created for the axes. In that case, instead of using xlabel and ylabel to set the names of the axes, it is better to use option label, which allows more flexibility. Option yx_ratio is used to change the default rectangular shape of the plot; in this example the plot will fill a square.

\[
\text{(i1) plot2d (x^2 - 1, [x, -3, 3], [box, false], grid2d, [yx_ratio, 1], [axes, solid], [xtics, -2, 4, 2], [ytics, 2, 2, 6], [label, \text{"x"}, 2.9, -0.3], ["x^2-1", 0.1, 8]], [title, \text{"A parabola"]})}
\]

A plot with a logarithmic scale in the vertical axis:

\[
\text{(i1) plot2d (sec(x), [x, -2, 2], [y, -20, 20])}
\]
(%i1) plot2d (exp(3*s), [s, -2, 2], logy)$

Plotting functions by name:

(%i1) F(x) := x^2 $
(%i2) :lisp (defun |$g| (x) (m* x x x))
$g$
(%i2) H(x) := if x < 0 then x^4 - 1 else 1 - x^5 $
(%i3) plot2d ([F, G, H], [u, -1, 1], [y, -1.5, 1.5])$

A plot of the butterfly curve, defined parametrically:

(%i1) r: (exp(cos(t))-2*cos(4*t)-sin(t/12)^5)$
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(%i2) plot2d([parametric, r*sin(t), r*cos(t), [t, -8*%pi, 8*%pi]])$

The circle will only look like a circle if the scale in the two axes is the same, which is done with the option `same_xy`.

(%i1) plot2d([[parametric, cos(t), sin(t), [t, 0, 2*%pi]], -abs(x)],
[x, -sqrt(2), sqrt(2)], same_xy)$

A plot of 200 random numbers between 0 and 9:
A plot of a discrete set of points, defining x and y coordinates separately:

\begin{verbatim}
(%i1) plot2d (discrete, makelist (random(10), 200));
\end{verbatim}

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{chart1.png}
\caption{A plot of a discrete set of points, defining x and y coordinates separately.}
\end{figure}

In the next example a table with three columns is saved in a file “data.txt” which is then read and the second and third column are plotted on the two axes:

\begin{verbatim}
(%i1) with_stdout ("data.txt", for x:0 thru 10 do
    print (x, x^2, x^3));
(%i2) data: read_matrix ("data.txt")
\end{verbatim}

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{chart2.png}
\caption{A plot of the second and third column of a table with three columns.}
\end{figure}
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(%i3) plot2d ([[discrete, transpose(data)[2], transpose(data)[3]],
            [style, points], [point_type, diamond], [color, red])$  

A plot of discrete data points together with a continuous function:

(%i1) xy: [[10, .6], [20, .9], [30, 1.1], [40, 1.3], [50, 1.4]]$
(%i2) plot2d([[discrete, xy], 2*%pi*sqrt(l/980)], [l,0,50],
            [style, points, lines], [color, red, blue],
            [point_type, asterisk],
            [legend, "experiment", "theory"],
            [xlabel, "pendulum's length (cm)"],
            [ylabel, "period (s)"])$

See also the section about Plotting Options.

plot3d

plot3d (expr, x_range, y_range, ..., options, ...)
plot3d ([expr_1, ..., expr_n], x_range, y_range, ..., options, ...)
Displays a plot of one or more surfaces defined as functions of two variables or in parametric form.
The functions to be plotted may be specified as expressions or function names. The mouse can be used to rotate the plot looking at the surface from different sides.

**Examples:**

Plot of a function of two variables:

```maxima
(%i1) plot3d (u^2 - v^2, [u, -2, 2], [v, -3, 3], [grid, 100, 100],
            [mesh_lines_color,false])$
```

Use of the \texttt{z} option to limit a function that goes to infinity (in this case the function is minus infinity on the x and y axes); this also shows how to plot with only lines and no shading:

```maxima
(%i1) plot3d ( log ( x^2*y^2 ), [x, -2, 2], [y, -2, 2], [z, -8, 4],
            [palette, false], [color, magenta])$
```

The infinite values of \texttt{z} can also be avoided by choosing a grid that does not fall on any points where the function is undefined, as in the next example, which also shows how to change the palette and how to include a color bar that relates colors to values of the \texttt{z} variable:
Two surfaces in the same plot. Ranges specific to one of the surfaces can be given by placing each expression and its ranges in a separate list; global ranges for the complete plot are also given after the functions definitions.

Plot of a Klein bottle, defined parametrically:

(%i1) expr_1: 5*cos(x)*(cos(x/2)*cos(y)+sin(x/2)*sin(2*y)+3)-10$
(%i2) expr_2: -5*sin(x)*(cos(x/2)*cos(y)+sin(x/2)*sin(2*y)+3)$
(%i3) expr_3: 5*(-sin(x/2)*cos(y)+cos(x/2)*sin(2*y))$
(%i4) plot3d ([expr_1, expr_2, expr_3], [x, -%pi, %pi],
[y, -%pi, %pi], [grid, 50, 50])$

Plot of a “spherical harmonic” function, using the predefined transformation, 
\texttt{spherical\_to\_xyz} to transform from spherical coordinates to rectangular 
coordinates. See the documentation for \texttt{spherical\_to\_xyz}.

(%i1) plot3d (\sin(2*\theta) * \cos(\phi), [\theta, 0, %pi],
[\phi, 0, 2*%pi],
[\text{transform}\_xy, \text{spherical}\_to\_xyz], [\text{grid}, 30, 60],
[\text{legend}, \text{false}])$

Use of the pre-defined function \texttt{polar\_to\_xy} to transform from cylindrical to rectangular 
coordinates. See the documentation for \texttt{polar\_to\_xy}.
(%i1) plot3d (r^.33*cos(th/3), [r,0,1], [th,0,6*%pi], [box, false],
[grid, 12, 80], [transform_xy, polar_to_xy], [legend, false])$

Plot of a sphere using the transformation from spherical to rectangular coordinates. Option same_xyz is used to get the three axes scaled in the same proportion. When transformations are used, it is not convenient to eliminate the mesh lines, because Gnuplot will not show the surface correctly.

(%i1) plot3d ( 5, [theta, 0, %pi], [phi, 0, 2*%pi], same_xyz,
[transform_xy, spherical_to_xyz], [mesh_lines_color,blue],
[palette,[gradient,"#1b1b4e", "#8c8cf8"]], [legend, false])$

Definition of a function of two-variables using a matrix. Notice the single quote in the definition of the function, to prevent plot3d from failing when it realizes that the matrix will require integer indices.

(%i1) M: matrix([1,2,3,4], [1,2,3,2], [1,2,3,4], [1,2,3,3])$
(%i2) f(x, y) := float('M [round(x), round(y)])$
(%i3) plot3d (f(x,y), [x,1,4],[y,1,4],[grid,3,3],[legend,false])$

By setting the elevation equal to zero, a surface can be seen as a map in which each color represents a different level.

(%i1) plot3d (cos (-x^2 + y^3/4), [x,-4,4], [y,-4,4], [zlabel,""], [mesh_lines_color,false], [elevation,0], [azimuth,0], color_bar, [grid,80,80], [ztics,false], [color_bar_tics,1])$

See also the section about Plotting Options.

plot_options [System variable]
This option is being kept for compatibility with older versions, but its use is deprecated. To set global plotting options, see their current values or remove options, use set_plot_option, get_plot_option and remove_plot_option.

remove_plot_option (name) [Function]
Removes the default value of an option. The name of the option must be given.

See also set_plot_option, get_plot_option and the section on Plotting Options.


**set_plot_option (option)**

Accepts any of the options listed in the section Plotting Options, and saves them for use in plotting commands. The values of the options set in each plotting command will have precedence, but if those options are not given, the default values set with this function will be used.

**set_plot_option** evaluates its argument and returns the complete list of options (after modifying the option given). If called without any arguments, it will simply show the list of current default options.

See also **remove_plot_option**, **get_plot_option** and the section on Plotting Options.

Example:

Modification of the **grid** values.

```plaintext
(%i1) set_plot_option ([grid, 30, 40]);
(%o1) [[plot_format, gnuplot_pipes], [grid, 30, 40], [run_viewer, true], [axes, true], [nticks, 29], [adapt_depth, 5], [color, blue, red, green, magenta, black, cyan], [point_type, bullet, box, triangle, plus, times, asterisk], [palette, [gradient, green, cyan, blue, violet], [gradient, magenta, violet, blue, cyan, green, yellow, orange, red, brown, black]], [gnuplot_preamble, ], [gnuplot_term, default]]
```

**spherical_to_xyz**

It can be given as value for the **transform_xy** option of **plot3d**. Its effect will be to interpret the two independent variables and the function in **plot3d** as the spherical coordinates of a point (first, the angle with the z axis, then the angle of the xy projection with the x axis and finally the distance from the origin) and transform them into x, y and z coordinates.

### 12.4 Plotting Options

All options consist of a list starting with one of the keywords in this section, followed by one or more values. Some of the options may have different effects in different plotting commands as it will be pointed out in the following list. The options that accept among their possible values true or false, can also be set to true by simply writing their names. For instance, typing logx as an option is equivalent to writing [logx, true].

**adapt_depth [adapt_depth, integer]**

Default value: 5

The maximum number of splittings used by the adaptive plotting routine.

**axes [axes, symbol]**

Default value: true

Where **symbol** can be either **true**, **false**, **x**, **y** or **solid**. If **false**, no axes are shown; if equal to **x** or **y** only the x or y axis will be shown; if it is equal to **true**, both axes will be shown and **solid** will show the two axes with a solid line, rather than the default broken line. This option does not have any effect in the 3 dimensional plots.
azimuth [azimuth, number] [Plot option]
Default value: 30
A plot3d plot can be thought of as starting with the x and y axis in the horizontal and vertical axis, as in plot2d, and the z axis coming out of the screen. The z axis is then rotated around the x axis through an angle equal to elevation and then the new xy plane is rotated around the new z axis through an angle azimuth. This option sets the value for the azimuth, in degrees.
See also elevation.

box [box, symbol] [Plot option]
Default value: true
If set to true, a bounding box will be drawn for the plot; if set to false, no box will be drawn.

color [color, color_1, ..., color_n] [Plot option]
In 2d plots it defines the color (or colors) for the various curves. In plot3d, it defines the colors used for the mesh lines of the surfaces, when no palette is being used.
If there are more curves or surfaces than colors, the colors will be repeated in sequence. The valid colors are red, green, blue, magenta, cyan, yellow, orange, violet, brown, gray, black, white, or a string starting with the character # and followed by six hexadecimal digits: two for the red component, two for green component and two for the blue component. If the name of a given color is unknown color, black will be used instead.

color_bar [color_bar, symbol] [Plot option]
Default value: false in plot3d, true in mandelbrot and julia
Where symbol can be either true or false. If true, whenever plot3d, mandelbrot or julia use a palette to represent different values, a box will be shown on the right, showing the corresponding between colors and values.

color_bar_tics [color_bar_tics, x1, x2, x3] [Plot option]
Defines the values at which a mark and a number will be placed in the color bar. The first number is the initial value, the second the increments and the third is the last value where a mark is placed. The second and third numbers can be omitted. When only one number is given, it will be used as the increment from an initial value that will be chosen automatically.

elevation [elevation, number] [Plot option]
Default value: 60
A plot3d plot can be thought of as starting with the x and y axis in the horizontal and vertical axis, as in plot2d, and the z axis coming out of the screen. The z axis is then rotated around the x axis through an angle equal to elevation and then the new xy plane is rotated around the new z axis through an angle azimuth. This option sets the value for the azimuth, in degrees.
See also azimuth.
grid [grid, integer, integer]  
Default value: 30, 30 
Sets the number of grid points to use in the x- and y-directions for three-dimensional plotting or for the julia and mandelbrot programs.

grid2d [grid, value]  
Default value: false  
Shows a grid of lines on the xy plane. The points where the grid lines are placed are the same points where tics are marked in the x and y axes, which can be controlled with the xtics and ytics options.

iterations [grid, value]  
Default value: 9  
Number of iterations made by the programs mandelbrot and julia.

label [label, [string, x, y], . . .]  
Wrote one or several labels in the points with x, y coordinates indicated after each label.

legend  

legend [legend, string_1, . . ., string_n]  
legend [legend, false]  
It specifies the labels for the plots when various plots are shown. If there are more plots than the number of labels given, they will be repeated. If given the value false, no legends will be shown. By default, the names of the expressions or functions will be used, or the words discrete1, discrete2, . . ., for discrete sets of points.

logx [logx, value]  
Makes the horizontal axes to be scaled logarithmically. It can be either true or false.

logy [logy, value]  
Makes the vertical axes to be scaled logarithmically. It can be either true or false.

mesh_lines_color [mesh_lines_color, color]  
Default value: black  
It sets the color used by plot3d to draw the mesh lines, when a palette is being used. It accepts the same colors as for the option color (see the list of allowed colors in color). It can also be given a value false to eliminate completely the mesh lines.

nticks [nticks, integer]  
Default value: 29  
When plotting functions with plot2d, it gives the initial number of points used by the adaptive plotting routine for plotting functions. When plotting parametric functions with plot3d, it sets the number of points that will be shown for the plot.

palette  

palette [palette, [palette_1], . . ., [palette_n]]  
palette [palette, false]  
It can consist of one palette or a list of several palettes. Each palette is a list with a keyword followed by values. If the keyword is gradient, it should be followed by a list of valid colors.
If the keyword is hue, saturation or value, it must be followed by 4 numbers. The first three numbers, which must be between 0 and 1, define the hue, saturation and value of a basic color to be assigned to the minimum value of z. The keyword specifies which of the three attributes (hue, saturation or value) will be increased according to the values of z. The last number indicates the increase corresponding to the maximum value of z. That last number can be bigger than 1 or negative; the corresponding values of the modified attribute will be rounded modulo 1.

Gnuplot only uses the first palette in the list; xmaxima will use the palettes in the list sequentially, when several surfaces are plotted together; if the number of palettes is exhausted, they will be repeated sequentially.

The color of the mesh lines will be given by the option `mesh_lines_color`. If `palette` is given the value `false`, the surfaces will not be shaded but represented with a mesh of curves only. In that case, the colors of the lines will be determined by the option `color`.

`plot_format [plot_format, format]`  
[Plot option]  
Default value: `gnuplot`, in Windows systems, or `gnuplot_pipes` in other systems.

Where `format` is one of the following: `gnuplot`, `xmaxima`, `mgnuplot`, `gnuplot_pipes` or `geomview`.

It sets the format to be used for plotting.

`plot_realpart [plot_realpart, symbol]`  
[Plot option]  
Default value: `false`

If set to `true`, the functions to be plotted will be considered as complex functions whose real value should be plotted; this is equivalent to plotting `realpart(function)`. If set to `false`, nothing will be plotted when the function does not give a real value. For instance, when `x` is negative, `log(x)` gives a complex value, with real value equal to `log(abs(x))`; if `plot_realpart` were `true`, `log(-5)` would be plotted as `log(5)`, while nothing would be plotted if `plot_realpart` were `false`.

`point_type [point_type, type_1, ..., type_n]`  
[Plot option]

In gnuplot, each set of points to be plotted with the style “points” or “linespoints” will be represented with objects taken from this list, in sequential order. If there are more sets of points than objects in this list, they will be repeated sequentially. The possible objects that can be used are: `bullet`, `circle`, `plus`, `times`, `asterisk`, `box`, `square`, `triangle`, `delta`, `wedge`, `nabla`, `diamond`, `lozenge`.

`pdf_file [pdf_file, file_name]`  
[Plot option]

Saves the plot into a PDF file with name equal to `file_name`, rather than showing it in the screen. By default, the file will be created in the directory defined by the variable `maxima_tempdir`, unless `file_name` contains the character “/”, in which case it will be assumed to contain the complete path where the file should be created. The value of `maxima_tempdir` can be changed to save the file in a different directory. When the option `gnuplot_pdf_term_command` is also given, it will be used to set up Gnuplot’s PDF terminal; otherwise, Gnuplot’s pdfcairo terminal will be used with solid colored lines of width 3, plot size of 17.2 cm by 12.9 cm and font of 18 points.
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**png_file [png_file, file_name]**  
[Plot option]  
Saves the plot into a PNG graphics file with name equal to file_name, rather than showing it in the screen. By default, the file will be created in the directory defined by the variable maxima_tempdir, unless file_name contains the character “/”, in which case it will be assumed to contain the complete path where the file should be created. The value of maxima_tempdir can be changed to save the file in a different directory. When the option gnuplot_png_term_command is also given, it will be used to set up Gnuplot’s PNG terminal; otherwise, Gnuplot’s pngcairo terminal will be used, with a font of size 12.

**ps_file [ps_file, file_name]**  
[Plot option]  
Saves the plot into a Postscript file with name equal to file_name, rather than showing it in the screen. By default, the file will be created in the directory defined by the variable maxima_tempdir, unless file_name contains the character “/”, in which case it will be assumed to contain the complete path where the file should be created. The value of maxima_tempdir can be changed to save the file in a different directory. When the option gnuplot_ps_term_command is also given, it will be used to set up Gnuplot’s Postscript terminal; otherwise, Gnuplot’s postscript terminal will be used with the EPS option, solid colored lines of width 2, plot size of 16.4 cm by 12.3 cm and font of 24 points.

**run_viewer [run_viewer, symbol]**  
[Plot option]  
This option is only used when the plot format is gnuplot and the terminal is default or when the Gnuplot terminal is set to dumb (see gnuplot_term) and can have a true or false value.

If the terminal is default, a file maxout_xxx.gnuplot (or other name specified with gnuplot_out_file) is created with the gnuplot commands necessary to generate the plot. Option run_viewer controls whether or not Gnuplot will be launched to execute those commands and show the plot.

If the terminal is default, gnuplot is run to execute the commands in maxout_xxx.gnuplot, producing another file maxplot.txt (or other name specified with gnuplot_out_file). Option run_viewer controls whether or not that file, with an ASCII representation of the plot, will be shown in the Maxima or Xmaxima console.

The default value for this option is true, making the plots to be shown in either the console or a separate graphics window.

**same_xy [same_xy, value]**  
[Plot option]  
It can be either true or false. If true, the scales used in the x and y axes will be the same, in either 2d or 3d plots. See also yx_ratio.

**same_xyz [same_xyz, value]**  
[Plot option]  
It can be either true or false. If true, the scales used in the 3 axes of a 3d plot will be the same.

**style**  
[Plot option]  

```
style [style, type_1, ..., type_n]
style [style, [style_1], ..., [style_n]]
```

The styles that will be used for the various functions or sets of data in a 2d plot. The word style must be followed by one or more styles. If there are more functions and
data sets than the styles given, the styles will be repeated. Each style can be either
lines for line segments, points for isolated points, linespoints for segments and points,
or dots for small isolated dots. Gnuplot accepts also an impulses style.

Each of the styles can be enclosed inside a list with some additional parameters. lines
accepts one or two numbers: the width of the line and an integer that identifies a
color. The default color codes are: 1: blue, 2: red, 3: magenta, 4: orange, 5: brown,
6: lime and 7: aqua. If you use Gnuplot with a terminal different than X11, those
colors might be different; for example, if you use the option [gnuplot_term, ps], color
index 4 will correspond to black, instead of orange.

points accepts one or two parameters; the first parameter is the radius of the
points, the second parameter is an integer that selects the color, using the same
code used for lines and the third parameter is currently used only by Gnuplot and
it corresponds to several objects instead of points. The default types of objects are:
squares, 8: filled triangles, 9: open triangles, 10: filled inverted triangles, 11: open
inverted triangles, 12: filled lozenges and 13: open lozenges.

linespoints accepts up to four parameters: line width, points radius, color and type
of object to replace the points.

See also color and point_type.

svg_file [svg_file, file_name] [Plot option]
Saves the plot into an SVG file with name equal to file_name, rather than showing
it in the screen. By default, the file will be created in the directory defined by the
variable maxima_tempdir, unless file_name contains the character “/”, in which case
it will be assumed to contain the complete path where the file should be created.
The value of maxima_tempdir can be changed to save the file in a different directory.
When the option gnuplot_svg_term_command is also given, it will be used to set up
Gnuplot’s SVG terminal; otherwise, Gnuplot’s svg terminal will be used with font of
14 points.

t [t, min, max] [Plot option]
Default range for parametric plots.

title [title, text] [Plot option]
Defines a title that will be written at the top of the plot.

transform_xy [transform_xy, symbol] [Plot option]
Where symbol is either false or the result obtained by using the function transform_xy.
If different from false, it will be used to transform the 3 coordinates in plot3d.
See make_transform, polar_to_xy and spherical_to_xyz.

x [x, min, max] [Plot option]
When used as the first option in a plot2d command (or any of the first two in plot3d),
it indicates that the first independent variable is x and it sets its range. It can also
be used again after the first option (or after the second option in plot3d) to define
the effective horizontal domain that will be shown in the plot.
xlabel \[xlabel, \text{string}\] \hspace{1em} \text{[Plot option]}\n
Specifies the string that will label the first axis; if this option is not used, that label will be the name of the independent variable, when plotting functions with \texttt{plot2d} or \texttt{implicit_plot}, or the name of the first variable, when plotting surfaces with \texttt{plot3d} or contours with \texttt{contour_plot}, or the first expression in the case of a parametric plot. It can not be used with \texttt{set_plot_option}.

xtics \[xtics, x1, x2, x3\] \hspace{1em} \text{[Plot option]}\n
Defines the values at which a mark and a number will be placed in the x axis. The first number is the initial value, the second the increments and the third is the last value where a mark is placed. The second and third numbers can be omitted. When only one number is given, it will be used as the increment from an initial value that will be chosen automatically.

xy_scale \[xy\_scale, sx, sy\] \hspace{1em} \text{[Plot option]}\n
In a 2d plot, it defines the ratio of the total size of the Window to the size that will be used for the plot. The two numbers given as arguments are the scale factors for the x and y axes.

y \[y, \text{min, max}\] \hspace{1em} \text{[Plot option]}\n
When used as one of the first two options in \texttt{plot3d}, it indicates that one of the independent variables is \texttt{y} and it sets its range. Otherwise, it defines the effective domain of the second variable that will be shown in the plot.

ylabel \[ylabel, \text{string}\] \hspace{1em} \text{[Plot option]}\n
Specifies the string that will label the second axis; if this option is not used, that label will be “y”, when plotting functions with \texttt{plot2d} or \texttt{implicit_plot}, or the name of the second variable, when plotting surfaces with \texttt{plot3d} or contours with \texttt{contour_plot}, or the second expression in the case of a parametric plot. It can not be used with \texttt{set_plot_option}.

ytic \[ytic, y1, y2, y3\] \hspace{1em} \text{[Plot option]}\n
Defines the values at which a mark and a number will be placed in the y axis. The first number is the initial value, the second the increments and the third is the last value where a mark is placed. The second and third numbers can be omitted. When only one number is given, it will be used as the increment from an initial value that will be chosen automatically.

yx_ratio \[yx\_ratio, r\] \hspace{1em} \text{[Plot option]}\n
In a 2d plot, the ratio between the vertical and the horizontal sides of the rectangle used to make the plot. See also \texttt{same_xy}.

z \[z, \text{min, max}\] \hspace{1em} \text{[Plot option]}\n
Used in \texttt{plot3d} to set the effective range of values of \texttt{z} that will be shown in the plot.

zlabel \[zlabel, \text{string}\] \hspace{1em} \text{[Plot option]}\n
Specifies the string that will label the third axis, when using \texttt{plot3d}. If this option is not used, that label will be “z”, when plotting surfaces, or the third expression in the case of a parametric plot. It can not be used with \texttt{set_plot_option} and it will be ignored by \texttt{plot2d} and \texttt{implicit_plot}.
**zmin** \([z\text{min}, z]\)  
In 3d plots, the value of \(z\) that will be at the bottom of the plot box.

### 12.5 Gnuplot Options

There are several plot options specific to gnuplot. All of them consist of a keyword (the name of the option), followed by a string that should be a valid gnuplot command, to be passed directly to gnuplot. In most cases, there exist a corresponding plotting option that will produce a similar result and whose use is more recommended than the gnuplot specific option.

**\texttt{gnuplot_term}** \([\texttt{gnuplot_term, terminal\_name}]\)  
[Plot option]  
Sets the output terminal type for gnuplot. The argument \(\text{terminal\_name}\) can be a string or one of the following 3 special symbols

- **default** (default value)  
  Gnuplot output is displayed in a separate graphical window and the gnuplot terminal used will be specified by the value of the option \texttt{gnuplot\_default\_term\_command}.

- **dumb**  
  Gnuplot output is saved to a file \texttt{maxout\_xxx.gnuplot} using "ASCII art" approximation to graphics. If the option \texttt{gnuplot\_out\_file} is set to \texttt{filename}, the plot will be saved there, instead of the default \texttt{maxout\_xxx.gnuplot}. The settings for the "dumb" terminal of Gnuplot are given by the value of option \texttt{gnuplot\_dumb\_term\_command}. If option \texttt{run\_viewer} is set to true and the plot format is gnuplot that ASCII representation will also be shown in the Maxima or Xmaxima console.

- **ps**  
  Gnuplot generates commands in the PostScript page description language. If the option \texttt{gnuplot\_out\_file} is set to \texttt{filename}, gnuplot writes the PostScript commands to \texttt{filename}. Otherwise, it is saved as \texttt{maxplot.ps} file. The settings for this terminal are given by the value of the option \texttt{gnuplot\_dumb\_term\_command}.

- A string representing any valid gnuplot term specification  
  Gnuplot can generate output in many other graphical formats such as png, jpeg, svg etc. To use those formats, option \texttt{gnuplot\_term} can be set to any supported gnuplot term name (which must be a symbol) or even a full gnuplot term specification with any valid options (which must be a string). For example \texttt{[gnuplot\_term, png]} creates output in PNG (Portable Network Graphics) format while \texttt{[gnuplot\_term, "png size 1000,1000"]} creates PNG of 1000 x 1000 pixels size. If the option \texttt{gnuplot\_out\_file} is set to \texttt{filename}, gnuplot writes the output to \texttt{filename}. Otherwise, it is saved as \texttt{maxplot.\_term} file, where \texttt{term} is gnuplot terminal name.

**\texttt{gnuplot\_out\_file}** \([\texttt{gnuplot\_out\_file, file\_name}]\)  
[Plot option]  
It can be used to replace the default name for the file that contains the commands that will interpreted by gnuplot, when the terminal is set to \texttt{default}, or to replace the default name of the graphic file that gnuplot creates, when the terminal is different
from default. If it contains one or more slashes, "/", the name of the file will be left
as it is; otherwise, it will be appended to the path of the temporary directory. The
complete name of the files created by the plotting commands is always sent as output
of those commands so they can be seen if the command is ended by semi-colon.
When used in conjunction with the gnuplot_term option, it can be used to save the
plot in a file, in one of the graphic formats supported by Gnuplot. To create PNG,
PDF, Postscript or SVG, it is easier to use options png_file, pdf_file, ps_file, or
svg_file.

gnuplot_pm3d [gnuplot_pm3d, value]  [Plot option]
With a value of false, it can be used to disable the use of PM3D mode, which is
enabled by default.

gnuplot_preamble [gnuplot_preamble, string]  [Plot option]
This option inserts gnuplot commands before any other commands sent to Gnuplot.
Any valid gnuplot commands may be used. Multiple commands should be separated
with a semi-colon. See also gnuplot_postamble.

gnuplot_postamble [gnuplot_postamble, string]  [Plot option]
This option inserts gnuplot commands after other commands sent to Gnuplot
and right before the plot command is sent. Any valid gnuplot commands may
be used. Multiple commands should be separated with a semi-colon. See also
gnuplot_preamble.

gnuplot_default_term_command
[gnuplot_default_term_command, command]  [Plot option]
The gnuplot command to set the terminal type for the default terminal. If this option
is not set, the command used will be: "set term wxt size 640,480 font ",12";
set term pop".

gnuplot_dumb_term_command
[gnuplot_dumb_term_command, command]  [Plot option]
The gnuplot command to set the terminal type for the dumb terminal. If this option
is not set, the command used will be: "set term dumb 79 22", which makes the text
output 79 characters by 22 characters.

gnuplot_pdf_term_command [gnuplot_pdf_term_command, command]  [Plot option]
The gnuplot command to set the terminal type for the PDF terminal. If this option
is not set, the command used will be: "set term pdfcairo color solid lw 3 size
17.2 cm, 12.9 cm font ",18". See the gnuplot documentation for more information.

gnuplot_png_term_command [gnuplot_png_term_command, command]  [Plot option]
The gnuplot command to set the terminal type for the PNG terminal. If this option
is not set, the command used will be: "set term pngcairo font ",12". See the
gnuplot documentation for more information.
The gnuplot command to set the terminal type for the PostScript terminal. If this option is not set, the command used will be: "set term postscript eps color solid lw 2 size 16.4 cm, 12.3 cm font ",24". See the gnuplot documentation for set term postscript for more information.

The gnuplot command to set the terminal type for the SVG terminal. If this option is not set, the command used will be: "set term svg font ",14". See the gnuplot documentation for more information.

This is an obsolete option that has been replaced legend described above.

This is an obsolete option that has been replaced by style.

12.6 Gnuplot_pipes Format Functions

Opens the pipe to gnuplot used for plotting with the gnuplot_pipes format. Is not necessary to manually open the pipe before plotting.

Closes the pipe to gnuplot which is used with the gnuplot_pipes format.

Closes the pipe to gnuplot which is used with the gnuplot_pipes format and opens a new pipe.

Updates the gnuplot window. If gnuplot_replot is called with a gnuplot command in a string s, then s is sent to gnuplot before reploting the window.

Resets the state of gnuplot used with the gnuplot_pipes format. To update the gnuplot window call gnuplot_replot after gnuplot_reset.
13 File Input and Output

13.1 Comments

A comment in Maxima input is any text between /* and */.

The Maxima parser treats a comment as whitespace for the purpose of finding tokens in the input stream; a token always ends at a comment. An input such as a/*foo*/b contains two tokens, a and b, and not a single token ab. Comments are otherwise ignored by Maxima; neither the content nor the location of comments is stored in parsed input expressions.

Comments can be nested to arbitrary depth. The /* and */ delimiters form matching pairs. There must be the same number of /* as there are */.

Examples:

```lisp
(%i1) /* aa is a variable of interest */ aa : 1234;
(%o1) 1234
(%i2) /* Value of bb depends on aa */ bb : aa^2;
(%o2) 1522756
(%i3) /* User-defined infix operator */ infix ("b");
(%o3) b
(%i4) /* Parses same as a b c, not abc */ a/* foo */b/* bar */c;
(%o4) a b c
(%i5) /* Comments */ can be nested /* to any depth */ */ */ */ 1 + xyz;
(%o5) xyz + 1
```

13.2 Files

A file is simply an area on a particular storage device which contains data or text. Files on the disks are figuratively grouped into "directories". A directory is just a list of files. Commands which deal with files are:

```lisp
appendfile    batch    batchload
closefile     file_output_append filename_merge
file_search   file_search_maxima file_search_lisp
file_search_demo file_search_usage file_search_tests
file_type     file_type_lisp file_type_maxima
load          load_pathname loadfile
loadprint     pathname_directory pathname_name
pathname_type printfile save
stringout    with_stdout writefile
```

When a file name is passed to functions like plot2d, save, or writefile and the file name does not include a path, Maxima stores the file in the current working directory. The current working directory depends on the system like Windows or Linux and on the installation.
13.3 Functions and Variables for File Input and Output

appendfile (filename)  [Function]
Appends a console transcript to filename. appendfile is the same as writefile, except that the transcript file, if it exists, is always appended.
closefile closes the transcript file opened by appendfile or writefile.

batch  [Function]
batch (filename)
batch (filename, option)
batch(filename) reads Maxima expressions from filename and evaluates them. batch searches for filename in the list file_search_maxima. See also file_search.
batch(filename, demo) is like demo(filename). In this case batch searches for filename in the list file_search_demo. See demo.
batch(filename, test) is like run_testsuite with the option display_all=true. For this case batch searches filename in the list file_search_maxima and not in the list file_search_tests like run_testsuite. Furthermore, run_testsuite runs tests which are in the list testsuite_files. With batch it is possible to run any file in a test mode, which can be found in the list file_search_maxima. This is useful, when writing a test file.
filename comprises a sequence of Maxima expressions, each terminated with ; or $.
The special variable % and the function %th refer to previous results within the file.
The file may include :lisp constructs. Spaces, tabs, and newlines in the file are ignored. A suitable input file may be created by a text editor or by the stringout function.
batch reads each input expression from filename, displays the input to the console, computes the corresponding output expression, and displays the output expression. Input labels are assigned to the input expressions and output labels are assigned to the output expressions. batch evaluates every input expression in the file unless there is an error. If user input is requested (by asksign or askinteger, for example) batch pauses to collect the requisite input and then continue. It may be possible to halt batch by typing control-C at the console. The effect of control-C depends on the underlying Lisp implementation.
batch has several uses, such as to provide a reservoir for working command lines, to give error-free demonstrations, or to help organize one’s thinking in solving complex problems.
batch evaluates its argument. batch returns the path of filename as a string, when called with no second argument or with the option demo. When called with the option test, the return value is a an empty list [] or a list with filename and the numbers of the tests which have failed. See also load, batchload, and demo.

batchload (filename)  [Function]
Reads Maxima expressions from filename and evaluates them, without displaying the input or output expressions and without assigning labels to output expressions. Printed output (such as produced by print or describe)) is displayed, however.
Chapter 13: File Input and Output

The special variable \% and the function \%th refer to previous results from the interactive interpreter, not results within the file. The file cannot include :lisp constructs.

batchload returns the path of filename, as a string. batchload evaluates its argument.

See also batch, and load.

closefile ()

[Function]
Closes the transcript file opened by writefile or appendfile.

file_output_append

[Option variable]
Default value: false

file_output_append governs whether file output functions append or truncate their output file. When file_output_append is true, such functions append to their output file. Otherwise, the output file is truncated.

save, stringout, and with_stdout respect file_output_append. Other functions which write output files do not respect file_output_append. In particular, plotting and translation functions always truncate their output file, and tex and appendfile always append.

filename_merge (path, filename)

[Function]
Constructs a modified path from path and filename. If the final component of path is of the form ###.something, the component is replaced with filename.something. Otherwise, the final component is simply replaced by filename.

The result is a Lisp pathname object.

file_search

[Function]
file_search (filename)
file_search (filename, pathlist)

file_search searches for the file filename and returns the path to the file (as a string) if it can be found; otherwise file_search returns false. file_search (filename) searches in the default search directories, which are specified by the file_search_maxima, file_search_lisp, and file_search_demo variables.

file_search first checks if the actual name passed exists, before attempting to match it to “wildcard” file search patterns. See file_search_maxima concerning file search patterns.

The argument filename can be a path and file name, or just a file name, or, if a file search directory includes a file search pattern, just the base of the file name (without an extension). For example,

file_search ("/home/wfs/special/zeta.mac");
file_search ("zeta.mac");
file_search ("zeta");

all find the same file, assuming the file exists and /home/wfs/special/###.mac is in file_search_maxima.

file_search (filename, pathlist) searches only in the directories specified by pathlist, which is a list of strings. The argument pathlist supersedes the default search directories, so if the path list is given, file_search searches only the ones
specified, and not any of the default search directories. Even if there is only one
directory in pathlist, it must still be given as a one-element list.
The user may modify the default search directories. See file_search_maxima.
file_search is invoked by load with file_search_maxima and file_search_lisp
as the search directories.

file_search_maxima [Option variable]
file_search_lisp [Option variable]
file_search_demo [Option variable]
file_search_usage [Option variable]
file_search_tests [Option variable]

These variables specify lists of directories to be searched by load, demo, and some
other Maxima functions. The default values of these variables name various directories
in the Maxima installation.
The user can modify these variables, either to replace the default values or to append
additional directories. For example,

```
file_search_maxima: ["/usr/local/foo/###.mac",
                   "/usr/local/bar/###.mac"]$
```

replaces the default value of file_search_maxima, while

```
file_search_maxima: append (file_search_maxima,
                   ["/usr/local/foo/###.mac", "/usr/local/bar/###.mac"])$
```

appends two additional directories. It may be convenient to put such an expression
in the file maxima-init.mac so that the file search path is assigned automatically
when Maxima starts. See also Section 32.1 [Introduction for Runtime Environment],
page 537.

Multiple filename extensions and multiple paths can be specified by special “wildcard”
constructions. The string ### expands into the sought-after name, while a comma-
separated list enclosed in curly braces {foo,bar,baz} expands into multiple strings.
For example, supposing the sought-after name is neumann,

```
"/home/{wfs,gcj}/###.{lisp,mac}"
```

expands into /home/wfs/neumann.lisp, /home/gcj/neumann.lisp, /home/wfs/neumann.mac, and /home/gcj/neumann.mac.

file_type (filename) [Function]

Returns a guess about the content of filename, based on the filename extension.
filename need not refer to an actual file; no attempt is made to open the file and
inspect the content.
The return value is a symbol, either object, lisp, or maxima. If the extension is
matches one of the values in file_type_maxima, file_type returns maxima. If the
extension matches one of the values in file_type_lisp, file_type returns lisp. If
none of the above, file_type returns object.
See also pathname_type.
See file_type_maxima and file_type_lisp for the default values.
Examples:

```
(%i2) map('file_type,
```
["test.lisp", "test.mac", "test.dem", "test.txt"]

(%o2) [lisp, maxima, maxima, object]

file_type_lisp
[Option variable]
Default value: [l, lsp, lisp]

file_type_lisp is a list of file extensions that maxima recognizes as denoting a Lisp source file.
See also file_type.

file_type_maxima
[Option variable]
Default value: [mac, mc, demo, dem, dm1, dm2, dm3, dmt, wxm]

file_type_maxima is a list of file extensions that maxima recognizes as denoting a Maxima source file.
See also file_type.

load (filename)
[Function]
Evaluates expressions in filename, thus bringing variables, functions, and other objects into Maxima. The binding of any existing object is clobbered by the binding recovered from filename. To find the file, load calls file_search with file_search_maxima and file_search_lisp as the search directories. If load succeeds, it returns the name of the file. Otherwise load prints an error message.
load works equally well for Lisp code and Maxima code. Files created by save, translate_file, and compile_file, which create Lisp code, and stringout, which creates Maxima code, can all be processed by load. load calls loadfile to load Lisp files and batchload to load Maxima files.
load does not recognize :lisp constructs in Maxima files, and while processing filename, the global variables _, __, %, and %th have whatever bindings they had when load was called.
It is also to note that structures will only be read back as structures if they have been defined by defstruct before the load command is called.
See also loadfile, batch, batchload, and demo. loadfile processes Lisp files; batch, batchload, and demo process Maxima files.
See file_search for more detail about the file search mechanism.
load evaluates its argument.

load_pathname
[System variable]
Default value: false

When a file is loaded with the functions load, loadfile or batchload the system variable load_pathname is bound to the pathname of the file which is processed.
The variable load_pathname can be accessed from the file during the loading.
Example:
Suppose we have a batchfile test.mac in the directory
"/home/dieter/workspace/mymaxima/temp/" with the following commands
   print("The value of load_pathname is: ", load_pathname)$
   print("End of batchfile")$
then we get the following output

```lisp
(%i1) load("/home/dieter/workspace/mymaxima/temp/test.mac")$
The value of load_pathname is:

``/home/dieter/workspace/mymaxima/temp/test.mac``

End of batchfile

**loadfile** (*filename*)  
[Function]  
Evaluates Lisp expressions in *filename*. **loadfile** does not invoke **file_search**, so *filename* must include the file extension and as much of the path as needed to find the file.  
*loadfile* can process files created by **save**, **translate_file**, and **compile_file**. The user may find it more convenient to use **load** instead of **loadfile**.

**loadprint**  
[Option variable]  
Default value: **true**  
**loadprint** tells whether to print a message when a file is loaded.  
- When **loadprint** is **true**, always print a message.  
- When **loadprint** is `'loadfile`, print a message only if a file is loaded by the function **loadfile**.  
- When **loadprint** is `'autoload`, print a message only if a file is automatically loaded. See **setup_autoload**.  
- When **loadprint** is **false**, never print a message.

**directory** (*path*)  
[Function]  
Returns a list of the files and directories found in *path* in the file system.  
*path* may contain wildcard characters (i.e., characters which represent unspecified parts of the path), which include at least the asterisk on most systems, and possibly other characters, depending on the system.  
**directory** relies on the Lisp function **DIRECTORY**, which may have implementation-specific behavior.

**pathname_directory** (*pathname*)  
[Function]  
**pathname_name** (*pathname*)  
[Function]  
**pathname_type** (*pathname*)  
[Function]  
These functions return the components of *pathname*.  
Examples:

```lisp
(%i1) pathname_directory("/home/dieter/maxima/changelog.txt");
(%o1) /home/dieter/maxima/
(%i2) pathname_name("/home/dieter/maxima/changelog.txt");
(%o2) changelog
(%i3) pathname_type("/home/dieter/maxima/changelog.txt");
(%o3) txt
```

**printfile** (*path*)  
[Function]  
Prints the file named by *path* to the console. *path* may be a string or a symbol; if it is a symbol, it is converted to a string.
If \( path \) names a file which is accessible from the current working directory, that file is printed to the console. Otherwise, \texttt{printfile} attempts to locate the file by appending \( path \) to each of the elements of \texttt{file_search_usage} via \texttt{filename_merge}.

\texttt{printfile} returns \( path \) if it names an existing file, or otherwise the result of a successful \texttt{filename_merge}.

\texttt{save} \hspace{1cm} [Function]

\begin{verbatim}
  save (filename, name_1, name_2, name_3, ...)
  save (filename, values, functions, labels, ...)
  save (filename, [m, n])
  save (filename, name_1=expr_1, ...)
  save (filename, all)
  save (filename, name_1=expr_1, name_2=expr_2, ...)
\end{verbatim}

Stores the current values of \( name_1, name_2, name_3, \ldots \), in \texttt{filename}. The arguments are the names of variables, functions, or other objects. If a name has no value or function associated with it, it is ignored. \texttt{save} returns \texttt{filename}.

\texttt{save} stores data in the form of Lisp expressions. If \texttt{filename} ends in \texttt{.lisp} the data stored by \texttt{save} may be recovered by \texttt{load (filename)}. See \texttt{load}.

The global flag \texttt{file_output_append} governs whether \texttt{save} appends or truncates the output file. When \texttt{file_output_append} is \texttt{true}, \texttt{save} appends to the output file. Otherwise, \texttt{save} truncates the output file. In either case, \texttt{save} creates the file if it does not yet exist.

The special form \texttt{save (filename, values, functions, labels, ...)} stores the items named by \texttt{values}, \texttt{functions}, \texttt{labels}, etc. The names may be any specified by the variable \texttt{infolists}. \texttt{values} comprises all user-defined variables.

The special form \texttt{save (filename, [m, n])} stores the values of input and output labels \( m \) through \( n \). Note that \( m \) and \( n \) must be literal integers. Input and output labels may also be stored one by one, e.g., \texttt{save ("foo.1", \%i42, \%o42)}. \texttt{save (filename, labels)} stores all input and output labels. When the stored labels are recovered, they clobber existing labels.

The special form \texttt{save (filename, name_1=expr_1, name_2=expr_2, ...)} stores the values of \( expr_1, expr_2, \ldots \), with names \( name_1, name_2, \ldots \). It is useful to apply this form to input and output labels, e.g., \texttt{save ("foo.1", aa=\%o88)}. The right-hand side of the equality in this form may be any expression, which is evaluated. This form does not introduce the new names into the current Maxima environment, but only stores them in \texttt{filename}.

These special forms and the general form of \texttt{save} may be mixed at will. For example, \texttt{save (filename, aa, bb, cc=42, functions, [11, 17])}.

The special form \texttt{save (filename, all)} stores the current state of Maxima. This includes all user-defined variables, functions, arrays, etc., as well as some automatically defined items. The saved items include system variables, such as \texttt{file_search_maxima} or \texttt{showtime}, if they have been assigned new values by the user; see \texttt{myoptions}.

\texttt{save} evaluates \texttt{filename} and quotes all other arguments.
stringout

- stringout (filename, expr_1, expr_2, expr_3, ...)
- stringout (filename, [m, n])
- stringout (filename, input)
- stringout (filename, functions)
- stringout (filename, values)

stringout writes expressions to a file in the same form the expressions would be typed for input. The file can then be used as input for the `batch` or `demo` commands, and it may be edited for any purpose. stringout can be executed while `writefile` is in progress.

The global flag `file_output_append` governs whether stringout appends or truncates the output file. When `file_output_append` is true, stringout appends to the output file. Otherwise, stringout truncates the output file. In either case, stringout creates the file if it does not yet exist.

The general form of stringout writes the values of one or more expressions to the output file. Note that if an expression is a variable, only the value of the variable is written and not the name of the variable. As a useful special case, the expressions may be input labels (%i1, %i2, %i3, ...) or output labels (%o1, %o2, %o3, ...).

If `grind` is true, stringout formats the output using the `grind` format. Otherwise the `string` format is used. See `grind` and `string`.

The special form stringout (filename, [m, n]) writes the values of input labels m through n, inclusive.

The special form stringout (filename, input) writes all input labels to the file.

The special form stringout (filename, functions) writes all user-defined functions (named by the global list `functions`) to the file.

The special form stringout (filename, values) writes all user-assigned variables (named by the global list `values`) to the file. Each variable is printed as an assignment statement, with the name of the variable, a colon, and its value. Note that the general form of stringout does not print variables as assignment statements.

with_stdout

- with_stdout (f, expr_1, expr_2, expr_3, ...)
- with_stdout (s, expr_1, expr_2, expr_3, ...)

Evaluates expr_1, expr_2, expr_3, ... and writes any output thus generated to a file f or output stream s. The evaluated expressions are not written to the output. Output may be generated by `print`, `display`, `grind`, among other functions.

The global flag `file_output_append` governs whether with_stdout appends or truncates the output file f. When `file_output_append` is true, with_stdout appends to the output file. Otherwise, with_stdout truncates the output file. In either case, with_stdout creates the file if it does not yet exist.

with_stdout returns the value of its final argument.

See also `writefile`.

```
(%i1) with_stdout ("tmp.out", for i:5 thru 10 do
   print (i, "! yields", i!))$
(%i2) printfile ("tmp.out")$
```
5 ! yields 120  
6 ! yields 720  
7 ! yields 5040  
8 ! yields 40320  
9 ! yields 362880  
10 ! yields 3628800

**writefile (filename)**  
Begins writing a transcript of the Maxima session to `filename`. All interaction between the user and Maxima is then recorded in this file, just as it appears on the console.

As the transcript is printed in the console output format, it cannot be reloaded into Maxima. To make a file containing expressions which can be reloaded, see `save` and `stringout`. `save` stores expressions in Lisp form, while `stringout` stores expressions in Maxima form.

The effect of executing `writefile` when `filename` already exists depends on the underlying Lisp implementation; the transcript file may be clobbered, or the file may be appended. `appendfile` always appends to the transcript file.

It may be convenient to execute `playback` after `writefile` to save the display of previous interactions. As `playback` displays only the input and output variables (%i1, %o1, etc.), any output generated by a print statement in a function (as opposed to a return value) is not displayed by `playback`.

`closefile` closes the transcript file opened by `writefile` or `appendfile`.

### 13.4 Functions and Variables for TeX Output

Note that the built-in TeX output functionality of wxMaxima makes no use of the functions described here but uses its own implementation instead.

**tex**  

\[
\begin{align*}
tex (expr) \\
tex (expr, destination) \\
tex (expr, false) \\
tex (label) \\
tex (label, destination) \\
tex (label, false)
\end{align*}
\]

Prints a representation of an expression suitable for the TeX document preparation system. The result is a fragment of a document, which can be copied into a larger document but not processed by itself.

`tex (expr)` prints a TeX representation of `expr` on the console.

`tex (label)` prints a TeX representation of the expression named by `label` and assigns it an equation label (to be displayed to the left of the expression). The TeX equation label is the same as the Maxima label.

`destination` may be an output stream or file name. When `destination` is a file name, `tex` appends its output to the file. The functions `openw` and `opena` create output streams.

`tex (expr, false)` and `tex (label, false)` return their TeX output as a string.
\texttt{tex} evaluates its first argument after testing it to see if it is a label. Quote-quote \texttt{''} forces evaluation of the argument, thereby defeating the test and preventing the label.

See also \texttt{texput}.

Examples:

\begin{verbatim}
(%i1) integrate (1/(1+x^3), x);
\end{verbatim}
\begin{verbatim}
2 x - 1
\log(x - x + 1) \atan(--------)
\sqrt(3) \sqrt(3) \log(x + 1)
\hline
6 \sqrt(3) 3
\end{verbatim}
\begin{verbatim}
(%o1) \hline
\end{verbatim}
\begin{verbatim}
(%i2) tex (%o1);
\end{verbatim}
\begin{verbatim}
\hline
\end{verbatim}
\begin{verbatim}
(%i3) tex (integrate (sin(x), x));
\end{verbatim}
\begin{verbatim}
\hline
\end{verbatim}
\begin{verbatim}
(%o3) \hline
\end{verbatim}
\begin{verbatim}
(%i4) tex (%o1, "foo.tex");
\end{verbatim}
\begin{verbatim}
\hline
\end{verbatim}
\begin{verbatim}
(%o4) \hline
\end{verbatim}

\texttt{tex (expr, false)} returns its TeX output as a string.

\begin{verbatim}
(%i1) S : tex (x * y * z, false);
\end{verbatim}
\begin{verbatim}
\hline
\end{verbatim}
\begin{verbatim}
(%i2) S;
\end{verbatim}
\begin{verbatim}
\hline
\end{verbatim}
\begin{verbatim}
(%o2) \hline
\end{verbatim}
\begin{verbatim}
(%i3) tex (integrate (sin(x), x));
\end{verbatim}
\begin{verbatim}
\hline
\end{verbatim}
\begin{verbatim}
(%o3) \hline
\end{verbatim}
\begin{verbatim}
(%i4) tex (%o1, "foo.tex");
\end{verbatim}
\begin{verbatim}
\hline
\end{verbatim}
\begin{verbatim}
(%o4) \hline
\end{verbatim}

\texttt{tex1 (e)} \hfill [Function]

Returns a string which represents the TeX output for the expressions \texttt{e}. The TeX output is not enclosed in delimiters for an equation or any other environment.

Examples:

\begin{verbatim}
(%i1) tex1 (sin(x) + cos(x));
\end{verbatim}
\begin{verbatim}
\hline
\end{verbatim}
\begin{verbatim}
(%o1) \hline
\end{verbatim}

\texttt{texput} \hfill [Function]

\begin{verbatim}
texput (a, s)
texput (a, f)
texput (a, s, operator_type)
texput (a, [s_1, s_2], matchfix)
texput (a, [s_1, s_2, s_3], matchfix)
\end{verbatim}

Assign the TeX output for the atom \texttt{a}, which can be a symbol or the name of an operator.

\texttt{texput (a, s)} causes the \texttt{tex} function to interpolate the string \texttt{s} into the TeX output in place of \texttt{a}.

\texttt{texput (a, f)} causes the \texttt{tex} function to call the function \texttt{f} to generate TeX output. \texttt{f} must accept one argument, which is an expression which has operator \texttt{a}, and must
return a string (the TeX output). \( f \) may call \texttt{tex1} to generate TeX output for the arguments of the input expression.

\texttt{texput (a, s, operator\_type)}, where \texttt{operator\_type} is \texttt{prefix}, \texttt{infix}, \texttt{postfix}, \texttt{nary}, or \texttt{nofix}, causes the \texttt{tex} function to interpolate \( s \) into the TeX output in place of \( a \), and to place the interpolated text in the appropriate position.

\texttt{texput (a, [s\_1, s\_2], matchfix)} causes the \texttt{tex} function to interpolate \( s\_1 \) and \( s\_2 \) into the TeX output on either side of the arguments of \( a \). The arguments (if more than one) are separated by commas.

\texttt{texput (a, [s\_1, s\_2, s\_3], matchfix)} causes the \texttt{tex} function to interpolate \( s\_1 \) and \( s\_2 \) into the TeX output on either side of the arguments of \( a \), with \( s\_3 \) separating the arguments.

Examples:

Assign TeX output for a variable.

\begin{verbatim}
(%i1) texput (me, "\mu_e");
(%o1) \mu_e
(%i2) tex (me);
$$\mu_e$$
(%o2) false
\end{verbatim}

Assign TeX output for an ordinary function (not an operator).

\begin{verbatim}
(%i1) texput (lcm, "\mathrm{lcm}");
(%o1) \mathrm{lcm}
(%i2) tex (lcm (a, b));
$$\mathrm{lcm}(a, b)$$
(%o2) false
\end{verbatim}

Call a function to generate TeX output.

\begin{verbatim}
(%i1) texfoo (e) := block ([a, b], [a, b] : args (e),
   concat("\left[\stackrel{\texttt{tex1}(b)}{\texttt{tex1}(a)}\right]\))
(%i2) texput (foo, texfoo);
(%o2) texfoo
(%i3) tex (foo (2^x, %pi));
$$\left[\stackrel{\pi}{2^{x}}\right]\$$
(%o3) false
\end{verbatim}

Assign TeX output for a prefix operator.

\begin{verbatim}
(%i1) prefix ("grad");
(%o1) grad
(%i2) texput ("grad", " \nabla ", prefix);
(%o2) \nabla
(%i3) tex (grad f);
$$ \nabla f$$
(%o3) false
\end{verbatim}

Assign TeX output for an infix operator.

\begin{verbatim}
(%i1) infix ("\~");
(%o1) ~
(%i2) texput ("\~", " \times ", infix);
\end{verbatim}
Assign TeX output for a postfix operator.

(%i1) postfix ("##");
(%o1) ##
(%i2) texput ("##", "!!", postfix);
(%o2) !!
(%i3) tex (x ##);
$$x!!$$
(%o3) false

Assign TeX output for a nary operator.

(%i1) nary ("@@");
(%o1) @@
(%i2) texput ("@@", " \circ ", nary);
(%o2) \circ
(%i3) tex (a @@ b @@ c @@ d);
$$a \circ b \circ c \circ d$$
(%o3) false

Assign TeX output for a nofix operator.

(%i1) nofix ("foo");
(%o1) foo
(%i2) texput ("foo", "\mathsc{foo}", nofix);
(%o2) \mathsc{foo}
(%i3) tex (foo);
$$\mathsc{foo}$$
(%o3) false

Assign TeX output for a matchfix operator.

(%i1) matchfix ("<<", ">>");
(%o1) <<
(%i2) texput ("<<", [" \langle ", " \rangle "], matchfix);
(%o2) [ \langle , \rangle ]
(%i3) tex (<<a>>);
$$ \langle a \rangle $$
(%o3) false
(%i4) tex (<<a, b>>);
$$ \langle a , b \rangle $$
(%o4) false
(%i5) texput ("<<", [" \langle ", " \rangle ", " \, | \, "], matchfix);
(%o5) [ \langle , \rangle , \, | \, ]
(%i6) tex (<<a>>);
$$ \langle a \rangle $$
(%o6) false
(\%i7) tex (<<a, b>>);
$$ \langle a \mid , \, b \rangle $$
(\%o7) false

get_tex_environment (op)  [Function]
set_tex_environment (op, before, after)  [Function]

Customize the TeX environment output by tex. As maintained by these functions, the TeX environment comprises two strings: one is printed before any other TeX output, and the other is printed after.

Only the TeX environment of the top-level operator in an expression is output; TeX environments associated with other operators are ignored.

get_tex_environment returns the TeX environment which is applied to the operator op; returns the default if no other environment has been assigned.

set_tex_environment assigns the TeX environment for the operator op.

Examples:

(\%i1) get_tex_environment ("=");
(\%o1) [\begin{verbatim}
\, ;
\end{verbatim}]
(\%i2) tex (f (x) := 1 - x);
\begin{verbatim}
f(x):=1-x;
\end{verbatim}
(\%o2) false

(\%i3) set_tex_environment ("="; "\$$", "$\$$");
(\%o3) [$$, $$]
(\%i4) tex (f (x) := 1 - x);
$$f(x):=1-x$$
(\%o4) false

get_tex_environment_default ()  [Function]
set_tex_environment_default (before, after)  [Function]

Customize the TeX environment output by tex. As maintained by these functions, the TeX environment comprises two strings: one is printed before any other TeX output, and the other is printed after.

get_tex_environment_default returns the TeX environment which is applied to expressions for which the top-level operator has no specific TeX environment (as assigned by set_tex_environment).

set_tex_environment_default assigns the default TeX environment.

Examples:

(\%i11) get_tex_environment_default ();
\(\%o1\) \[$$, $$\]
\(\%i2\) \(\text{tex}\ (f(x) + g(x))\);
\[g(x) + f(x)\]
\(\%o2\) \false{}
\(\%i3\) \(\text{set\_tex\_environment\_default}\ ("\begin{equation}\"
"","\end{equation}"));
\(\%o3\) \[\begin{equation}\],\end{equation}\]
\(\%i4\) \(\text{tex}\ (f(x) + g(x))\);
\begin{equation}
g(x) + f(x)\end{equation}
\(\%o4\) \false{}

13.5 Functions and Variables for Fortran Output

\texttt{fortindent} \quad [\text{Option variable}]

Default value: 0

\texttt{fortindent} controls the left margin indentation of expressions printed out by the \texttt{fortran} command. 0 gives normal printout (i.e., 6 spaces), and positive values will cause the expressions to be printed farther to the right.

\texttt{fortran (expr)} \quad [\text{Function}]

Prints \texttt{expr} as a Fortran statement. The output line is indented with spaces. If the line is too long, \texttt{fortran} prints continuation lines. \texttt{fortran} prints the exponentiation operator \(^\) as \(*\), and prints a complex number \(a + b\ %i\) in the form \((a, b)\).

\texttt{expr} may be an equation. If so, \texttt{fortran} prints an assignment statement, assigning the right-hand side of the equation to the left-hand side. In particular, if the right-hand side of \texttt{expr} is the name of a matrix, then \texttt{fortran} prints an assignment statement for each element of the matrix.

If \texttt{expr} is not something recognized by \texttt{fortran}, the expression is printed in \texttt{grind} format without complaint. \texttt{fortran} does not know about lists, arrays, or functions. \texttt{fortindent} controls the left margin of the printed lines. 0 is the normal margin (i.e., indented 6 spaces). Increasing \texttt{fortindent} causes expressions to be printed further to the right.

When \texttt{fortspaces} is \texttt{true}, \texttt{fortran} fills out each printed line with spaces to 80 columns.

\texttt{fortran} evaluates its arguments; quoting an argument defeats evaluation. \texttt{fortran} always returns \texttt{done}.

See also the function [\texttt{function\_f90}, page 899, for printing one or more expressions as a Fortran 90 program.

Examples:

\(\%i1\) \texttt{expr: (a + b)^12}
(\%i2) fortran (expr);
    (b+a)**12
(\%o2) done
(\%i3) fortran ('x=expr);
    x = (b+a)**12
(\%o3) done
(\%i4) fortran ('x=expand (expr));
    x = b**12+12*a*b**11+66*a**2*b**10+220*a**3*b**9+495*a**4*b**8+792
        *a**5*b**7+924*a**6*b**6+792*a**7*b**5+495*a**8*b**4+220*a**9*b
        2 **3+66*a**10*b**2+12*a**11*b+a**12
(\%o4) done
(\%i5) fortran ('x=7+5*\%i);
    x = (7,5)
(\%o5) done
(\%i6) fortran ('x=[1,2,3,4]);
    x = [1,2,3,4]
(\%o6) done
(\%i7) f(x) := x^2$
(\%i8) fortran (f);
    f
(\%o8) done

**fortspaces**

[Option variable]

Default value: false

When **fortspaces** is true, fortran fills out each printed line with spaces to 80 columns.
14 Polynomials

14.1 Introduction to Polynomials

Polynomials are stored in Maxima either in General Form or as Canonical Rational Expressions (CRE) form. The latter is a standard form, and is used internally by operations such as factor, ratsimp, and so on.

Canonical Rational Expressions constitute a kind of representation which is especially suitable for expanded polynomials and rational functions (as well as for partially factored polynomials and rational functions when RATFAC is set to true). In this CRE form an ordering of variables (from most to least main) is assumed for each expression. Polynomials are represented recursively by a list consisting of the main variable followed by a series of pairs of expressions, one for each term of the polynomial. The first member of each pair is the exponent of the main variable in that term and the second member is the coefficient of that term which could be a number or a polynomial in another variable again represented in this form. Thus the principal part of the CRE form of 3*X^2-1 is (X 2 3 0 -1) and that of 2*X*Y+X-3 is (Y 1 (X 1 2) 0 (X 1 1 0 -3)) assuming Y is the main variable, and is (X 1 (Y 1 2 0 1) 0 -3) assuming X is the main variable. "Main"-ness is usually determined by reverse alphabetical order. The "variables" of a CRE expression needn’t be atomic. In fact any subexpression whose main operator is not + - * / or ^ with integer power will be considered a "variable" of the expression (in CRE form) in which it occurs. For example the CRE variables of the expression X+SIN(X+1)+2*SQRT(X)+1 are X, SQRT(X), and SIN(X+1). If the user does not specify an ordering of variables by using the RATVARS function Maxima will choose an alphabetic one. In general, CRE’s represent rational expressions, that is, ratios of polynomials, where the numerator and denominator have no common factors, and the denominator is positive. The internal form is essentially a pair of polynomials (the numerator and denominator) preceded by the variable ordering list. If an expression to be displayed is in CRE form or if it contains any subexpressions in CRE form, the symbol /R/ will follow the line label. See the RAT function for converting an expression to CRE form. An extended CRE form is used for the representation of Taylor series. The notion of a rational expression is extended so that the exponents of the variables can be positive or negative rational numbers rather than just positive integers and the coefficients can themselves be rational expressions as described above rather than just polynomials. These are represented internally by a recursive polynomial form which is similar to and is a generalization of CRE form, but carries additional information such as the degree of truncation. As with CRE form, the symbol /T/ follows the line label of such expressions.

14.2 Functions and Variables for Polynomials

algebraic

Default value: false

algebraic must be set to true in order for the simplification of algebraic integers to take effect.

berlefact

Default value: true
When `berlefact` is `false` then the Kronecker factoring algorithm will be used otherwise the Berlekamp algorithm, which is the default, will be used.

**bezout** $(p1, p2, x)$  
[Function]
an alternative to the `resultant` command. It returns a matrix. `determinant` of this matrix is the desired resultant.

Examples:

```lisp
(%i1) bezout(a*x+b, c*x^2+d, x);  
    [ b c - a d ]  
    [           ]  
    [ a b     ]  
(%o1)      

(%i2) determinant(%);  
2 2  
2   
(%o2)  a d + b c  

(%i3) resultant(a*x+b, c*x^2+d, x);  
2 2  
2   
(%o3)  a d + b c  
```

**bothcoef** $(expr, x)$  
[Function]
Returns a list whose first member is the coefficient of $x$ in $expr$ (as found by `ratcoef` if $expr$ is in CRE form otherwise by `coeff`) and whose second member is the remaining part of $expr$. That is, $[A, B]$ where $expr = A*x + B$.

Example:

```lisp
(%i1) islinear (expr, x) := block ([c],  
    c: bothcoef (rat (expr, x), x),  
    is (freeof (x, c) and c[1] # 0))$  
(%i2) islinear ((r^2 - (x - r)^2)/x, x);  
(%o2) true  
```

**coeff** $(expr, x, n)$  
[Function]
Returns the coefficient of $x^n$ in $expr$, where $expr$ is a polynomial or a monomial term in $x$. Other than `ratcoef` `coeff` is a strictly syntactical operation and will only find literal instances of $x^n$ in the internal representation of $expr$.

`coeff(expr, x^n)` is equivalent to `coeff(expr, x, n)`. `coeff(expr, x, 0)` returns the remainder of $expr$ which is free of $x$. If omitted, $n$ is assumed to be 1.

$x$ may be a simple variable or a subscripted variable, or a subexpression of $expr$ which comprises an operator and all of its arguments.

It may be possible to compute coefficients of expressions which are equivalent to $expr$ by applying `expand` or `factor`. `coeff` itself does not apply `expand` or `factor` or any other function.

`coeff` distributes over lists, matrices, and equations.

See also `ratcoef`.

Examples:
coeff returns the coefficient $x^n$ in expr.

```
(%i1) coeff (b^3*a^3 + b^2*a^2 + b*a + 1, a^3);
    3
(%o1) b
```

coeff(expr, $x^n$) is equivalent to coeff(expr, $x$, $n$).

```
(%i1) coeff (c[4]*z^4 - c[3]*z^3 - c[2]*z^2 + c[1]*z, z, 3);
    3
(%o1) c
(%i2) coeff (c[4]*z^4 - c[3]*z^3 - c[2]*z^2 + c[1]*z, z^3);
    3
(%o2) c
```

coeff(expr, $x$, 0) returns the remainder of expr which is free of $x$.

```
(%i1) coeff (a*u + b^2*u^2 + c^3*u^3, b, 0);
    3 3
(%o1) c u + a u
```

$x$ may be a simple variable or a subscripted variable, or a subexpression of expr which comprises an operator and all of its arguments.

```
(%i1) coeff (h^4 - 2*%pi*h^2 + 1, h, 2);
    2
(%o1) - 2 %pi
(%i2) coeff ((d - a)^2*(b + c)^3 + (a + b)^4*(c - d), a + b, 4);
    4
(%o2) c - d
```

coeff itself does not apply expand or factor or any other function.

```
(%i1) coeff (c*(a + b)^3, a);
    0
(%o1) 0
(%i2) expand (c*(a + b)^3);
    2 2 3
(%o2) b c + 3 a b c + 3 a b c + a c
(%i3) coeff (%o2, a);
    2
(%o3) 3 b c
(%i4) coeff (b^3*c + 3*a*b^2*c + 3*a^2*b*c + a^3*c, (a + b)^3);
    0
(%o4) 0
(%i5) factor (b^3*c + 3*a*b^2*c + 3*a^2*b*c + a^3*c);
    3
(%o5) (b + a) c
(%i6) coeff (%o5, (a + b)^3);
    0
(%o6) c
```

coeff distributes over lists, matrices, and equations.

```
(%i1) coeff ([[4*a, -3*a], 2*a], a);
    [4, - 3, 2]
```
(%i2) coeff (matrix ([a*x, b*x], [-c*x, -d*x]), x);
   [ a  b ]
   [    ]
   [ - c - d ]
(%o2)
(%i3) coeff (a*u - b*v = 7*u + 3*v, u);
   a = 7

content (p_1, x_1, ..., x_n) [Function]
Returns a list whose first element is the greatest common divisor of the coefficients of
the terms of the polynomial p_1 in the variable x_n (this is the content) and whose
second element is the polynomial p_1 divided by the content.
Examples:
(%i1) content (2*x*y + 4*x^2*y^2, y);
   2
(%o1) [2 x, 2 x y + y]

denom (expr) [Function]
Returns the denominator of the rational expression expr.
See also num
(%i1) g1:(x+2)*(x+1)/((x+3)^2);
   (x + 1) (x + 2)
   ---------------
   2
   (x + 3)
(%o1)
(%i2) denom(g1);
   2
(%o2) (x + 3)
(%i3) g2:sin(x)/10*cos(x)/y;
   cos(x) sin(x)
   -------------
   10 y
(%o3)
(%i4) denom(g2);
   10 y

divide (p_1, p_2, x_1, ..., x_n) [Function]
computes the quotient and remainder of the polynomial p_1 divided by the polynomial
p_2, in a main polynomial variable, x_n. The other variables are as in the ratvars
function. The result is a list whose first element is the quotient and whose second
element is the remainder.
Examples:
(%i1) divide (x + y, x - y, x);
   [1, 2 y]
   (%o1)
(%i2) divide (x + y, x - y);
   [- 1, 2 x]
(%o2)
Note that y is the main variable in the second example.
\textbf{eliminate ([eqn_1, \ldots, eqn_n], [x_1, \ldots, x_k])} \quad \text{[Function]}

Eliminates variables from equations (or expressions assumed equal to zero) by taking successive resultants. This returns a list of \( n - k \) expressions with the \( k \) variables \( x_1, \ldots, x_k \) eliminated. First \( x_1 \) is eliminated yielding \( n - 1 \) expressions, then \( x_2 \) is eliminated, etc. If \( k = n \) then a single expression in a list is returned free of the variables \( x_1, \ldots, x_k \). In this case \texttt{solve} is called to solve the last resultant for the last variable.

Example:

\begin{verbatim}
(%i1) expr1: 2*x^2 + y*x + z;
   %o1
2
   (%o1) z + x y + 2 x
(%i2) expr2: 3*x + 5*y - z - 1;
   %o2
   (%o2) - z + 5 y + 3 x - 1
(%i3) expr3: z^2 + x - y^2 + 5;
   2 2
   7 6 5 4
   (%o3) [7425 x - 1170 x + 1299 x + 12076 x + 22887 x
     - 5154 x - 1291 x + 7688 x + 15376]
(%i4) eliminate ([expr3, expr2, expr1], [y, z]);
6 5 4
8 7 6 5 4
2
2
2
2
2
4
2
2
2
2
2

\textbf{ezgcd (p_1, p_2, p_3, \ldots)} \quad \text{[Function]}

Returns a list whose first element is the greatest common divisor of the polynomials \( p_1, p_2, p_3, \ldots \) and whose remaining elements are the polynomials divided by the greatest common divisor. This always uses the \texttt{ezgcd} algorithm.

See also \texttt{gcd}, \texttt{gcdex}, \texttt{gcddivide}, and \texttt{poly_gcd}.

Examples:

The three polynomials have the greatest common divisor \( 2x - 3 \). The \texttt{gcd} is first calculated with the function \texttt{gcd} and then with the function \texttt{ezgcd}.

\begin{verbatim}
(%i1) p1 : 6*x^3-17*x^2+14*x-3;
   3 2
   (%o1) 6 x - 17 x + 14 x - 3
(%i2) p2 : 4*x^4-14*x^3+12*x^2+2*x-3;
   4 3 2
   (%o2) 4 x - 14 x + 12 x + 2 x - 3
(%i3) p3 : -8*x^3+14*x^2-x-3;
   3 2
   (%o3) - 8 x + 14 x - x - 3
(%i4) gcd(p1, gcd(p2, p3));
   (%o4) 2 x - 3
(%i5) ezgcd(p1, p2, p3);
   2 3 2 2
\end{verbatim}
(%o5) [2 x - 3, 3 x - 4 x + 1, 2 x - 4 x + 1, - 4 x + x + 1]

facexpand

[Option variable]

Default value: true

facexpand controls whether the irreducible factors returned by factor are in expanded (the default) or recursive (normal CRE) form.

factor

[Function]

factor (expr)
factor (expr, p)

Factors the expression expr, containing any number of variables or functions, into factors irreducible over the integers. factor (expr, p) factors expr over the field of rationals with an element adjoined whose minimum polynomial is p.

factor uses ifactors function for factoring integers.

factorflag if false suppresses the factoring of integer factors of rational expressions.

dontfactor may be set to a list of variables with respect to which factoring is not to occur. (It is initially empty). Factoring also will not take place with respect to any variables which are less important (using the variable ordering assumed for CRE form) than those on the dontfactor list.

savefactors if true causes the factors of an expression which is a product of factors to be saved by certain functions in order to speed up later factorizations of expressions containing some of the same factors.

berlefact if false then the Kronecker factoring algorithm will be used otherwise the Berlekamp algorithm, which is the default, will be used.

intfaclim if true maxima will give up factorization of integers if no factor is found after trial divisions and Pollard’s rho method. If set to false (this is the case when the user calls factor explicitly), complete factorization of the integer will be attempted. The user’s setting of intfaclim is used for internal calls to factor. Thus, intfaclim may be reset to prevent Maxima from taking an inordinately long time factoring large integers.

See also collectterms.

Examples:

(%i1) factor (2^63 - 1);
2
(%o1) 7 73 127 337 92737 649657

(%i2) factor (-8*y - 4*x + z^2*(2*y + x));
2 2
(%o2) (2 y + x) (z - 2) (z + 2)

(%i3) -1 - 2*x - x^2 + y^2 + 2*x*y^2 + x^2*y^2;
2 2 2 2 2
(%o3) x y + 2 x y + y - x - 2 x - 1

(%i4) block ([dontfactor: [x]], factor (%/36/(1 + 2*y + y^2)));
2
(%o4) (x + 2 x + 1) (y - 1)

-------------
36 (y + 1)
(%i5) factor (1 + %e^(3*x));
   x
(%o5) (%e + 1)(%e - %e + 1)

(%i6) factor (1 + x^4, a^2 - 2);
   2  2
(%o6) (x - a x + 1)(x + a x + 1)

(%i7) factor (-y^2*z^2 - x*z^2 + x^2*y^2 + x^3);
   2 2
(%o7) -(y + x)(z - x)(z + x)

(%i8) (2 + x)/(3 + x)/(b + x)/(c + x)^2;
   x + 2
(%o8) ------------------------
   2
   (x + 3)(x + b)(x + c)

(%i9) ratsimp (%);

(%i10) partfrac (%); 4  3
   2
(2 + (c + (2 b + 6) c + 3 b) x + ((b + 3) c + 6 b c) x + 3 b c)

(%i11) map ('factor, %);
\[\frac{2}{c - 4} \frac{c - b + 6}{c - 2} - \frac{2}{(c - 3) (c - b) (x + c)} - \frac{2}{(c - 3) (c - b) (x + c)}\]

\[\frac{b - 2}{(b - 3) (c - b) (x + b)} + \frac{1}{(b - 3) (c - b) (x + b)}\]

\[\text{ratsimp} \left(\frac{x^5 - 1}{x - 1}\right)\]

\[x^4 + x^3 + x^2 + x + 1\]

\[\text{subst} (a, x, \%)\]

\[a^4 + a^3 + a^2 + a + 1\]

\[\text{factor} (\text{th}(2), \%)\]

\[a^2 (x - a) (x - a) (x + a + a + a + 1)\]

\[\text{factor} (1 + x^{12})\]

\[x^4 (x^8 - 1)\]

\[\text{factor} (1 + x^{99})\]

\[x^2 (x - x + 1)^6 (x - x + 1)\]

\[x^{10} - x^9 + x^8 - x^7 + x^6 - x^5 + x^4 - x^3 + x^2 - x + 1\]

\[x^{20} - x^{19} + x^{17} - x^{16} + x^{14} - x^{13} + x^{11} - x^9 + 6\]

\[x^{40} - x^{39} + x^{37} - x^{36} + x^{34} - x^{33} + x^{31} - x^{30} + x^{29} - x^{28} + x^{26} - x^{25} + x^{23} - x^{22} + x^{20} - x^{19} + x^{17} - x^{16} + x^{14} - x^{13} + x^{11} - x^9 + x^7 - x^6 + x^5 - x^4 + x^3 - x^2 + x - 1\]

\[\text{factorflag} \quad \text{[Option variable]}\]

Default value: false

When factorflag is false, suppresses the factoring of integer factors of rational expressions.

\[\text{factorout} \ (\text{expr}, x_1, x_2, \ldots) \quad \text{[Function]}\]

Rearranges the sum expr into a sum of terms of the form \(f (x_1, x_2, \ldots) g\) where \(g\) is a product of expressions not containing any \(x_i\) and \(f\) is factored.

Note that the option variable keepfloat is ignored by factorout.
Example:

\[
\begin{align*}
\text{(i1)} & \quad \text{expand} \ (a*(x+1)*(x-1)*(u+1)^2); \\
& \quad 2 \quad 2 \quad 2 \quad 2 \quad 2 \\
& \text{(o1)} \quad a \ u \ x + 2 a \ u \ x + a x - a \ u - 2 a \ u - a \\
\text{(i2)} & \quad \text{factorout}(%,x); \\
& \quad 2 \\
& \text{(o2)} \quad a \ u \ (x - 1) \ (x + 1) + 2 a \ u \ (x - 1) \ (x + 1) \\
& \quad + a \ (x - 1) \ (x + 1)
\end{align*}
\]

**factorsum (expr)**

Tries to group terms in factors of expr which are sums into groups of terms such that their sum is factorable. factorsum can recover the result of expand \( ((x + y)^2 + (z + w)^2) \) but it can’t recover expand \( ((x + 1)^2 + (x + y)^2) \) because the terms have variables in common.

Example:

\[
\begin{align*}
\text{(i1)} & \quad \text{expand} \ ((x + 1)*((u + v)^2 + a*(w + z)^2)); \\
& \quad 2 \quad 2 \quad 2 \quad 2 \\
& \text{(o1)} \quad a \ x \ z + a \ z + 2 a w x z + 2 a w z + a \ w \ x + v \ x \\
& \quad + 2 u \ v \ x + u \ x + a \ w + v + 2 u \ v + u \\
\text{(i2)} & \quad \text{factorsum} \ (%); \\
& \quad 2 \quad 2 \\
& \text{(o2)} \quad (x + 1) \ (a \ (z + w) + (v + u))
\end{align*}
\]

**fasttimes (p_1, p_2)**

Returns the product of the polynomials p_1 and p_2 by using a special algorithm for multiplication of polynomials. p_1 and p_2 should be multivariate, dense, and nearly the same size. Classical multiplication is of order \( n_1 n_2 \) where \( n_1 \) is the degree of p_1 and \( n_2 \) is the degree of p_2. fasttimes is of order \( \max(n_1, n_2)^{1.585} \).

**fullratsimp (expr)**

fullratsimp repeatedly applies ratsimp followed by non-rational simplification to an expression until no further change occurs, and returns the result.

When non-rational expressions are involved, one call to ratsimp followed as is usual by non-rational ("general") simplification may not be sufficient to return a simplified result. Sometimes, more than one such call may be necessary. fullratsimp makes this process convenient.

fullratsimp (expr, x_1, ..., x_n) takes one or more arguments similar to ratsimp and rat.

Example:

\[
\begin{align*}
\text{(i1)} & \quad \text{expr:} \ (x^{(a/2)} + 1)^2*(x^{(a/2)} - 1)^2/(x^a - 1); \\
& \quad a/2 \quad 2 \quad a/2 \quad 2 \\
& \quad (x \ - 1) \ (x \ + 1) \\
& \quad \text{(o1)} \quad \frac{-----------------------------}{a}
\end{align*}
\]
\( x - 1 \)

(%i2) ratsimp (expr);
\[
\frac{2 a - a}{x - 2 x + 1}
\]

(%o2) \[
-----------------
\]

(%i3) fullratsimp (expr);
\[
a
\]

(%o3) \[
x - 1
\]

(%i4) rat (expr);
\[
\frac{\frac{a}{2} 4 \quad \frac{a}{2} 2}{\frac{x}{x} - 2 \left(\frac{x}{x}\right) + 1}
\]

(%o4)/R/
\[
-----------------
\]

fullratsubst (a, b, c) [Function]
is the same as ratsubst except that it calls itself recursively on its result until that result stops changing. This function is useful when the replacement expression and the replaced expression have one or more variables in common.

fullratsubst will also accept its arguments in the format of lratsubst. That is, the first argument may be a single substitution equation or a list of such equations, while the second argument is the expression being processed.

load ("lrats") loads fullratsubst and lratsubst.

Examples:

(%i1) load ("lrats")$

• subst can carry out multiple substitutions. lratsubst is analogous to subst.

(%i2) subst ([a = b, c = d], a + c);
\[
d + b
\]

(%o2) \[
\]

(%i3) lratsubst ([a^2 = b, c^2 = d], (a + e)*c*(a + c));
\[
(d + a c) e + a d + b c
\]

(%o3) \[
\]

• If only one substitution is desired, then a single equation may be given as first argument.

(%i4) lratsubst (a^2 = b, a^3);
\[
a b
\]

(%o4) \[
\]

• fullratsubst is equivalent to ratsubst except that it recurses until its result stops changing.

(%i5) ratsubst (b*a, a^2, a^3);
\[
2
\]

(%o5) \[
\]

(%i6) fullratsubst (b*a, a^2, a^3);
\[
2
\]

(%o6) \[
\]
fullratsubst also accepts a list of equations or a single equation as first argument.

(%i7) fullratsubst ([a^2 = b, b^2 = c, c^2 = a], a^3*b*c);
(%o7) b

(%i8) fullratsubst (a^2 = b*a, a^3);
(%o8) a b

fullratsubst may cause an indefinite recursion.

(%i9) errcatch (fullratsubst (b*a^2, a^2, a^3));
*** - Lisp stack overflow. RESET

gcd (p_1, p_2, x_1, ...)  [Function]

Returns the greatest common divisor of \( p_1 \) and \( p_2 \). The flag gcd determines which algorithm is employed. Setting gcd to ez, subres, red, or spmod selects the ezgcd, subresultant prs, reduced, or modular algorithm, respectively. If gcd false then gcd \((p_1, p_2, x)\) always returns 1 for all \( x \). Many functions (e.g. ratsimp, factor, etc.) cause gcd’s to be taken implicitly. For homogeneous polynomials it is recommended that gcd equal to subres be used. To take the gcd when an algebraic is present, e.g.,
\[
gcd (x^2 - 2*sqrt(2) * x + 2, x - sqrt(2))
\]
cause gcd’s to be taken implicitly. For homogeneous polynomials it is recommended that gcd equal to subres be used. To take the gcd when an algebraic is present, e.g.,
\[
gcd (x^2 - 2*sqrt(2) * x + 2, x - sqrt(2))
\]
cause gcd’s to be taken implicitly. For homogeneous polynomials it is recommended that gcd equal to subres be used. To take the gcd when an algebraic is present, e.g.,
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gcd (x^2 - 2*sqrt(2) * x + 2, x - sqrt(2))
\]
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\[
gcd (x^2 - 2*sqrt(2) * x + 2, x - sqrt(2))
\]
cause gcd’s to be taken implicitly. For homogeneous polynomials it is recommended that gcd equal to subres be used. To take the gcd when an algebraic is present, e.g.,
\[
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\]
cause gcd’s to be taken implicitly. For homogeneous polynomials it is recommended that gcd equal to subres be used. To take the gcd when an algebraic is present, e.g.,
\[
gcd (x^2 - 2*sqrt(2) * x + 2, x - sqrt(2))
\]
cause gcd’s to be taken implicitly. For homogeneous polynomials it is recommended that gcd equal to subres be used. To take the gcd when an algebraic is present, e.g.,
\[
gcd (x^2 - 2*sqrt(2) * x + 2, x - sqrt(2))
\]
cause gcd’s to be taken implicitly. For homogeneous polynomials it is recommended that gcd equal to subres be used. To take the gcd when an algebraic is present, e.g.,
\[
gcd (x^2 - 2*sqrt(2) * x + 2, x - sqrt(2))
\]
cause gcd’s to be taken implicitly. For homogeneous polynomials it is recommended that gcd equal to subres be used. To take the gcd when an algebraic is present, e.g.,
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gcd (x^2 - 2*sqrt(2) * x + 2, x - sqrt(2))
\]
cause gcd’s to be taken implicitly. For homogeneous polynomials it is recommended that gcd equal to subres be used. To take the gcd when an algebraic is present, e.g.,
\[
gcd (x^2 - 2*sqrt(2) * x + 2, x - sqrt(2))
\]
cause gcd’s to be taken implicitly. For homogeneous polynomials it is recommended that gcd equal to subres be used. To take the gcd when an algebraic is present, e.g.,
\[
gcd (x^2 - 2*sqrt(2) * x + 2, x - sqrt(2))
\]
cause gcd’s to be taken implicitly. For homogeneous polynomials it is recommended that gcd equal to subres be used. To take the gcd when an algebraic is present, e.g.,
\[
gcd (x^2 - 2*sqrt(2) * x + 2, x - sqrt(2))
\]
gcdex

\texttt{gcdex} (\texttt{f, g})
\texttt{gcdex} (\texttt{f, g, x})

Returns a list \([a, b, u]\) where \(u\) is the greatest common divisor (gcd) of \(f\) and \(g\), and \(u\) is equal to \(af + bg\). The arguments \(f\) and \(g\) should be univariate polynomials, or else polynomials in \(x\) a supplied main variable since we need to be in a principal ideal domain for this to work. The gcd means the gcd regarding \(f\) and \(g\) as univariate polynomials with coefficients being rational functions in the other variables.

gcdex implements the Euclidean algorithm, where we have a sequence of \(L_i: [a_i, b_i, r_i]\) which are all perpendicular to \([f, g, -1]\) and the next one is built as if \(q = \text{quotient}(r_i/r_{i+1})\) then \(L_{i+2} = L_i - q L_{i+1}\), and it terminates at \(L_{i+1}\) when the remainder \(r_{i+2}\) is zero.

The arguments \(f\) and \(g\) can be integers. For this case the function \texttt{igcdex} is called by \texttt{gcdex}.

See also \texttt{ezgcd, gcd, gcdivide}, and \texttt{poly_gcd}.

Examples:
\begin{verbatim}
(%i1) gcdex (x^2 + 1, x^3 + 4);
2
  x + 4 x - 1 x + 4
(%o1)/R/ [- ----------------, -----, 1]
       17  17

(%i2) % . [x^2 + 1, x^3 + 4, -1];
0

Note that the gcd in the following is 1 since we work in \(k(y)[x]\), not the \(y+1\) we would expect in \(k[y, x]\).

(%i1) gcdex (x*(y + 1), y^2 - 1, x);
1
(%o1)/R/ [0, ------, 1]
    2
    y - 1
\end{verbatim}


gcfactor (n)

Factors the Gaussian integer \(n\) over the Gaussian integers, i.e., numbers of the form \(a + b \%i\) where \(a\) and \(b\) are rational integers (i.e., ordinary integers). Factors are normalized by making \(a\) and \(b\) non-negative.

gfactor (expr)

Factors the polynomial \(\text{expr}\) over the Gaussian integers (that is, the integers with the imaginary unit \(\%i\) adjoined). This is like \texttt{factor (expr, a^2+1)} where \(a\) is \(\%i\).

Example:
\begin{verbatim}
(%i1) gfactor (x^4 - 1);
(%o1) (x - 1) (x + 1) (x - %i) (x + %i)
\end{verbatim}

gfactorsum (expr)

is similar to \texttt{factorsum} but applies \texttt{gfactor} instead of \texttt{factor}.
hipow (expr, x)  
Returns the highest explicit exponent of x in expr. x may be a variable or a general expression. If x does not appear in expr, hipow returns 0.

hipow does not consider expressions equivalent to expr. In particular, hipow does not expand expr, so hipow (expr, x) and hipow (expand (expr, x)) may yield different results.

Examples:

(%i1) hipow (y^3 * x^2 + x * y^4, x);
(%o1) 2

(%i2) hipow ((x + y)^5, x);
(%o2) 1

(%i3) hipow (expand ((x + y)^5), x);
(%o3) 5

(%i4) hipow ((x + y)^5, x + y);
(%o4) 5

(%i5) hipow (expand ((x + y)^5), x + y);
(%o5) 0

intfaclim  
Default value: true

If true, maxima will give up factorization of integers if no factor is found after trial divisions and Pollard’s rho method and factorization will not be complete.

When intfaclim is false (this is the case when the user calls factor explicitly), complete factorization will be attempted. intfaclim is set to false when factors are computed in divisors, divsum and totient.

Internal calls to factor respect the user-specified value of intfaclim. Setting intfaclim to true may reduce the time spent factoring large integers.

keepfloat  
Default value: false

When keepfloat is true, prevents floating point numbers from being rationalized when expressions which contain them are converted to canonical rational expression (CRE) form.

Note that the function solve and those functions calling it (eigenvalues, for example) currently ignore this flag, converting floating point numbers anyway.

Examples:

(%i1) rat(x/2.0);
rat: replaced 0.5 by 1/2 = 0.5

x
(%o1)/R/ -

(%i2) rat(x/2.0), keepfloat;
(%o2)/R/ 0.5 x

solve ignores keepfloat:
(%i1) solve(1.0-x,x), keepfloat;
rat: replaced 1.0 by 1/1 = 1.0
(%o1) [x = 1]

lopow (expr, x) [Function]
Returns the lowest exponent of x which explicitly appears in expr. Thus
(%i1) lopow ((x+y)^2 + (x+y)^a, x+y);
(%o1) min(a, 2)

lratsubst (L, expr) [Function]
is analogous to subst (L, expr) except that it uses ratsubst instead of subst.
The first argument of lratsubst is an equation or a list of equations identical in
format to that accepted by subst. The substitutions are made in the order given by
the list of equations, that is, from left to right.
load ("lrats") loads fullratsubst and lratsubst.
Examples:
(%i1) load ("lrats")$
• subst can carry out multiple substitutions. lratsubst is analogous to subst.
(%i2) subst ([a = b, c = d], a + c);
(%o2) b + d
(%i3) lratsubst ([a^2 = b, c^2 = d], (a + c)*c*(a + c));
(%o3) (d + a c) e + a d + b c
• If only one substitution is desired, then a single equation may be given as first
argument.
(%i4) lratsubst (a^2 = b, a^3);
(%o4) a b

modulus [Option variable]
Default value: false
When modulus is a positive number p, operations on rational numbers (as returned by
rat and related functions) are carried out modulo p, using the so-called "balanced"
modulus system in which n modulo p is defined as an integer k in \(-(p-1)/2, \ldots,
0, \ldots, (p-1)/2\) when p is odd, or \(-(p/2 - 1), \ldots, 0, \ldots, p/2\) when p is
even, such that a p + k equals n for some integer a.
If expr is already in canonical rational expression (CRE) form when modulus is reset,
then you may need to re-rat expr, e.g., expr: rat (ratdisrep (expr)), in order to
get correct results.
Typically modulus is set to a prime number. If modulus is set to a positive non-prime
integer, this setting is accepted, but a warning message is displayed. Maxima signals
an error, when zero or a negative integer is assigned to modulus.
Examples:
(%i1) modulus:7;
(%o1) 7
(%i2) polymod([0,1,2,3,4,5,6,7]);
(%o2) [0, 1, 2, 3, - 3, - 2, - 1, 0]
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(%i3) modulus:false;
false
(%i4) poly:x^6+x^2+1;
6 2
x + x + 1
(%i5) factor(poly);
6 2
x + x + 1
(%i6) modulus:13;
13
(%i7) factor(poly);
2 4 2
(x + 6) (x - 6 x - 2)
(%i8) polymod(%);
6 2
x + x + 1

num(expr) [Function]
Returns the numerator of expr if it is a ratio. If expr is not a ratio, expr is returned.
num evaluates its argument.

See also denom

(%i1) g1:(x+2)*(x+1)/((x+3)^2);
(x + 1) (x + 2)
-----------
2
(x + 3)
(%i2) num(g1);
(x + 1) (x + 2)
(%i3) g2:sin(x)/10*cos(x)/y;
cos(x) sin(x)
-----------
10 y
(%i4) num(g2);
cos(x) sin(x)

polydecomp(p, x) [Function]
Decomposes the polynomial p in the variable x into the functional composition of polynomials in x. polydecomp returns a list \([p_1, \ldots, p_n]\) such that
\[
\text{lambda } ([x], p_1) (\text{lambda } ([x], p_2) (\ldots (\text{lambda } ([x], p_n) (x)) \ldots))
\]
is equal to \(p\). The degree of \(p_i\) is greater than 1 for \(i\) less than \(n\).
Such a decomposition is not unique.

Examples:

(%i1) polydecomp(x^210, x);
7 5 3 2
[x , x , x , x ]
(%i2) p : expand (subst (x^3 - x - 1, x, x^2 - a));
   6  4  3  2
(%o2) x - 2 x - 2 x + x + 2 x - a + 1
(%i3) polydecomp (p, x);
      2  3
(%o3) [x - a, x - x - 1]
The following function composes $L = [e_1, \ldots, e_n]$ as functions in $x$; it is the inverse of polydecomp:

(%i1) compose (L, x) :=
   block ([r : x], for e in L do r : subst (e, x, r), r) $

Re-express above example using compose:

(%i1) polydecomp (compose ([x^2 - a, x^3 - x - 1], x), x);
      2  3
(%o1) [compose([x - a, x - x - 1], x)]

Note that though compose (polydecomp (p, x), x) always returns $p$ (unexpanded),
polydecomp (compose ([p_1, \ldots, p_n], x), x) does not necessarily return $[p_1, \ldots, p_n]$:

(%i1) polydecomp (compose ([x^2 + 2*x + 3, x^2 - x - 1], x), x);
      2
 (%o1) [compose([x + 2 x + 3], x)]
(%i2) polydecomp (compose ([x^2 + x + 1, x^2 + x + 1], x), x);
      2
 (%o2) [compose([x + x + 1], x)]

polymod

polymod (p)
polymod (p, m)

Converts the polynomial $p$ to a modular representation with respect to the current modulus which is the value of the variable modulus.
polymod (p, m) specifies a modulus $m$ to be used instead of the current value of modulus.

See modulus.

quotient

quotient (p_1, p_2)
quotient (p_1, p_2, x_1, \ldots, x_n)

Returns the polynomial $p_1$ divided by the polynomial $p_2$. The arguments $x_1, \ldots, x_n$ are interpreted as in ratvars.

quotient returns the first element of the two-element list returned by divide.

rat

rat (expr)
rat (expr, x_1, \ldots, x_n)

Converts $expr$ to canonical rational expression (CRE) form by expanding and combining all terms over a common denominator and cancelling out the greatest common
divisor of the numerator and denominator, as well as converting floating point numbers to rational numbers within a tolerance of \texttt{ratepsilon}. The variables are ordered according to the \(x_1, \ldots, x_n\), if specified, as in \texttt{ratvars}.

\texttt{rat} does not generally simplify functions other than addition +, subtraction -, multiplication *, division /, and exponentiation to an integer power, whereas \texttt{ratsimp} does handle those cases. Note that atoms (numbers and variables) in CRE form are not the same as they are in the general form. For example, \texttt{rat(x)-x} yields \texttt{rat(0)} which has a different internal representation than 0.

When \texttt{ratfac} is \texttt{true}, \texttt{rat} yields a partially factored form for CRE. During rational operations the expression is maintained as fully factored as possible without an actual call to the factor package. This should always save space and may save some time in some computations. The numerator and denominator are still made relatively prime (e.g., \texttt{rat(((x^2 - 1)^4/(x + 1)^2) yields (x - 1)^4 (x + 1)^2 when ratfac is true}), but the factors within each part may not be relatively prime.

\texttt{ratprint} if \texttt{false} suppresses the printout of the message informing the user of the conversion of floating point numbers to rational numbers.

\texttt{keepfloat} if \texttt{true} prevents floating point numbers from being converted to rational numbers.

See also \texttt{ratexpand} and \texttt{ratsimp}.

Examples:

\begin{verbatim}
(%i1) ((x - 2*y)^4/(x^2 - 4*y^2)^2 + 1)*(y + a)*(2*y + x) /
   (4*y^2 + x^2);
   4
(x - 2 y)
   (y + a) (2 y + x) (- --------- + 1)
   2 2
   (x - 4 y )

(%o1) ------------------------------
   2 2
   4 y + x

(%i2) rat (%i1, y, a, x);
   2 a + 2 y

(%o2)/R/
   --------
   x + 2 y
\end{verbatim}

\texttt{ratalgdenom} 

Default value: \texttt{true}

When \texttt{ratalgdenom} is \texttt{true}, allows rationalization of denominators with respect to radicals to take effect. \texttt{ratalgdenom} has an effect only when canonical rational expressions (CRE) are used in algebraic mode.

\texttt{ratcoef} 

\begin{verbatim}
ratcoef (expr, x, n)
ratcoef (expr, x)
\end{verbatim}

Returns the coefficient of the expression \(x^n\) in the expression \texttt{expr}. If omitted, \(n\) is assumed to be 1.
The return value is free (except possibly in a non-rational sense) of the variables in \( x \). If no coefficient of this type exists, 0 is returned.

`ratcoef` expands and rationally simplifies its first argument and thus it may produce answers different from those of `coeff` which is purely syntactic. Thus `ratcoef ((x + 1)/y + x, x)` returns \((y + 1)/y\) whereas `coeff` returns 1.

`ratcoef (expr, x, 0)`, viewing `expr` as a sum, returns a sum of those terms which do not contain \( x \). Therefore if \( x \) occurs to any negative powers, `ratcoef` should not be used.

Since `expr` is rationally simplified before it is examined, coefficients may not appear quite the way they were envisioned.

Example:

\[
(%i1) s: a*x + b*x + 5$
\]

\[
(%i2) ratcoef (s, a + b); \quad x
\]

`ratdenom (expr)`

Returns the denominator of `expr`, after coercing `expr` to a canonical rational expression (CRE). The return value is a CRE.

`expr` is coerced to a CRE by `rat` if it is not already a CRE. This conversion may change the form of `expr` by putting all terms over a common denominator.

`denom` is similar, but returns an ordinary expression instead of a CRE. Also, `denom` does not attempt to place all terms over a common denominator, and thus some expressions which are considered ratios by `ratdenom` are not considered ratios by `denom`.

`ratdenomdivide`

Default value: `true`

When `ratdenomdivide` is `true`, `ratexpand` expands a ratio in which the numerator is a sum into a sum of ratios, all having a common denominator. Otherwise, `ratexpand` collapses a sum of ratios into a single ratio, the numerator of which is the sum of the numerators of each ratio.

Examples:

\[
(%i1) expr: (x^2 + x + 1)/(y^2 + 7);
\]

\[
\frac{x + x + 1}{2}
\]

\[
(%i1) \quad \frac{x + x + 1}{y + 7}
\]

\[
(%i2) ratdenomdivide: true$
\]

\[
(%i2) \quad ratexpand (expr);
\]

\[
\frac{x \; \frac{1}{2} \; \frac{1}{2}}{y + 7 \; y + 7 \; y + 7}
\]
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(%i4) ratdenomdivide: false$
(%i5) ratexpand (expr);

2
 x + x + 1
-------
2
 y + 7
(%i6) expr2: a^-2/(b^-2 + 3) + b/(b^-2 + 3);

2
 b a
------ + ------
2 2
 b + 3 b + 3
(%i7) ratexpand (expr2);

2
 b + a
------
2
 b + 3

ratdiff (expr, x) [Function]
Differentiates the rational expression expr with respect to x. expr must be a ratio of polynomials or a polynomial in x. The argument x may be a variable or a subexpression of expr.

The result is equivalent to diff, although perhaps in a different form. ratdiff may be faster than diff, for rational expressions.

ratdiff returns a canonical rational expression (CRE) if expr is a CRE. Otherwise, ratdiff returns a general expression.

ratdiff considers only the dependence of expr on x, and ignores any dependencies established by depends.

Example:

(%i11) expr: (4*x^3 + 10*x - 11)/(x^5 + 5);

3
 4 x + 10 x - 11
-------
5
 x + 5
(%i12) ratdiff (expr, x);

7
 8 x + 40 x - 55 x - 60 x - 50
--------
10
 x + 10 x + 25
(%i13) expr: f(x)^3 - f(x)^2 + 7;

3
 f (x) - f (x) + 7
(%i4) ratdiff (expr, f(x));

(%o4) 3 f(x) - 2 f(x)

(%i5) expr: (a + b)^3 + (a + b)^2;

(%o5) (b + a)^3 + (b + a)^2

(%i6) ratdiff (expr, a + b);

(%o6) 3 b + (6 a + 2) b + 3 a + 2 a

ratdisrep (expr)

[Function]

Returns its argument as a general expression. If expr is a general expression, it is returned unchanged.

Typically ratdisrep is called to convert a canonical rational expression (CRE) into a general expression. This is sometimes convenient if one wishes to stop the "contagion", or use rational functions in non-rational contexts.

See also totaldisrep.

ratexpand (expr)

[Option variable]

ratexpand

Expands expr by multiplying out products of sums and exponentiated sums, combining fractions over a common denominator, cancelling the greatest common divisor of the numerator and denominator, then splitting the numerator (if a sum) into its respective terms divided by the denominator.

The return value of ratexpand is a general expression, even if expr is a canonical rational expression (CRE).

The switch ratexpand if true will cause CRE expressions to be fully expanded when they are converted back to general form or displayed, while if it is false then they will be put into a recursive form. See also ratsimp.

When ratdenomdivide is true, ratexpand expands a ratio in which the numerator is a sum into a sum of ratios, all having a common denominator. Otherwise, ratexpand collapses a sum of ratios into a single ratio, the numerator of which is the sum of the numerators of each ratio.

When keepfloat is true, prevents floating point numbers from being rationalized when expressions which contain them are converted to canonical rational expression (CRE) form.

Examples:

(%i11) ratexpand ((2*x - 3*y)^3);

(%o11) - 27 y + 54 x y - 36 x y + 8 x

(%i12) expr: (x - 1)/(x + 1)^2 + 1/(x - 1);

(%o12) (x - 1)/(x + 1)^2 + 1/(x - 1)

(%i13) expand (expr);
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(\%o3) \[\frac{x}{2} - \frac{1}{x - 1} + \frac{1}{x + 2x + 1} + \frac{1}{x + 2x + 1}\]

(%i4) \text{ratexpand (expr)};

\[\frac{2x^2}{3} + \frac{2}{3x + x - x - 1}\]

\text{ratfac} \hspace{1cm} \text{[Option variable]}

Default value: \text{false}

When \text{ratfac} is \text{true}, canonical rational expressions (CRE) are manipulated in a partially factored form.

During rational operations the expression is maintained as fully factored as possible without calling \text{factor}. This should always save space and may save time in some computations. The numerator and denominator are made relatively prime, for example \text{factor} \((x^2 - 1)^4/(x + 1)^2\) yields \((x - 1)^4 (x + 1)^2\), but the factors within each part may not be relatively prime.

In the \text{ctensr} (Component Tensor Manipulation) package, Ricci, Einstein, Riemann, and Weyl tensors and the scalar curvature are factored automatically when \text{ratfac} is \text{true}. \text{ratfac should only be set for cases where the tensorial components are known to consist of few terms.}

The \text{ratfac} and \text{ratweight} schemes are incompatible and may not both be used at the same time.

\text{ratnumer (expr)} \hspace{1cm} \text{[Function]}

Returns the numerator of \text{expr}, after coercing \text{expr} to a canonical rational expression (CRE). The return value is a CRE.

\text{expr} is coerced to a CRE by \text{rat} if it is not already a CRE. This conversion may change the form of \text{expr} by putting all terms over a common denominator.

\text{num} is similar, but returns an ordinary expression instead of a CRE. Also, \text{num} does not attempt to place all terms over a common denominator, and thus some expressions which are considered ratios by \text{ratnumer} are not considered ratios by \text{num}.

\text{ratp (expr)} \hspace{1cm} \text{[Function]}

Returns \text{true} if \text{expr} is a canonical rational expression (CRE) or extended CRE, otherwise \text{false}.

CRE are created by \text{rat} and related functions. Extended CRE are created by \text{taylor} and related functions.

\text{ratprint} \hspace{1cm} \text{[Option variable]}

Default value: \text{true}

When \text{ratprint} is \text{true}, a message informing the user of the conversion of floating point numbers to rational numbers is displayed.
**ratsimp (expr)**

**ratsimp (expr, x_1, ..., x_n)**

Simplifies the expression expr and all of its subexpressions, including the arguments to non-rational functions. The result is returned as the quotient of two polynomials in a recursive form, that is, the coefficients of the main variable are polynomials in the other variables. Variables may include non-rational functions (e.g., \( \sin (x^2 + 1) \)) and the arguments to any such functions are also rationally simplified.

**ratsimp (expr, x_1, ..., x_n)** enables rational simplification with the specification of variable ordering as in *ratvars*.

When **ratsimpexpons** is **true**, **ratsimp** is applied to the exponents of expressions during simplification.

See also **ratexpand**. Note that **ratsimp** is affected by some of the flags which affect **ratexpand**.

Examples:

\[
\begin{align*}
\text{(i1)} & \quad \sin \left( \frac{x}{x^2 + x} \right) = \exp \left( (\log(x) + 1)^2 - \log(x)^2 \right); \\
\text{(o1)} & \quad \sin(\frac{1}{x^2 + 1}) = e^x \\
\text{(i2)} & \quad \text{ratsimp (i1)}; \\
\text{(o2)} & \quad \sin(\frac{1}{x + 1}) = e^x \\
\text{(i3)} & \quad \frac{(x - 1)^{3/2} - (x + 1)\sqrt{x - 1}}{\sqrt{(x - 1)(x + 1)}}; \\
\text{(o3)} & \quad -\frac{2\sqrt{x - 1}}{\sqrt{x + 1}} \\
\text{(i4)} & \quad \text{ratsimp (i3)}; \\
\text{(o4)} & \quad \frac{2\sqrt{x - 1}}{\sqrt{x + 1}} \\
\text{(i5)} & \quad x^{a + 1/a}, \text{ratsimpexpons: true}; \\
\text{(o5)} & \quad x^{a + 1/a}
\end{align*}
\]

**ratsimpexpons**

Default value: **false**

When **ratsimpexpons** is **true**, **ratsimp** is applied to the exponents of expressions during simplification.
radsubstflag

[Option variable]
Default value: false
radsubstflag, if true, permits ratsubst to make substitutions such as u for sqrt (x) in x.

ratsubst (a, b, c)

[Function]
Substitutes a for b in c and returns the resulting expression. b may be a sum, product, power, etc.
ratsubst knows something of the meaning of expressions whereas subst does a purely syntactic substitution. Thus subst (a, x + y, x + y + z) returns x + y + z whereas ratsubst returns z + a.
When radsubstflag is true, ratsubst makes substitutions for radicals in expressions which don’t explicitly contain them.
ratsubst ignores the value true of the option variable keepfloat.

Examples:
(%i1) ratsubst (a, x*y^2, x^4*y^3 + x^4*y^8);
  3  4
(%o1) a x y + a
(%i2) cos(x)^4 + cos(x)^3 + cos(x)^2 + cos(x) + 1;
   4  3  2
(%o2) cos (x) + cos (x) + cos (x) + cos(x) + 1
(%i3) ratsubst (1 - sin(x)^2, cos(x)^2, %);
   4  2
(%o3) sin (x) - 3 sin (x) + cos(x) (2 - sin (x)) + 3
(%i4) ratsubst (1 - cos(x)^2, sin(x)^2, sin(x)^4);
   2
(%o4) cos (x) - 2 cos (x) + 1
(%i5) radsubstflag: false$
(%i6) ratsubst (u, sqrt(x), x);
(%o6) x
(%i7) radsubstflag: true$
(%i8) ratsubst (u, sqrt(x), x);
   2
(%o8) u

ratvars (x_1, ..., x_n)

[Function]
ratvars ()

[Function]
ratvars

[System variable]
Declares main variables x_1, ..., x_n for rational expressions. x_n, if present in a rational expression, is considered the main variable. Otherwise, x_[n-1] is considered the main variable if present, and so on through the preceding variables to x_1, which is considered the main variable only if none of the succeeding variables are present.
If a variable in a rational expression is not present in the ratvars list, it is given a lower priority than x_1.
The arguments to ratvars can be either variables or non-rational functions such as sin(x).
The variable \texttt{ratvars} is a list of the arguments of the function \texttt{ratvars} when it was called most recently. Each call to the function \texttt{ratvars} resets the list. \texttt{ratvars()} clears the list.

\texttt{ratvarswitch} \hspace{1cm} \text{[Option variable]}
Default value: \texttt{true}

Maxima keeps an internal list in the Lisp variable \texttt{VARLIST} of the main variables for rational expressions. If \texttt{ratvarswitch} is \texttt{true}, every evaluation starts with a fresh list \texttt{VARLIST}. This is the default behavior. Otherwise, the main variables from previous evaluations are not removed from the internal list \texttt{VARLIST}.

The main variables, which are declared with the function \texttt{ratvars} are not affected by the option variable \texttt{ratvarswitch}.

Examples:

If \texttt{ratvarswitch} is \texttt{true}, every evaluation starts with a fresh list \texttt{VARLIST}.

\begin{verbatim}
(%i1) ratvarswitch:true$
(%i2) rat(2*x+y^2);
  2
(%o2)/R/ y + 2 x
(%i3) :lisp varlist
($X $Y)
(%i3) rat(2*a+b^2);
  2
(%o3)/R/ b + 2 a
(%i4) :lisp varlist
($A $B)
\end{verbatim}

If \texttt{ratvarswitch} is \texttt{false}, the main variables from the last evaluation are still present.

\begin{verbatim}
(%i4) ratvarswitch:false$
(%i5) rat(2*x+y^2);
  2
(%o5)/R/ y + 2 x
(%i6) :lisp varlist
($X $Y)
(%i6) rat(2*a+b^2);
  2
(%o6)/R/ b + 2 a
(%i7) :lisp varlist
($A $B $X $Y)
\end{verbatim}
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ratweight  

\[
\text{ratweight} \ (x_1, w_1, \ldots, x_n, w_n)  
\]
\[
\text{ratweight} ()  
\]

Assigns a weight \( w_i \) to the variable \( x_i \). This causes a term to be replaced by 0 if its weight exceeds the value of the variable \( \text{ratwtlvl} \) (default yields no truncation). The weight of a term is the sum of the products of the weight of a variable in the term times its power. For example, the weight of \( 3 \ x_1^2 \ x_2 \) is \( 2 \ w_1 + w_2 \). Truncation according to \( \text{ratwtlvl} \) is carried out only when multiplying or exponentiating canonical rational expressions (CRE).

\( \text{ratweight} () \) returns the cumulative list of weight assignments.

Note: The \( \text{ratfac} \) and \( \text{ratweight} \) schemes are incompatible and may not both be used at the same time.

Examples:

\[
\begin{align*}
\%(i1) \ &\text{ratweight} \ (a, 1, b, 1); \\
\%(o1) \ &\{a, 1, b, 1\} \\
\%(i2) \ &\text{expr1: rat}(a + b + 1) \\
\%(i3) \ &\text{expr1}^2; \\
\%(o3)/R/ \ &b + (2 \ a + 2) \ b + a + 2 \ a + 1 \\
\%(i4) \ &\text{ratwtlvl}: 1$ \\
\%(i5) \ &\text{expr1}^2; \\
\%(o5)/R/ \ &2 \ b + 2 \ a + 1 
\end{align*}
\]

ratweights  

[System variable]

Default value: []

\( \text{ratweights} \) is the list of weights assigned by \( \text{ratweight} \). The list is cumulative: each call to \( \text{ratweight} \) places additional items in the list.

\( \text{kill} (\text{ratweights}) \) and \( \text{save} (\text{ratweights}) \) both work as expected.

ratwtlvl  

[Option variable]

Default value: false

\( \text{ratwtlvl} \) is used in combination with the \( \text{ratweight} \) function to control the truncation of canonical rational expressions (CRE). For the default value of false, no truncation occurs.

remainder  

[Function]

\[
\text{remainder} \ (p_1, p_2)  
\]
\[
\text{remainder} \ (p_1, p_2, x_1, \ldots, x_n)  
\]

Returns the remainder of the polynomial \( p_1 \) divided by the polynomial \( p_2 \). The arguments \( x_1, \ldots, x_n \) are interpreted as in \( \text{ratvars} \).

\( \text{remainder} \) returns the second element of the two-element list returned by \( \text{divide} \).

resultant \( (p_1, p_2, x) \)  

[Function]

The function \( \text{resultant} \) computes the resultant of the two polynomials \( p_1 \) and \( p_2 \), eliminating the variable \( x \). The resultant is a determinant of the coefficients of \( x \) in
and \( p_1 \) and \( p_2 \), which equals zero if and only if \( p_1 \) and \( p_2 \) have a non-constant factor in common. If \( p_1 \) or \( p_2 \) can be factored, it may be desirable to call \texttt{factor} before calling \texttt{resultant}.

The option variable \texttt{resultant} controls which algorithm will be used to compute the resultant. See the option variable \texttt{[option_resultant], page 272},.

The function \texttt{bezout} takes the same arguments as \texttt{resultant} and returns a matrix. The determinant of the return value is the desired resultant.

Examples:

\begin{verbatim}
(%i1) resultant(2*x^2+3*x+1, 2*x^2+x+1, x);  
(%o1) 8  
(%i2) resultant(x+1, x+1, x);  
(%o2) 0  
(%i3) resultant((x+1)*x, (x+1), x);  
(%o3) 0  
(%i4) resultant(a*x^2+b*x+1, c*x + 2, x);  
2                      
(%o4) c - 2 b c + 4 a  

(%i5) bezout(a*x^2+b*x+1, c*x+2, x);  
\begin{bmatrix}
 2 a & 2 b - c \\
  c  & 2
\end{bmatrix}  

(%i6) determinant(%);  
(%o6) 4 a - (2 b - c) c
\end{verbatim}

\texttt{resultant}  
[Option variable]
Default value: \texttt{subres}

The option variable \texttt{resultant} controls which algorithm will be used to compute the resultant with the function \texttt{resultant}. The possible values are:

- \texttt{subres} for the subresultant polynomial remainder sequence (PRS) algorithm,
- \texttt{mod} (not enabled) for the modular resultant algorithm, and
- \texttt{red} for the reduced polynomial remainder sequence (PRS) algorithm.

On most problems the default value \texttt{subres} should be best.

\texttt{savefactors}  
[Option variable]
Default value: \texttt{false}

When \texttt{savefactors} is \texttt{true}, causes the factors of an expression which is a product of factors to be saved by certain functions in order to speed up later factorizations of expressions containing some of the same factors.

\texttt{showratvars (expr)}  
[Function]
Returns a list of the canonical rational expression (CRE) variables in expression \texttt{expr}. See also \texttt{ratvars}.
**sqfr** (expr)  
[Function]  
is similar to **factor** except that the polynomial factors are "square-free." That is, they have factors only of degree one. This algorithm, which is also used by the first stage of **factor**, utilizes the fact that a polynomial has in common with its n'th derivative all its factors of degree greater than n. Thus by taking greatest common divisors with the polynomial of the derivatives with respect to each variable in the polynomial, all factors of degree greater than 1 can be found.

Example:

(%i1) sqfr (4*x^4 + 4*x^3 - 3*x^2 - 4*x - 1);
     2   2
(%o1) (2 x + 1) (x - 1)

**tellrat**  
[Function]

tellrat (p_1, ..., p_n)  
tellrat ()

Adds to the ring of algebraic integers known to Maxima the elements which are the solutions of the polynomials p_1, ..., p_n. Each argument p_i is a polynomial with integer coefficients.

tellrat (x) effectively means substitute 0 for x in rational functions.

tellrat () returns a list of the current substitutions.

**algebraic** must be set to true in order for the simplification of algebraic integers to take effect.

Maxima initially knows about the imaginary unit %i and all roots of integers.

There is a command **untellrat** which takes kernels and removes **tellrat** properties.

When **tellrat**'ing a multivariate polynomial, e.g., **tellrat** (x^2 - y^2), there would be an ambiguity as to whether to substitute y^2 for x^2 or vice versa. Maxima picks a particular ordering, but if the user wants to specify which, e.g. **tellrat** (y^2 = x^2) provides a syntax which says replace y^2 by x^2.

Examples:

(%i1) 10*(%i + 1)/(%i + 3^(1/3));
      10 (%i + 1)
(%o1) -------------------
      1/3

(%i2) ev (ratdisrep (rat(%)), algebraic);
      1/3
(4 3 - 2 3 - 4) %i + 2 3 + 4 3 - 2
(%o2) (2/3 1/3 2/3 1/3)

(%i3) tellrat (1 + a + a^2);
     2
(%o3) [a + a + 1]

(%i4) 1/(a*sqrt(2) - 1) + a/(sqrt(3) + sqrt(2));
      1 a
(%o4) ------------------- + -------------------
    sqrt(2) a - 1 sqrt(3) + sqrt(2)

(%i5) ev (ratdisrep (rat(%)), algebraic);
\[
(7 \sqrt{3} - 10 \sqrt{2} + 2) a - 2 \sqrt{2} - 1
\]

\[
(%o5) \frac{7}{7}
\]

\[
(%i6) \text{tellrat} (y^2 = x^2);
\]

\[
2 2 2
\]

\[
(%o6) [y - x, a + a + 1]
\]

totaldisrep (expr) [Function]
Converting every subexpression of \(expr\) from canonical rational expressions (CRE) to general form and returns the result. If \(expr\) is itself in CRE form then totaldisrep is identical to ratdisrep.
totaldisrep may be useful for ratdisrepping expressions such as equations, lists, matrices, etc., which have some subexpressions in CRE form.

untellrat (x_1, \ldots, x_n) [Function]
Removes \text{tellrat} properties from \(x_1, \ldots, x_n\).
15 Special Functions

15.1 Introduction to Special Functions

Special function notation follows:

- \texttt{bessel\_j (index, expr)}: Bessel function, 1st kind
- \texttt{bessel\_y (index, expr)}: Bessel function, 2nd kind
- \texttt{bessel\_i (index, expr)}: Modified Bessel function, 1st kind
- \texttt{bessel\_k (index, expr)}: Modified Bessel function, 2nd kind
- \texttt{hankel\_1 (v,z)}: Hankel function of the 1st kind
- \texttt{hankel\_2 (v,z)}: Hankel function of the 2nd kind
- \texttt{struve\_h (v,z)}: Struve H function
- \texttt{struve\_l (v,z)}: Struve L function
- \texttt{assoc\_legendre\_p[v,u] (z)}: Legendre function of degree \(v\) and order \(u\)
- \texttt{assoc\_legendre\_q[v,u] (z)}: Legendre function, 2nd kind
- \texttt{%f[p,q] ([], [], expr)}: Generalized Hypergeometric function
- \texttt{gamma (z)}: Gamma function
- \texttt{gamma\_incomplete\_lower (a,z)}: Lower incomplete gamma function
- \texttt{gamma\_incomplete (a,z)}: Tail of incomplete gamma function
- \texttt{hypergeometric (l1, l2, z)}: Hypergeometric function
- \texttt{slommel}:
- \texttt{%m[u,k] (z)}: Whittaker function, 1st kind
- \texttt{%w[u,k] (z)}: Whittaker function, 2nd kind
- \texttt{erfc (z)}: Complement of the erf function
- \texttt{expintegral\_e (v,z)}: Exponential integral E
- \texttt{expintegral\_el (z)}: Exponential integral E1
- \texttt{expintegral\_ei (z)}: Exponential integral Ei
- \texttt{expintegral\_li (z)}: Logarithmic integral Li
- \texttt{expintegral\_si (z)}: Exponential integral Si
- \texttt{expintegral\_ci (z)}: Exponential integral Ci
- \texttt{expintegral\_shi (z)}: Exponential integral Shi
- \texttt{expintegral\_chi (z)}: Exponential integral Chi
- \texttt{kelliptic (z)}: Complete elliptic integral of the first kind (\(K\))
- \texttt{parabolic\_cylinder\_d (v,z)}: Parabolic cylinder D function

15.2 Bessel Functions

\texttt{bessel\_j (v, z)}

The Bessel function of the first kind of order \(v\) and argument \(z\).

\texttt{bessel\_j} is defined as
\[
\sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{z}{2}\right)^{v+2k}}{k! \Gamma(v+k+1)}
\]

although the infinite series is not used for computations.

**bessel_y (v, z)**

The Bessel function of the second kind of order \(v\) and argument \(z\).

*bessel_y* is defined as

\[
\cos(\pi v) J_v(z) - J_{-v}(z)
\]

\[
\sin(\pi v)
\]

when \(v\) is not an integer. When \(v\) is an integer \(n\), the limit as \(v\) approaches \(n\) is taken.

**bessel_i (v, z)**

The modified Bessel function of the first kind of order \(v\) and argument \(z\).

*bessel_i* is defined as

\[
\sum_{k=0}^{\infty} \frac{1}{k! \Gamma(v+k+1)} \left(\frac{z}{2}\right)^{v+2k}
\]

although the infinite series is not used for computations.

**bessel_k (v, z)**

The modified Bessel function of the second kind of order \(v\) and argument \(z\).

*bessel_k* is defined as

\[
\pi \csc(\pi v) \left( I_{-v}(z) - I_v(z) \right)
\]

when \(v\) is not an integer. If \(v\) is an integer \(n\), then the limit as \(v\) approaches \(n\) is taken.

**hankel_1 (v, z)**

The Hankel function of the first kind of order \(v\) and argument \(z\) (A&S 9.1.3). hankel_1 is defined as

\[
bessel_j(v,z) + %i * bessel_y(v,z)
\]

Maxima evaluates hankel_1 numerically for a complex order \(v\) and complex argument \(z\) in float precision. The numerical evaluation in bigfloat precision is not supported.

When besselexpand is true, hankel_1 is expanded in terms of elementary functions when the order \(v\) is half of an odd integer. See besselexpand.

Maxima knows the derivative of hankel_1 wrt the argument \(z\).

Examples:

Numerical evaluation:

(%i1) hankel_1(1, 0.5);
(%o1) 0.24226845767487 - 1.471472392670243 %i
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(%i2) hankel_1(1,0.5+%i);
(%o2) - 0.25582879948621 %i - 0.23957560188301

Expansion of hankel_1 when besselexpand is true:
(%i1) hankel_1(1/2,z),besselexpand:true;
   sqrt(2) sin(z) - sqrt(2) %i cos(z)
(%o1) ----------------------------------
   sqrt(%pi) sqrt(z)

Derivative of hankel_1 wrt the argument z. The derivative wrt the order v is not supported. Maxima returns a noun form:
(%i1) diff(hankel_1(v,z),z);
   hankel_1(v - 1, z) - hankel_1(v + 1, z)
(%o1) ---------------------------------------
          2
(%i2) diff(hankel_1(v,z),v);
   d
(%o2) -- (hankel_1(v, z))
   dv

hankel_2 (v, z) [Function]

The Hankel function of the second kind of order v and argument z (A&S 9.1.4). hankel_2 is defined as
bessel_j(v,z) - %i * bessel_y(v,z)

Maxima evaluates hankel_2 numerically for a complex order v and complex argument z in float precision. The numerical evaluation in bigfloat precision is not supported.

When besselexpand is true, hankel_2 is expanded in terms of elementary functions when the order v is half of an odd integer. See besselexpand.

Maxima knows the derivative of hankel_2 wrt the argument z.

For examples see hankel_1.

besselexpand [Option variable]

Default value: false

Controls expansion of the Bessel functions when the order is half of an odd integer. In this case, the Bessel functions can be expanded in terms of other elementary functions. When besselexpand is true, the Bessel function is expanded.

(%i1) besselexpand: false$
(%i2) bessel_j (3/2, z);
   3
(%o2) bessel_j(-, z)
    2
(%i3) besselexpand: true$
(%i4) bessel_j (3/2, z);
   sin(z)   cos(z)
  sqrt(2) sqrt(z) (-- --- - ---)
       2     z

(%i5) besselexpand: false$
(%i6) bessel_j (3/2, z);
   3
(%o6) bessel_j(-, z)
    2
sqrt(%pi)

scaled_bessel_i (v, z)

The scaled modified Bessel function of the first kind of order \( v \) and argument \( z \). That is, \( \text{scaled}\text{\textunderscore}bessel\_i(v, z) = \exp(-\text{abs}(z)) \times \text{bessel}\_i(v, z) \). This function is particularly useful for calculating \text{bessel}\_i for large \( z \), which is large. However, maxima does not otherwise know much about this function. For symbolic work, it is probably preferable to work with the expression \( \exp(-\text{abs}(z)) \times \text{bessel}\_i(v, z) \).

scaled_bessel_i0 (z)

Identical to \( \text{scaled}\text{\textunderscore}bessel\_i(0,z) \).

scaled_bessel_i1 (z)

Identical to \( \text{scaled}\text{\textunderscore}bessel\_i(1,z) \).

%\text{s} [u,v] (z)

Lommel's little \( s[u,v](z) \) function. Probably Gradshteyn & Ryzhik 8.570.1.

### 15.3 Airy Functions

The Airy functions \( \text{Ai}(x) \) and \( \text{Bi}(x) \) are defined in Abramowitz and Stegun, *Handbook of Mathematical Functions*, Section 10.4.

\[ y = \text{Ai}(x) \text{ and } y = \text{Bi}(x) \text{ are two linearly independent solutions of the Airy differential equation } \text{diff}(y(x), x, 2) - x \, y(x) = 0. \]

If the argument \( x \) is a real or complex floating point number, the numerical value of the function is returned.

airy_ai (x)

The Airy function \( \text{Ai}(x) \). (A&S 10.4.2)

The derivative \( \text{diff}(\text{airy}\_ai(x), x) \) is \( \text{airy}\_dai(x) \).

See also \( \text{airy}\_bi, \text{airy}\_dai, \text{airy}\_dbi \).

airy_dai (x)

The derivative of the Airy function \( \text{Ai} \) \( \text{airy}\_ai(x) \).

See \( \text{airy}\_ai \).

airy_bi (x)

The Airy function \( \text{Bi}(x) \). (A&S 10.4.3)

The derivative \( \text{diff}(\text{airy}\_bi(x), x) \) is \( \text{airy}\_dbi(x) \).

See \( \text{airy}\_ai, \text{airy}\_dbi \).

airy_db (x)

The derivative of the Airy Bi function \( \text{airy}\_bi(x) \).

See \( \text{airy}\_ai \) and \( \text{airy}\_bi \).
15.4 Gamma and factorial Functions

The gamma function and the related beta, psi and incomplete gamma functions are defined in Abramowitz and Stegun, *Handbook of Mathematical Functions*, Chapter 6.

**bffac (expr, n)**
Bigfloat version of the factorial (shifted gamma) function. The second argument is how many digits to retain and return, it’s a good idea to request a couple of extra.

**bfpsi (n, z, fpprec)**
**bfpsi0 (z, fpprec)**
`bfpsi` is the polygamma function of real argument `z` and integer order `n`. `bfpsi0` is the digamma function. `bfpsi0 (z, fpprec)` is equivalent to `bfpsi (0, z, fpprec)`. These functions return bigfloat values. `fpprec` is the bigfloat precision of the return value.

**cbffac (z, fpprec)**
Complex bigfloat factorial.
`load ("bffac")` loads this function.

**gamma (z)**
The basic definition of the gamma function (A&S 6.1.1) is

$$
\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt
$$

Maxima simplifies `gamma` for positive integer and positive and negative rational numbers. For half integral values the result is a rational number times `sqrt(%pi)`. The simplification for integer values is controlled by `factlim`. For integers greater than `factlim` the numerical result of the factorial function, which is used to calculate `gamma`, will overflow. The simplification for rational numbers is controlled by `gammalim` to avoid internal overflow. See `factlim` and `gammalim`.

For negative integers `gamma` is not defined.

Maxima can evaluate `gamma` numerically for real and complex values in float and bigfloat precision.

`gamma` has mirror symmetry.

When `gamma_expand` is `true`, Maxima expands `gamma` for arguments `z+n` and `z-n` where `n` is an integer.

Maxima knows the derivate of `gamma`.

Examples:

Simplification for integer, half integral, and rational numbers:

```
(%i1) map('gamma,[1,2,3,4,5,6,7,8,9]);
(%o1) [1, 1, 2, 6, 24, 120, 720, 5040, 40320]
(%i2) map('gamma,[1/2,3/2,5/2,7/2]);
```

```
(%o2) [sqrt(%pi), 3 sqrt(%pi), 15 sqrt(%pi), 2 sqrt(%pi)]
```

```
   2  4  8
```
(%i3) map('gamma,[2/3,5/3,7/3]);
2 1
2 gamma(-) 4 gamma(-)  
2 3 3
(%o3) [gamma(-), ----------, ----------]
3 3 9

Numerical evaluation for real and complex values:

(%i4) map('gamma,[2.5,2.5b0]);
(%o4) [1.329340388179137, 1.3293403881791370205b0]

(%i5) map('gamma,[1.0+%i,1.0b0+%i]);
(%o5) [0.498015668118356 - .1549498283018107 %i,
  4.9801566811835604272b-1 - 1.5494982830181068513b-1 %i]

gamma has mirror symmetry:

(%i6) declare(z,complex)$
(%i7) conjugate(gamma(z));
gamma(conjugate(z))

Maxima expands \( \gamma(z+n) \) and \( \gamma(z-n) \), when \( \gamma\_\text{expand} \) is true:

(%i8) gamma\_expand:true$

(%i9) [gamma(z+1),gamma(z-1),gamma(z+2)/gamma(z+1)];
   gamma(z)  
   [z gamma(z), ----------, z + 1]
   z - 1

The derivative of gamma:

(%i10) diff(gamma(z),z);  
(%o10) psi (z) gamma(z)
0

See also makegamma.

The Euler-Mascheroni constant is \( \gamma \).

log\_gamma (z) [Function]
The natural logarithm of the gamma function.

gamma\_incomplete\_lower (a, z) [Function]
The lower incomplete gamma function (A&S 6.5.2):

\[
\gamma(a,z) = \int_0^z t^{a-1} e^{-t} \, dt
\]

See also gamma\_incomplete (upper incomplete gamma function).

gamma\_incomplete (a, z) [Function]
The incomplete upper gamma function (A&S 6.5.3):

\[
\Gamma(a,z) = \int_z^\infty t^{a-1} e^{-t} \, dt
\]
See also \texttt{gamma\_expand} for controlling how \texttt{gamma\_incomplete} is expressed in terms of elementary functions and \texttt{erfc}.

Also see the related functions \texttt{gamma\_incomplete\_regularized} and \texttt{gamma\_incomplete\_generalized}.

\textbf{gamma\_incomplete\_regularized} \((a, z)\) \hfill \textbf{[Function]}

The regularized incomplete upper gamma function (A&S 6.5.1):

\[
Q(a, z) = \frac{\Gamma(a, z)}{\Gamma(a)}
\]

See also \texttt{gamma\_expand} for controlling how \texttt{gamma\_incomplete} is expressed in terms of elementary functions and \texttt{erfc}.

Also see \texttt{gamma\_incomplete}.

\textbf{gamma\_incomplete\_generalized} \((a, z1, z1)\) \hfill \textbf{[Function]}

The generalized incomplete gamma function.

\[
\Gamma(a, z1, z2) = \int_{z1}^{z2} t^{a-1} e^{-t} \, dt
\]

Also see \texttt{gamma\_incomplete} and \texttt{gamma\_incomplete\_regularized}.

\textbf{gamma\_expand} \hfill \textbf{[Option variable]}

Default value: \texttt{false}

\texttt{gamma\_expand} controls expansion of \texttt{gamma\_incomplete}. When \texttt{gamma\_expand} is \texttt{true}, \texttt{gamma\_incomplete}(v,z) is expanded in terms of \(z, \exp(z),\) and \texttt{erfc}(z) when possible.

\begin{verbatim}
(%i1) gamma_incomplete(2,z);
(%o1) gamma_incomplete(2, z)
(%i2) gamma_expand:true;
(%o2) true
(%i3) gamma_incomplete(2,z);
   - z
(%o3) (z + 1) %e
(%i4) gamma_incomplete(3/2,z);
   - z sqrt(%pi) erfc(sqrt(z))
(%o4) sqrt(z) %e + -----------------------
     2
\end{verbatim}

\textbf{gammalim} \hfill \textbf{[Option variable]}

Default value: 10000

\texttt{gammalim} controls simplification of the gamma function for integral and rational number arguments. If the absolute value of the argument is not greater than \texttt{gammalim}, then simplification will occur. Note that the \texttt{factlim} switch controls simplification of the result of \texttt{gamma} of an integer argument as well.
**makegamma** (expr)  
[Function]  
Transforms instances of binomial, factorial, and beta functions in expr into gamma functions.  
See also **makefact**.

**beta** (a, b)  
[Function]  
The beta function is defined as \( \frac{\text{gamma}(a) \cdot \text{gamma}(b)}{\text{gamma}(a+b)} \) (A&S 6.2.1). Maxima simplifies the beta function for positive integers and rational numbers, which sum to an integer. When **beta_args_sum_to_integer** is true, Maxima simplifies also general expressions which sum to an integer.  
For \( a \) or \( b \) equal to zero the beta function is not defined.  
In general the beta function is not defined for negative integers as an argument. The exception is for \( a=-n \), \( n \) a positive integer and \( b \) a positive integer with \( b<=n \), it is possible to define an analytic continuation. Maxima gives for this case a result.  
When **beta_expand** is true, expressions like \( \text{beta}(a+n,b) \) and \( \text{beta}(a-n,b) \) or \( \text{beta}(a,b+n) \) and \( \text{beta}(a,b-n) \) with \( n \) an integer are simplified.  
Maxima can evaluate the beta function for real and complex values in float and bigfloat precision. For numerical evaluation Maxima uses \( \log\_\gamma \):  
\[
- \log\_\gamma(b + a) + \log\_\gamma(b) + \log\_\gamma(a)
\]
\%e  
Maxima knows that the beta function is symmetric and has mirror symmetry. Maxima knows the derivatives of the beta function with respect to \( a \) or \( b \).  
To express the beta function as a ratio of gamma functions see **makegamma**.

Examples:

Simplification, when one of the arguments is an integer:

\[
(\text{i1}) \quad [\text{beta}(2,3),\text{beta}(2,1/3),\text{beta}(2,a)];
\]
\[
1 \quad 9 \quad 1
\]
\[
--- \quad -- \quad ---------
\]
\[
12 \quad 4 \quad a \quad (a + 1)
\]

Simplification for two rational numbers as arguments which sum to an integer:

\[
(\text{i2}) \quad [\text{beta}(1/2,5/2),\text{beta}(1/3,2/3),\text{beta}(1/4,3/4)];
\]
\[
3 \quad \%\pi \quad 2 \quad \%\pi
\]
\[
----- \quad ------- \quad sqrt(2) \quad \%\pi
\]
\[
8 \quad sqrt(3)
\]

When setting **beta_args_sum_to_integer** to true more general expression are simplified, when the sum of the arguments is an integer:

\[
(\text{i3}) \quad \text{beta_args_sum_to_integer}:\text{true}$
\]
\[
(\text{i4}) \quad \text{beta}(a+1,-a+2);
\]
\[
\%\pi \quad (a - 1) \quad a
\]
\[
-----------
\]
\[
2 \quad \sin(\%\pi \quad (2 - a))
\]

The possible results, when one of the arguments is a negative integer:

\[
(\text{i5}) \quad [\text{beta}(-3,1),\text{beta}(-3,2),\text{beta}(-3,3)];
\]
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beta(a+n,b) or beta(a-n) with n an integer simplifies when beta_expand is true:

(beta(a+1,b), beta(a-1,b), beta(a+1,b)/beta(a,b+1));

\[- \frac{a \beta(a, b) \beta(a, b) (b + a - 1)}{b + a} \quad \frac{a}{a - 1} \quad \frac{-}{b} \]

beta is not defined, when one of the arguments is zero:

beta(0,b);

-- an error. To debug this try debugmode(true);

Numerical evaluation for real and complex arguments in float or bigfloat precision:

beta(2.5,2.3);

beta(2.5,1.4+%i);

beta(2.5b0,2.3b0);

beta(2.5b0,1.4b0+%i);

Beta is symmetric and has mirror symmetry:

beta(a,b)-beta(b,a);

beta(conjugate(a), conjugate(b));

The derivative of the beta function wrt a:

diff(beta(a,b),a);

\[- \frac{\beta(a, b) (\psi(b + a) - \psi(a))}{b + a} \quad \frac{0}{0} \]

beta_incomplete (a, b, z)

The basic definition of the incomplete beta function (A&S 6.6.1) is

\[
\frac{z}{b - 1} \quad \frac{a - 1}{a} \quad \frac{I}{1 - t} \quad \frac{t}{dt} \quad \frac{0}{0}
\]
This definition is possible for \( \text{realpart}(a) > 0 \) and \( \text{realpart}(b) > 0 \) and \( \text{abs}(z) < 1 \).
For other values the incomplete beta function can be defined through a generalized hypergeometric function:

\[
\text{gamma}(a) \ \text{hypergeometric}_{\text{generalized}}([a, 1 - b], [a + 1], z) \ z
\]

(See functions.wolfram.com for a complete definition of the incomplete beta function.)

For negative integers \( a = -n \) and positive integers \( b = m \) with \( m \leq n \) the incomplete beta function is defined through

\[
\sum_{k=0}^{m-1} \frac{n-1}{k} \left( 1 - m \right) \frac{z^k}{k! \ (n-k)}
\]

Maxima uses this definition to simplify \( \text{beta}_\text{incomplete} \) for \( a \) a negative integer.

For \( a \) a positive integer, \( \text{beta}_\text{incomplete} \) simplifies for any argument \( b \) and \( z \) and for \( b \) a positive integer for any argument \( a \) and \( z \), with the exception of \( a \) a negative integer.

For \( z = 0 \) and \( \text{realpart}(a) > 0 \), \( \text{beta}_\text{incomplete} \) has the specific value zero. For \( z=1 \) and \( \text{realpart}(b) > 0 \), \( \text{beta}_\text{incomplete} \) simplifies to the beta function \( \text{beta}(a,b) \).

Maxima evaluates \( \text{beta}_\text{incomplete} \) numerically for real and complex values in float or bigfloat precision. For the numerical evaluation an expansion of the incomplete beta function in continued fractions is used.

When the option variable \( \text{beta}_\text{expand} \) is true, Maxima expands expressions like \( \text{beta}_\text{incomplete}(a+n,b,z) \) and \( \text{beta}_\text{incomplete}(a-n,b,z) \) where \( n \) is a positive integer.

Maxima knows the derivatives of \( \text{beta}_\text{incomplete} \) with respect to the variables \( a \), \( b \) and \( z \) and the integral with respect to the variable \( z \).

Examples:

Simplification for \( a \) a positive integer:

\[
\%i1) \ \text{beta}_\text{incomplete}(2,b,z); \text{\hspace{1cm}} b \\
\hspace{2cm} 1 - (1 - z) (b (z + 1)) \hspace{1cm} \text{\hspace{1cm}} b (b + 1) \\
\%o1) \hspace{2cm} \frac{1 - (1 - z) (b z + 1)}{b (b + 1)}
\]

Simplification for \( b \) a positive integer:

\[
\%i2) \ \text{beta}_\text{incomplete}(a,2,z); \text{\hspace{1cm}} a \\
\hspace{2cm} (a (1 - z) + 1) z \hspace{1cm} \text{\hspace{1cm}} a (a + 1) \\
\%o2) \hspace{2cm} \frac{(a (1 - z) + 1) z}{a (a + 1)}
\]

Simplification for \( a \) and \( b \) a positive integer:

\[
\%i3) \ \text{beta}_\text{incomplete}(3,2,z);
\]
\[
\frac{3}{12} \left( \frac{3 (1 - z) + 1}{z} \right)
\]

\(a\) is a negative integer and \(b \leq (-a)\), Maxima simplifies:

\((\%i4)\) \text{beta\_incomplete(-3, 1, z);}

\[
\frac{1}{3} \frac{1}{z}
\]

For the specific values \(z = 0\) and \(z = 1\), Maxima simplifies:

\((\%i5)\) assume(a>0, b>0)$

\((\%i6)\) \text{beta\_incomplete(a, b, 0);}

\((\%o6)\) 0

\((\%i7)\) \text{beta\_incomplete(a, b, 1);}

\((\%o7)\) \text{beta(a, b)}

Numerical evaluation in float or bigfloat precision:

\((\%i8)\) \text{beta\_incomplete(0.25, 0.50, 0.9);}

\((\%o8)\) 4.594959440269333

\((\%i9)\) fpprec:25$

\((\%i10)\) \text{beta\_incomplete(0.25, 0.50, 0.9b0);}

\((\%o10)\) 4.594959440269324086971203b0

For \(\text{abs}(z) > 1\) \text{beta\_incomplete} returns a complex result:

\((\%i11)\) \text{beta\_incomplete(0.25, 0.50, 1.7);}

\((\%o11)\) 5.244115108584249 - 1.4551807787844 %i

Results for more general complex arguments:

\((\%i14)\) \text{beta\_incomplete(0.25+\%i, 1.0+\%i, 1.7+\%i);}

\((\%o14)\) 2.726960675662536 - .3831753704269199 %i

\((\%i15)\) \text{beta\_incomplete(1/2, 5/4*\%i, 2.8+\%i);}

\((\%o15)\) 13.04649635168716 %i - 5.802067956270001

Expansion, when \text{beta\_expand} is true:

\((\%i23)\) \text{beta\_incomplete(a+1, b, z),beta\_expand:true;}

\[
\frac{b a}{b + a} \frac{a \text{beta\_incomplete(a, b, z)} (1 - z) z}{b + a} - \frac{b a}{b + a}
\]

\((\%i24)\) \text{beta\_incomplete(a-1, b, z),beta\_expand:true;}

\[
\frac{b a - 1}{1 - a} \frac{\text{beta\_incomplete(a, b, z)} (- b - a + 1) (1 - z) z}{1 - a} - \frac{b a - 1}{1 - a}
\]
Derivative and integral for \( \text{beta}_\text{incomplete} \):

\[
\text{(%i34) } \text{diff}(\text{beta}_\text{incomplete}(a, b, z), z); \\
\qquad b - 1 \quad a - 1 \\
\text{(%o34)} \quad (1 - z) \quad z \\
\text{(%i35) } \text{integrate}(\text{beta}_\text{incomplete}(a, b, z), z); \\
\qquad b \quad a \\
\qquad (1 - z) \quad z \\
\text{(%o35)} \quad \frac{b \cdot a}{(b + a)} + \text{beta}_\text{incomplete}(a, b, z) \cdot z \\
\quad \quad - \frac{a \text{ beta}_\text{incomplete}(a, b, z)}{b + a} \\
\text{(%i36) } \text{factor(diff(\%, z));} \\
\text{(%o36)} \quad \text{beta}_\text{incomplete}(a, b, z) \\
\text{beta}_\text{incomplete}_\text{regularized} (a, b, z) \quad \text{[Function]} \\
\text{The regularized incomplete beta function (A&S 6.6.2), defined as} \\
\text{beta}_\text{incomplete}_\text{regularized}(a, b, z) = \\
\quad \frac{\text{beta}_\text{incomplete}(a, b, z)}{\text{beta}(a, b)} \\
\text{As for \text{beta}_\text{incomplete} this definition is not complete. See functions.wolfram.com} \\
\text{for a complete definition of \text{beta}_\text{incomplete}_\text{regularized}.} \\
\text{beta}_\text{incomplete}_\text{regularized} \text{simplifies \( a \) or \( b \) a positive integer.} \\
\text{For } z = 0 \text{ and } \text{realpart}(a) > 0, \text{beta}_\text{incomplete}_\text{regularized} \text{ has the specific value} \\
\text{0. For } z = 1 \text{ and } \text{realpart}(b) > 0, \text{beta}_\text{incomplete}_\text{regularized} \text{ simplifies to } 1. \\
\text{Maxima can evaluate \text{beta}_\text{incomplete}_\text{regularized} for real and complex arguments in float and bigfloat precision.} \\
\text{When \text{beta}_\text{expand} is true, Maxima expands \text{beta}_\text{incomplete}_\text{regularized} for arguments} \\
\text{\( a + n \) or \( a - n \), where \( n \) is an integer.} \\
\text{Maxima knows the derivatives of \text{beta}_\text{incomplete}_\text{regularized} with respect to the} \\
\text{variables } a, b, \text{ and } z \text{ and the integral with respect to the variable } z. \\
\text{Examples:} \\
\text{Simplification for } a \text{ or } b \text{ a positive integer:} \\
\text{(%i1) } \text{beta}_\text{incomplete}_\text{regularized}(2,b,z); \\
\text{\quad } 1 - (1 - z) \cdot (b \cdot z + 1) \\
\text{(%o1)} \\
\text{(%i2) } \text{beta}_\text{incomplete}_\text{regularized}(a,2,z); \\
\text{\quad } (a \cdot (1 - z) + 1) \cdot z \\
\text{(%o2)} \\
\text{(%i3) } \text{beta}_\text{incomplete}_\text{regularized}(3,2,z); \\
\text{\quad } 3 \cdot (1 - z) + 1) \cdot z \\
\text{(%o3)}
For the specific values $z = 0$ and $z = 1$, Maxima simplifies:

```
(%i4) assume(a>0,b>0)$
(%i5) beta_incomplete_regularized(a,b,0);
   0
(%o5)
(%i6) beta_incomplete_regularized(a,b,1);
   1
(%o6)
```

Numerical evaluation for real and complex arguments in float and bigfloat precision:

```
(%i7) beta_incomplete_regularized(0.12,0.43,0.9);
   0.9114011367359802
(%o7) 9.1140113673598075519946998779975b-1
(%i8) fpprec:32$
(%i9) beta_incomplete_regularized(0.12,0.43,0.9b0);
   9.1140113673598075519946998779975b-1
(%o9) 9.1140113673598075519946998779975b-1
(%i10) beta_incomplete_regularized(1+%i,3/3,1.5*%i);
   0.2865367499935403 %i - 0.12299596334684
(%o10)
(%i11) fpprec:20$
(%i12) beta_incomplete_regularized(1+%i,3/3,1.5b0*%i);
   2.8653674999354036142b-1 %i - 1.2299596333468400163b-1
(%o12)
```

Expansion, when beta_expand is true:

```
(%i13) beta_incomplete_regularized(a+1,b,z);
   b a
1 - z z
(%o13) beta_incomplete_regularized(a, b, z) - ------------
   a beta(a, b)
(%i14) beta_incomplete_regularized(a-1,b,z);
   1 - z z
b a - 1
(%o14) beta_incomplete_regularized(a, b, z) - ----------------------
   beta(a, b) (b + a - 1)
```

The derivative and the integral wrt $z$:

```
(%i15) diff(beta_incomplete_regularized(a,b,z),z);              (1 - z) z
   b - 1 a - 1
(%o15) --------------------------------------------------------
   beta(a, b)
(%i16) integrate(beta_incomplete_regularized(a,b,z),z);         (1 - z) z
   b a
(%o16) beta_incomplete_regularized(a, b, z) z - ---------------
   a beta(a, b)
   b + a
```

The basic definition of the generalized incomplete beta function is
Maxima simplifies `beta_incomplete_regularized` for `a` and `b` a positive integer.

For `realpart(a) > 0` and `z1 = 0` or `z2 = 0`, Maxima simplifies `beta_incomplete_generalized` to `beta_incomplete`. For `realpart(b) > 0` and `z1 = 1` or `z2=1`, Maxima simplifies to an expression with `beta` and `beta_incomplete`.

Maxima evaluates `beta_incomplete_regularized` for real and complex values in float and bigfloat precision.

When `beta_expand` is `true`, Maxima expands `beta_incomplete_generalized` for `a + n` and `a - n`, `n` a positive integer.

Maxima knows the derivative of `beta_incomplete_generalized` with respect to the variables `a`, `b`, `z1`, and `z2` and the integrals with respect to the variables `z1` and `z2`.

Examples:

Maxima simplifies `beta_incomplete_generalized` for `a` and `b` a positive integer:

```
(%i1) beta_incomplete_generalized(2,b,z1,z2);
    b  b
(1 - z1) (b z1 + 1) - (1 - z2) (b z2 + 1)
(%o1) -----------------------------------------------------------
    b (b + 1)
(%i2) beta_incomplete_generalized(a,2,z1,z2);
    a  a
(a (1 - z2) + 1) z2 - (a (1 - z1) + 1) z1
(%o2) -----------------------------------------------------------
    a (a + 1)
(%i3) beta_incomplete_generalized(3,2,z1,z2);
    2  2  2  2
(1 - z1) (3 z1 + 2 z1 + 1) - (1 - z2) (3 z2 + 2 z2 + 1)
(%o3) -----------------------------------------------------------
    12
```

Simplification for specific values `z1 = 0`, `z2 = 0`, `z1 = 1`, or `z2 = 1`:

```
(%i4) assume(a > 0, b > 0)$
(%i5) beta_incomplete_generalized(a,b,z1,0);  
    b
(1 - z1) (b z1 + 1)
(%o5) -----------------------------------------------------------
    b (b + 1)
(%i6) beta_incomplete_generalized(a,b,0,z2);  
    a
(a (1 - z2) + 1) z2
(%o6) -----------------------------------------------------------
    a (a + 1)
(%i7) beta_incomplete_generalized(a,b,z1,1);  
    b
beta(a, b) - beta_incomplete(a, b, z1)
(%o7)
```
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(%i8) beta_incomplete_generalized(a,b,1,z2);
(%o8) beta_incomplete(a, b, z2) - beta(a, b)

Numerical evaluation for real arguments in float or bigfloat precision:

(%i9) beta_incomplete_generalized(1/2,3/2,0.25,0.31);
(%o9) 0.9638178086868676

(%i10) fpprec:32$
(%i10) beta_incomplete_generalized(1/2,3/2,0.25,0.31b0);
(%o10) 9.6381780863686935309170054689964b-2

Numerical evaluation for complex arguments in float or bigfloat precision:

(%i11) beta_incomplete_generalized(1/2+%i,3/2+%i,0.25,0.31);
(%o11) -0.9625463003205376 %i - 0.0033238477353769

(%i12) fpprec:20$
(%i13) beta_incomplete_generalized(1/2+%i,3/2+%i,0.25,0.31b0);
(%o13) -9.6254630032054178691b-2 %i - 3.323847735343591914b-3

Expansion for \( a + n \) or \( a - n \), \( n \) a positive integer, when \( \text{beta\_expand} \) is true:

(%i14) beta_expand:true$

(%i15) beta_incomplete_generalized(a+1,b,z1,z2);

\[
\begin{align*}
&\frac{(1 - z1) z1 - (1 - z2) z2}{b + a} \\
&\frac{\text{betainc}(a, b, z1, z2) + \frac{1}{b + a} \text{betainc}(a, b, z1)}{b + a}
\end{align*}
\]

(%o15)

(%i16) beta_incomplete_generalized(a-1,b,z1,z2);

\[
\frac{\text{betainc}(a, b, z1, z2) (- b - a + 1)}{1 - a}
\]

Derivative wrt the variable \( z1 \) and integrals wrt \( z1 \) and \( z2 \):

(%i17) diff(beta_incomplete_generalized(a,b,z1,z2),z1);
(%o17) b - 1 a - 1

(%i18) integrate(beta_incomplete_generalized(a,b,z1,z2),z1);
(%o18) beta_incomplete_generalized(a, b, z1, z2) z1

+ beta_incomplete(a + 1, b, z1)

(%i19) integrate(beta_incomplete_generalized(a,b,z1,z2),z2);
(%o19) beta_incomplete_generalized(a, b, z1, z2) z2
- beta_incomplete(a + 1, b, z2)

beta_expand
[Option variable]
Default value: false
When beta_expand is true, beta(a,b) and related functions are expanded for arguments like \(a + n\) or \(a - n\), where \(n\) is an integer.

beta_args_sum_to_integer
[Option variable]
Default value: false
When beta_args_sum_to_integer is true, Maxima simplifies beta(a,b), when the arguments \(a\) and \(b\) sum to an integer.

psi \[n\](x)
[Function]
The derivative of \(\log(\text{gamma}(x))\) of order \(n+1\). Thus, \(psi[0](x)\) is the first derivative, \(psi[1](x)\) is the second derivative, etc.
Maxima does not know how, in general, to compute a numerical value of psi, but it can compute some exact values for rational args. Several variables control what range of rational args psi will return an exact value, if possible. See maxpsiposint, maxpsinegint, maxpsifracnum, and maxpsifracdenom. That is, \(x\) must lie between maxpsinegint and maxpsiposint. If the absolute value of the fractional part of \(x\) is rational and has a numerator less than maxpsifracnum and has a denominator less than maxpsifracdenom, psi will return an exact value.
The function bfpsi in the bffac package can compute numerical values.

maxpsiposint
[Option variable]
Default value: 20
maxpsiposint is the largest positive value for which \(psi[n](x)\) will try to compute an exact value.

maxpsinegint
[Option variable]
Default value: -10
maxpsinegint is the most negative value for which \(psi[n](x)\) will try to compute an exact value. That is if \(x\) is less than maxnegint, \(psi[n](x)\) will not return simplified answer, even if it could.

maxpsifracnum
[Option variable]
Default value: 6
Let \(x\) be a rational number less than one of the form \(p/q\). If \(p\) is greater than maxpsifracnum, then \(psi[n](x)\) will not try to return a simplified value.

maxpsifracdenom
[Option variable]
Default value: 6
Let \(x\) be a rational number less than one of the form \(p/q\). If \(q\) is greater than maxpsifracdenom, then \(psi[n](x)\) will not try to return a simplified value.

makefact (expr)
[Function]
Transforms instances of binomial, gamma, and beta functions in expr into factorials.
See also makegamma.
numfactor (expr)
Returns the numerical factor multiplying the expression expr, which should be a single term.
content returns the greatest common divisor (gcd) of all terms in a sum.

(%i1) gamma (7/2);
15 sqrt(\pi)
---
8

(%i2) numfactor (%);
15
--
8

15.5 Exponential Integrals
The Exponential Integral and related functions are defined in Abramowitz and Stegun, *Handbook of Mathematical Functions*, Chapter 5

expintegral_e1 (z)
The Exponential Integral E1(z) (A&S 5.1.1) defined as
\[ E_1(z) = \int_z^\infty \frac{e^{-t}}{t} dt \]
with \(|\arg z| < \pi\).

expintegral_ei (z)
The Exponential Integral Ei(z) (A&S 5.1.2)

expintegral_li (z)
The Exponential Integral Li(z) (A&S 5.1.3)

expintegral_en (n, z)
The Exponential Integral En(z) (A&S 5.1.4) defined as
\[ E_n(z) = \int_1^\infty \frac{e^{-zt}}{t^n} dt \]
with \(\text{Re } z > 0 \text{ and } n = 0, 1, 2, \ldots\).

expintegral_si (z)
The Exponential Integral Si(z) (A&S 5.2.1) defined as
\[ Si(z) = \int_0^z \frac{\sin t}{t} dt \]

expintegral_ci (z)
The Exponential Integral Ci(z) (A&S 5.2.2) defined as
\[ Ci(z) = \gamma + \log z + \int_0^z \frac{\cos t - 1}{t} dt \]
with \(|\arg z| < \pi\).
expintegral_shi (z)

The Exponential Integral Shi(z) (A&S 5.2.3) defined as

\[ \text{Shi}(z) = \int_0^z \frac{\sinh t}{t} dt \]

expintegral_chi (z)

The Exponential Integral Chi(z) (A&S 5.2.4) defined as

\[ \text{Chi}(z) = \gamma + \log z + \int_0^z \frac{\cosh t - 1}{t} dt \]

with \(|\arg z| < \pi\).

expintrep

[Option variable]

Default value: false

Change the representation of one of the exponential integrals, expintegral_e(m, z), expintegral_e1, or expintegral_ei to an equivalent form if possible.

Possible values for expintrep are false, gamma_incomplete, expintegral_e1, expintegral_ei, expintegral_li, expintegral_trig, or expintegral_hyp.

false means that the representation is not changed. Other values indicate the representation is to be changed to use the function specified where expintegral_trig means expintegral_si, expintegral_ci, and expintegral_hyp means expintegral_shi or expintegral_ch

expintexpand

[Option variable]

Default value: false

Expand the Exponential Integral E[n](z) for half integral values in terms of Erfc or Erf and for positive integers in terms of Ei

15.6 Error Function

The Error function and related funtions are defined in Abramowitz and Stegun, *Handbook of Mathematical Functions*, Chapter 7

erf (z)

[Function]

The Error Function erf(z) (A&S 7.1.1)

See also flag erfflag.

erfc (z)

[Function]

The Complementary Error Function erfc(z) (A&S 7.1.2)

\[ \text{erfc}(z) = 1 - \text{erf}(z) \]

erfi (z)

[Function]

The Imaginary Error Function.

\[ \text{erfi}(z) = -i \text{erf}(i z) \]

erf_generalized (z1,z2)

[Function]

Generalized Error function Erf(z1,z2)
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fresnel_c (z) [Function]
The Fresnel Integral \( C(z) = \int_0^z \cos(\pi/2 t^2) dt \). (A&S 7.3.1)
The simplification fresnel_c(-x) = -fresnel_c(x) is applied when flag trigsign is true.
The simplification fresnel_c(%i*x) = %i*fresnel_c(x) is applied when flag %iargs is true.
See flags erf_representation and hypergeometric_representation.

fresnel_s (z) [Function]
The Fresnel Integral \( S(z) = \int_0^z \sin(\pi/2 t^2) dt \). (A&S 7.3.2)
The simplification fresnel_s(-x) = -fresnel_s(x) is applied when flag trigsign is true.
The simplification fresnel_s(%i*x) = -%i*fresnel_s(x) is applied when flag %iargs is true.
See flags erf_representation and hypergeometric_representation.

erf_representation [Option variable]
Default value: false
When T erfc, erfi, erf_generalized, fresnel_s and fresnel_c are transformed to erf.

hypergeometric_representation [Option variable]
Default value: false
Enables transformation to a Hypergeometric representation for fresnel_s and fresnel_c

15.7 Struve Functions
The Struve functions are defined in Abramowitz and Stegun, *Handbook of Mathematical Functions*, Chapter 12.

struve_h (v, z) [Function]
The Struve Function H of order \( v \) and argument \( z \). (A&S 12.1.1)

struve_l (v, z) [Function]
The Modified Struve Function L of order \( v \) and argument \( z \). (A&S 12.2.1)

15.8 Hypergeometric Functions
The Hypergeometric Functions are defined in Abramowitz and Stegun, *Handbook of Mathematical Functions*, Chapters 13 and 15.
Maxima has very limited knowledge of these functions. They can be returned from function hgfred.

%\text{m} [k,u] (z) [Function]
Whittaker M function \( M[k,u](z) = \exp(-z/2) z^{-1/2+u} M(1/2+u-k,1/2+2u,z) \).
(A&S 13.1.32)

%\text{w} [k,u] (z) [Function]
Whittaker W function. (A&S 13.1.33)
The \%f[p,q]([a],[b],z) hypergeometric function, where \( a \) a list of length \( p \) and \( b \) a list of length \( q \).

The \texttt{hypergeometric}([a1, ..., ap],[b1, ... ,bq],x) function. Unlike Maxima’s \%f hypergeometric function, the function \texttt{hypergeometric} is a simplifying function; also, \texttt{hypergeometric} supports complex double and big floating point evaluation. For the Gauss hypergeometric function, that is \( p = 2 \) and \( q = 1 \), floating point evaluation outside the unit circle is supported, but in general, it is not supported.

When the option variable \texttt{expand_hypergeometric} is true (default is false) and one of the arguments \( a1 \) through \( ap \) is a negative integer (a polynomial case), \texttt{hypergeometric} returns an expanded polynomial.

Examples:

\begin{verbatim}
(%i1) hypergeometric([],[],x);
(%o1) %e^x

Polynomial cases automatically expand when \texttt{expand_hypergeometric} is true:

(%i2) hypergeometric([-3],[7],x);
(%o2) hypergeometric([-3], [7], x)

(%i3) hypergeometric([-3],[7],x), expand_hypergeometric : true;
(%o3) -x^3/504 + 3*x^2/56 - 3*x/7 + 1

Both double float and big float evaluation is supported:

(%i4) hypergeometric([5.1],[7.1 + %i],0.42);
(%o4) 1.346250786375334 - 0.0559061414208204 %i

(%i5) hypergeometric([5,6],[8], 5.7 - %i);
(%o5) .00737582409774946 - .001049813688578674 %i

(%i6) hypergeometric([5,6],[8], 5.7b0 - %i), fpprec : 30;
(%o6) 7.3758240977494674506442010824b-3 - 1.04981368857867315858055393376b-3 %i
\end{verbatim}

15.9 Parabolic Cylinder Functions

The Parabolic Cylinder Functions are defined in Abramowitz and Stegun, \textit{Handbook of Mathematical Functions}, Chapter 19.

Maxima has very limited knowledge of these functions. They can be returned from function \texttt{hgfred}.

\texttt{parabolic_cylinder_d} (v, z)

The parabolic cylinder function \texttt{parabolic_cylinder_d(v,z)}. (A&S 19.3.1)

15.10 Functions and Variables for Special Functions

\texttt{specint} (exp(- s*t) * expr, t)

Compute the Laplace transform of \( expr \) with respect to the variable \( t \). The integrand \( expr \) may contain special functions.
The following special functions are handled by \texttt{specint}: incomplete gamma function, error functions (but not the error function \texttt{erfi}, it is easy to transform \texttt{erfi} e.g. to the error function \texttt{erf}), exponential integrals, bessel functions (including products of bessel functions), hankel functions, hermite and the laguerre polynomials.

Furthermore, \texttt{specint} can handle the hypergeometric function $\%f[p, q]([], [], z)$, the whittaker function of the first kind $\%m[u, k](z)$ and of the second kind $\%w[u, k](z)$.

The result may be in terms of special functions and can include unsimplified hypergeometric functions.

When \texttt{laplace} fails to find a Laplace transform, \texttt{specint} is called. Because \texttt{laplace} knows more general rules for Laplace transforms, it is preferable to use \texttt{laplace} and not \texttt{specint}.

demo(hypgeo) displays several examples of Laplace transforms computed by \texttt{specint}.

Examples:

(\%i1) assume (p > 0, a > 0)$
(\%i2) specint (t^(1/2) * exp(-a*t/4) * exp(-p*t), t);
\frac{\sqrt{\pi}}{a^{3/2}} 2 p - \frac{p}{4}
(\%o2)

(\%i3) specint (t^(1/2) * bessel_j(1, 2 * a^(1/2) * t^(1/2)) * exp(-p*t), t);
\frac{-a/p}{\sqrt{a} \%e^{2p}}
(\%o3)

Examples for exponential integrals:

(\%i4) assume(s>0,a>0,s-a>0)$
(\%i5) ratsimp(specint(%e^(-a*t) * (log(a)+expintegral_e1(a*t))*%e^(-s*t),t));
\frac{\log(s)}{s - a}
(\%o5)

(\%i6) logarc:true$
(\%i7) gamma_expand:true$

radcan(specint((cos(t)*expintegral_si(t) - sin(t)*expintegral_ci(t))*%e^(-s*t),t));
\frac{2}{\log(s)}
(\%o8) 2 s + 1
ratsimp(specint((2*t*log(a)+2/a*sin(a*t))
-2*t*expintegral_ci(a*t))*%e^(-s*t),t));

\[ \frac{\log(s + a)}{s} \]

Results when using the expansion of \texttt{gamma_incomplete} and when changing the representation to \texttt{expintegral_e1}:

\begin{verbatim}
(%i10) assume(s>0)$
(%i11) specint(1/sqrt(%pi*t)*unit_step(t-k)*%e^(-s*t),t);
                   1
                    \texttt{gamma_incomplete(-, k s)}
\frac{------------------------}{\sqrt{\pi} \sqrt{s}}
\texttt{sqrt(\pi) sqrt(s)}

(%i12) gamma_expand:true$
(%i13) specint(1/sqrt(%pi*t)*unit_step(t-k)*%e^(-s*t),t);
                   \texttt{erfc}\left(\sqrt{k} \sqrt{s}\right)
\frac{---------------------}{\sqrt{s}}
\texttt{sqrt(s)}

(%i14) expintrep:expintegral_e1$
(%i15) ratsimp(specint(1/(t+a)^2*%e^(-s*t),t));

\[ \frac{a s}{a} \texttt{expintegral_e1}(a s) - 1 \]
\[ \frac{- \frac{1}{2}}{a} \]
\end{verbatim}

\textbf{hypergeometric_simp (e)} [Function]

\texttt{hypergeometric_simp} simplifies hypergeometric functions by applying \texttt{hgfred} to the arguments of any hypergeometric functions in the expression \(e\).

Only instances of \texttt{hypergeometric} are affected; any \(\%f\), \(\%w\), and \(\%m\) in the expression \(e\) are not affected. Any unsimplified hypergeometric functions are returned unchanged (instead of changing to \(\%f\) as \texttt{hgfred} would).

\texttt{load(hypergeometric)}; loads this function.

See also \texttt{hgfred}.

Examples:

\begin{verbatim}
(%i1) load ("hypergeometric")$
(%i2) foo : [hypergeometric([[1,1], [2], z]), hypergeometric([[1/2], [1], z])];
(%o2) [hypergeometric([[1, 1], [2], z]),
     hypergeometric([[-], [1], z])]

(%i3) hypergeometric_simp (foo);
\end{verbatim}
\[ \frac{\log(1 - z)}{z^2} \]  
(%o3) \left[ - \frac{\text{bessel}_i(0, -)}{z} \right] \%e

(%i4) \text{bar} : \text{hypergeometric}([n], [m], z + 1);
(%o4) \text{hypergeometric}([n], [m], z + 1)

(%i5) \text{hypergeometric_simp (bar)};
(%o5) \text{hypergeometric}([n], [m], z + 1)

\text{hgfred} (a, b, t) \quad \text{[Function]}
Simplify the generalized hypergeometric function in terms of other, simpler, forms. \(a\) is a list of numerator parameters and \(b\) is a list of the denominator parameters.

If hgfred cannot simplify the hypergeometric function, it returns an expression of the form \(\%f[p,q](a, b, x)\) where \(p\) is the number of elements in \(a\), and \(q\) is the number of elements in \(b\). This is the usual \(pFq\) generalized hypergeometric function.

\(%\text{i1}\) assume(not(equal(z,0)));
(%o1) \{\text{notequal}(z, 0)\}

(%i2) hgfred([v+1/2],[2*v+1],2*%i*z);
(%o2) \frac{\frac{v}{2} \%i \ z}{4 \ \text{bessel}_j(v, z) \ \gamma(v + 1) \ \%e} \frac{v}{z}

(%i3) hgfred([1,1],[2],z);
(%o3) \frac{\log(1 - z)}{z}

(%i4) hgfred([a,a+1/2],[3/2],z^2);
(%o4) \frac{1 - 2 \ a}{2 \ (1 - 2 \ a) \ z} \left( z + 1 \right) \frac{1 - 2 \ a}{\left( 1 - z \right)}

It can be beneficial to load orthopoly too as the following example shows. Note that \(L\) is the generalized Laguerre polynomial.

(%i5) load(orthopoly)$
(%i6) hgfred([-2],[a],z);
(%o6) \frac{\left( a - 1 \right)}{a \ (a + 1)}

(%i7) \text{ev(%)};
\[
\frac{2}{a} - \frac{2}{a+1} + 1
\]

\[
(\%o7)
\]

\textbf{lambert\_w (z)} \quad \text{[Function]}

The principal branch of Lambert’s W function \( W(z) \), the solution of \( z = W(z) \times \exp(W(z)) \). (DLMF 4.13)

\textbf{generalized\_lambert\_w (k, z)} \quad \text{[Function]}

The \( k \)-th branch of Lambert’s W function \( W(z) \), the solution of \( z = W(z) \times \exp(W(z)) \). (DLMF 4.13)

The principal branch, denoted \( W_p(z) \) in DLMF, is \( \text{lambert\_w(z)} = \text{generalized\_lambert\_w(0,z)} \).

The other branch with real values, denoted \( W_m(z) \) in DLMF, is \( \text{generalized\_lambert\_w(-1,z)} \).

\textbf{nzeta (z)} \quad \text{[Function]}

The Plasma Dispersion Function \( \text{nzeta(z)} = \frac{\%i}{\sqrt{\%pi}} \exp(-z^2) \times (1-\text{erf}(-\frac{\%i}{z})) \)

\textbf{nzetar (z)} \quad \text{[Function]}

\( \text{realpart(nzeta(z))} \).

\textbf{nzetai (z)} \quad \text{[Function]}

\( \text{imagpart(nzeta(z))} \).
16 Elliptic Functions

16.1 Introduction to Elliptic Functions and Integrals

Maxima includes support for Jacobian elliptic functions and for complete and incomplete elliptic integrals. This includes symbolic manipulation of these functions and numerical evaluation as well. Definitions of these functions and many of their properties can be found in Abramowitz and Stegun, Chapter 16–17. As much as possible, we use the definitions and relationships given there.

In particular, all elliptic functions and integrals use the parameter $m$ instead of the modulus $k$ or the modular angle $\alpha$. This is one area where we differ from Abramowitz and Stegun who use the modular angle for the elliptic functions. The following relationships are true:

\[ m = k^2 \]

and

\[ k = \sin \alpha \]

The elliptic functions and integrals are primarily intended to support symbolic computation. Therefore, most of derivatives of the functions and integrals are known. However, if floating-point values are given, a floating-point result is returned.

Support for most of the other properties of elliptic functions and integrals other than derivatives has not yet been written.

Some examples of elliptic functions:

\[
\begin{align*}
%i1 & : \text{jacobi_sn}(u, m) \\
%o1 & : \text{jacobi_sn}(u, m) \\
%i2 & : \text{jacobi_sn}(u, 1) \\
%o2 & : \tanh(u) \\
%i3 & : \text{jacobi_sn}(u, 0) \\
%o3 & : \sin(u) \\
%i4 & : \text{diff(\text{jacobi_sn}(u, m), u)} \\
%o4 & : \text{jacobi_cn}(u, m) \text{jacobi_dn}(u, m) \\
%i5 & : \text{diff(\text{jacobi_sn}(u, m), m)} \\
%o5 & : \text{jacobi_cn}(u, m) \text{jacobi_dn}(u, m)
\end{align*}
\]

\[
\text{elliptic_e(asin(jacobi_sn(u, m)), m)} \\
\frac{(u - \text{asin(jacobi_sn(u, m))})}{(2 \ m)} \\
\frac{1 - m}{(2 \ m)} \\
\frac{2 \text{jacobi_cn}(u, m) \text{jacobi_sn}(u, m)}{2 (1 - m)}
\]

Some examples of elliptic integrals:

\[
\begin{align*}
%i1 & : \text{elliptic_f}(\phi, m) \\
%o1 & : \text{elliptic_f}(\phi, m)
\end{align*}
\]
(%i2) elliptic_f(phi, 0);
(%o2) phi

(%i3) elliptic_f(phi, 1);
(%o3) log(tan(-- + --))
     2     4

(%i4) elliptic_e(phi, 1);
(%o4) sin(phi)

(%i5) elliptic_e(phi, 0);
(%o5) phi

(%i6) elliptic_kc(1/2);
(%o6) elliptic_kc(-)

(%i7) makegamma(%);
(%o7) gamma(-)
     4

(%i8) diff(elliptic_f(phi, m), phi);
(%o8) ---------------------
     2

  1
sqrt(1 - m sin(phi))

(%i9) diff(elliptic_f(phi, m), m);
(%o9) ------------------------------
     m

  cos(phi) sin(phi)
- ------------------/(2 (1 - m))
     2

  sqrt(1 - m sin(phi))

Support for elliptic functions and integrals was written by Raymond Toy. It is placed under the terms of the General Public License (GPL) that governs the distribution of Maxima.

16.2 Functions and Variables for Elliptic Functions

jacobi_sn (u, m) [Function]
The Jacobian elliptic function \( sn(u, m) \).

jacobi_cn (u, m) [Function]
The Jacobian elliptic function \( cn(u, m) \).

jacobi_dn (u, m) [Function]
The Jacobian elliptic function \( dn(u, m) \).
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jacobi_ns (u, m)
The Jacobian elliptic function $ns(u, m) = 1/sn(u, m)$.

jacobi_sc (u, m)
The Jacobian elliptic function $sc(u, m) = sn(u, m)/cn(u, m)$.

jacobi_sd (u, m)
The Jacobian elliptic function $sd(u, m) = sn(u, m)/dn(u, m)$.

jacobi_nc (u, m)
The Jacobian elliptic function $nc(u, m) = 1/cn(u, m)$.

jacobi_cs (u, m)
The Jacobian elliptic function $cs(u, m) = cn(u, m)/sn(u, m)$.

jacobi_cd (u, m)
The Jacobian elliptic function $cd(u, m) = cn(u, m)/dn(u, m)$.

jacobi_nd (u, m)
The Jacobian elliptic function $nd(u, m) = 1/dn(u, m)$.

jacobi_ds (u, m)
The Jacobian elliptic function $ds(u, m) = dn(u, m)/sn(u, m)$.

jacobi_dc (u, m)
The Jacobian elliptic function $dc(u, m) = dn(u, m)/cn(u, m)$.

inverse_jacobi_sn (u, m)
The inverse of the Jacobian elliptic function $sn(u, m)$.

inverse_jacobi_cn (u, m)
The inverse of the Jacobian elliptic function $cn(u, m)$.

inverse_jacobi_dn (u, m)
The inverse of the Jacobian elliptic function $dn(u, m)$.

inverse_jacobi_ns (u, m)
The inverse of the Jacobian elliptic function $ns(u, m)$.

inverse_jacobi_sc (u, m)
The inverse of the Jacobian elliptic function $sc(u, m)$.

inverse_jacobi_sd (u, m)
The inverse of the Jacobian elliptic function $sd(u, m)$.

inverse_jacobi_nc (u, m)
The inverse of the Jacobian elliptic function $nc(u, m)$.

inverse_jacobi_cs (u, m)
The inverse of the Jacobian elliptic function $cs(u, m)$.

inverse_jacobi_cd (u, m)
The inverse of the Jacobian elliptic function $cd(u, m)$.
inverse_jacobi_nd (u, m)

The inverse of the Jacobian elliptic function \( nd(u, m) \).

inverse_jacobi_ds (u, m)

The inverse of the Jacobian elliptic function \( ds(u, m) \).

inverse_jacobi_dc (u, m)

The inverse of the Jacobian elliptic function \( dc(u, m) \).

16.3 Functions and Variables for Elliptic Integrals

elliptic_f (phi, m)

The incomplete elliptic integral of the first kind, defined as

\[
\int_{0}^{\phi} \frac{d\theta}{\sqrt{1 - m \sin^2 \theta}}
\]

See also [elliptic_e], page 302, and [elliptic_kc], page 302.

elliptic_e (phi, m)

The incomplete elliptic integral of the second kind, defined as

\[
\int_{0}^{\phi} \sqrt{1 - m \sin^2 \theta} d\theta
\]

See also [elliptic_f], page 302, and [elliptic_ec], page 303.

elliptic_eu (u, m)

The incomplete elliptic integral of the second kind, defined as

\[
\int_{0}^{u} dn(v, m) dv = \int_{0}^{\tau} \sqrt{\frac{1 - mt^2}{1 - t^2}} dt
\]

where \( \tau = sn(u, m) \).

This is related to \( \text{elliptic}_e \) by

\[
E(u, m) = E(\phi, m)
\]

where \( \phi = \sin^{-1} sn(u, m) \).

See also [elliptic_e], page 302.

elliptic_pi (n, phi, m)

The incomplete elliptic integral of the third kind, defined as

\[
\int_{0}^{\phi} \frac{d\theta}{(1 - n \sin^2 \theta)\sqrt{1 - m \sin^2 \theta}}
\]

elliptic_kc (m)

The complete elliptic integral of the first kind, defined as
\[
\int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1 - m \sin^2 \theta}}
\]

For certain values of \( m \), the value of the integral is known in terms of \( Gamma \) functions. Use \texttt{makegamma} to evaluate them.

\texttt{elliptic\_ec (m)}

The complete elliptic integral of the second kind, defined as

\[
\int_0^{\frac{\pi}{2}} \sqrt{1 - m \sin^2 \theta} d\theta
\]

For certain values of \( m \), the value of the integral is known in terms of \( Gamma \) functions. Use \texttt{makegamma} to evaluate them.
17 Limits

17.1 Functions and Variables for Limits

lhosipalllim [Option variable]
  Default value: 4
  lhosipalllim is the maximum number of times L'Hospital's rule is used in limit. This prevents infinite looping in cases like \( \lim (\cot(x)/\csc(x), x, 0) \).

limit [Function]
  \( \lim (expr, x, val, dir) \)
  \( \lim (expr, x, val) \)
  \( \lim (expr) \)
  Computes the limit of \( expr \) as the real variable \( x \) approaches the value \( val \) from the direction \( dir \). \( dir \) may have the value plus for a limit from above, minus for a limit from below, or may be omitted (implying a two-sided limit is to be computed).

limit uses the following special symbols: inf (positive infinity) and minf (negative infinity). On output it may also use und (undefined), ind (indefinite but bounded) and infinity (complex infinity).

infinity (complex infinity) is returned when the limit of the absolute value of the expression is positive infinity, but the limit of the expression itself is not positive infinity or negative infinity. This includes cases where the limit of the complex argument is a constant, as in \( \lim (\log(x), x, \text{minf}) \), cases where the complex argument oscillates, as in \( \lim((-2)^x, x, \text{inf}) \), and cases where the complex argument is different for either side of a two-sided limit, as in \( \lim(1/x, x, 0) \) and \( \lim(log(x), x, 0) \).

lhosipalllim is the maximum number of times L'Hospital's rule is used in limit. This prevents infinite looping in cases like \( \lim (\cot(x)/\csc(x), x, 0) \).

tlimswitch when true will allow the limit command to use Taylor series expansion when necessary.

limsubst prevents limit from attempting substitutions on unknown forms. This is to avoid bugs like \( \lim (f(n)/f(n+1), n, \text{inf}) \) giving 1. Setting limsubst to true will allow such substitutions.

limit with one argument is often called upon to simplify constant expressions, for example, limit (inf-1).

example (limit) displays some examples.


limsubst [Option variable]
  Default value: false
  prevents limit from attempting substitutions on unknown forms. This is to avoid bugs like \( \lim (f(n)/f(n+1), n, \text{inf}) \) giving 1. Setting limsubst to true will allow such substitutions.
tlimit

- tlimit (expr, x, val, dir)
- tlimit (expr, x, val)
- tlimit (expr)

Take the limit of the Taylor series expansion of \( \text{expr} \) in \( x \) at \( \text{val} \) from direction \( \text{dir} \).

ttlimswitch

[Option variable]

Default value: true

When \( \text{ttlimswitch} \) is true, the \text{limit} command will use a Taylor series expansion if the limit of the input expression cannot be computed directly. This allows evaluation of limits such as \( \text{limit}(x/(x-1)-1/\log(x), x, 1, \text{plus}) \). When \( \text{ttlimswitch} \) is false and the limit of input expression cannot be computed directly, \text{limit} will return an unevaluated limit expression.
18 Differentiation

18.1 Functions and Variables for Differentiation

antid (expr, x, u(x))  [Function]
Returns a two-element list, such that an antiderivative of expr with respect to x can be constructed from the list. The expression expr may contain an unknown function u and its derivatives.

Let L, a list of two elements, be the return value of antid. Then \( L[1] + \int \) (L[2], x) is an antiderivative of expr with respect to x.

When antid succeeds entirely, the second element of the return value is zero. Otherwise, the second element is nonzero, and the first element is nonzero or zero. If antid cannot make any progress, the first element is zero and the second nonzero.

load ("antid") loads this function. The antid package also defines the functions nonzero and freeof and linear.

antid is related to antidiff as follows. Let L, a list of two elements, be the return value of antid. Then the return value of antidiff is equal to \( L[1] + \int \) (L[2], x) where x is the variable of integration.

Examples:

```lisp
(%i1) load ("antid")$

(%i2) expr: exp (z(x)) * diff (z(x), x) * y(x);
   z(x) d
   y(x) %e (-- (z(x)))
   dx

(%i3) a1: antid (expr, x, z(x));
   z(x) z(x) d
   [y(x) %e , - %e (-- (y(x)))]
   dx

(%i4) a2: antideriving (expr, x, z(x));
   / z(x)
   [ z(x) d
   / z(x)
   y(x) %e - I %e (--- (y(x))) dx
   ]
   dx

(%i5) a2 - (first (a1) + \int \) (second (a1), x));
   0

(%i6) antid (expr, x, y(x));
   z(x) d
   [0, y(x) %e (-- (z(x)))]
   dx

(%i7) antideriving (expr, x, y(x));
   / [ z(x) d
   / [ z(x)
   I y(x) %e (--- (z(x))) dx
   ]
   dx
```
antidiff (expr, x, u(x))  [Function]
Returns an antiderivative of expr with respect to x. The expression expr may contain
an unknown function u and its derivatives.

When antidiff succeeds entirely, the resulting expression is free of integral signs
(that is, free of the integrate noun). Otherwise, antidiff returns an expression
which is partly or entirely within an integral sign. If antidiff cannot make any
progress, the return value is entirely within an integral sign.

load ("antid") loads this function. The antid package also defines the functions
nonzeroandfreeof and linear.

antidiff is related to antid as follows. Let L, a list of two elements, be the return
value of antid. Then the return value of antidiff is equal to L[1] + 'integrate
(L[2], x) where x is the variable of integration.

Examples:

(%i1) load ("antid")$
(%i2) expr: exp (z(x)) * diff (z(x), x) * y(x);
   z(x)    
   (%o2) y(x) %e (-- (z(x)))
        dx
(%i3) a1: antid (expr, x, z(x));
   z(x)    
   (%o3) [y(x) %e , - %e (-- (y(x)))]
        dx
(%i4) a2: antidiff (expr, x, z(x));
   /                  z(x)    
   (%o4) y(x) %e - I %e (-- (y(x))) dx
         /                   dx
(%i5) a2 - (first (a1) + 'integrate (second (a1), x));
   (%o5) 0
(%i6) antid (expr, x, y(x));
   z(x)    
   (%o6) [0, y(x) %e (-- (z(x)))]
        dx
(%i7) antidiff (expr, x, y(x));
   /                  z(x)    
   (%o7) I y(x) %e (-- (z(x))) dx
         /
at

\[ \text{at} \left( \text{expr}, [\text{eqn}_1, \ldots, \text{eqn}_n] \right) \]
\[ \text{at} \left( \text{expr}, \text{eqn} \right) \]

Evaluates the expression \( \text{expr} \) with the variables assuming the values as specified for them in the list of equations \([\text{eqn}_1, \ldots, \text{eqn}_n] \) or the single equation \( \text{eqn} \).

If a subexpression depends on any of the variables for which a value is specified but there is no \text{atvalue} specified and it can’t be otherwise evaluated, then a noun form of the \text{at} is returned which displays in a two-dimensional form.

\text{at} carries out multiple substitutions in parallel.

See also \text{atvalue}. For other functions which carry out substitutions, see also \text{subst} and \text{ev}.

Examples:

\begin{verbatim}
(%i1) atvalue (f(x,y), [x = 0, y = 1], a^2);
  2
(%o1) a

(%i2) atvalue ('diff (f(x,y), x), x = 0, 1 + y);
  \(\text{\(\partial\) + 1}

(%i3) printprops (all, atvalue);
\[ \frac{d}{d\theta_1} \left( f(\theta_1, \theta_2) \right) = \theta_2 + 1 \]
\[ \theta_1 = 0 \]

  \[ f(0, 1) = a \]

(%o3) done

(%i4) diff (4*f(x, y)^2 - u(x, y)^2, x);
\[ 8 f(x, y) \frac{d}{dx} (f(x, y)) - 2 u(x, y) \frac{d}{dx} (u(x, y)) \]

(%i5) at (%i4, [x = 0, y = 1]);
\[ 16 a - 2 u(0, 1) \frac{d}{dx} (u(x, 1)) \]
\[ x = 0 \]
\end{verbatim}

Note that in the last line \( y \) is treated differently to \( x \) as \( y \) isn’t used as a differentiation variable.

The difference between \text{subst}, \text{at} and \text{ev} can be seen in the following example:

\begin{verbatim}
(%i1) e1:I(t)=C*diff(U(t),t)$
(%i2) e2:U(t)=L*diff(I(t),t)$
\end{verbatim}
(\%i3) \text{at}(e1,e2);

\[
\begin{align*}
\frac{d}{dt} I(t) &= C \left( \frac{d}{dt} U(t) \right) \\
U(t) &= L \left( \frac{d}{dt} I(t) \right)
\end{align*}
\]

(\%o3) \text{I}(t) = C \left( \frac{d}{dt} U(t) \right)

(\%i4) \text{subst}(e2,e1);

\[
\begin{align*}
\frac{d}{dt} I(t) &= C \left( \frac{d}{dt} L \left( \frac{d}{dt} I(t) \right) \right)
\end{align*}
\]

(\%o4) \text{I}(t) = C \left( \frac{d}{dt} L \left( \frac{d}{dt} I(t) \right) \right)

(\%i5) \text{ev}(e1,e2,\text{diff});

\[
\begin{align*}
\frac{d}{dt} I(t) &= C L \left( \frac{d}{dt} I(t) \right)
\end{align*}
\]

(\%o5) \text{I}(t) = C \frac{d}{dt} L \left( \frac{d}{dt} I(t) \right)

\text{atomgrad} \quad \text{[Property]}

\text{atomgrad} \text{ is the atomic gradient property of an expression. This property is assigned by \text{gradef}.}

\text{atvalue} \quad \text{[Function]}

\text{atvalue} (\text{expr}, [x_1 = a_1, \ldots, x_m = a_m], c)

\text{atvalue} (\text{expr}, x_1 = a_1, c)

Assigns the value \(c\) to \text{expr} at the point \(x = a\). Typically boundary values are established by this mechanism.

\text{expr} is a function evaluation, \(f(x_1, \ldots, x_m)\), or a derivative, \(\text{diff} \ (f(x_1, \ldots, x_m), x_i, n_1, \ldots, x_n, n_m)\) in which the function arguments explicitly appear. \(n_i\) is the order of differentiation with respect to \(x_i\).

The point at which the \text{atvalue} is established is given by the list of equations \([x_1 = a_1, \ldots, x_m = a_m]\). If there is a single variable \(x_1\), the sole equation may be given without enclosing it in a list.

\text{printprops} ([f_1, f_2, \ldots], \text{atvalue}) \text{ displays the atvalues of the functions } f_1, f_2, \ldots \text{ as specified by calls to \text{atvalue}. \text{printprops} (f, \text{atvalue}) \text{ displays the atvalues of one function } f. \text{printprops} (\text{all, atvalue}) \text{ displays the atvalues of all functions for which atvalues are defined.}

The symbols \(\@1, \@2, \ldots\) represent the variables \(x_1, x_2, \ldots\) when atvalues are displayed.

\text{atvalue} \text{ evaluates its arguments. \text{atvalue} \text{ returns } c, \text{ the atvalue.}

See also \text{at}.

Examples:

(\%i11) \text{atvalue} (f(x,y), [x = 0, y = 1], a^2);

(\%o11)

\[
\begin{align*}
2a
\end{align*}
\]
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(%i2) \text{atvalue ('diff (f(x,y), x), x = 0, 1 + y)};

(%o2) \quad 2 + 1

(%i3) \text{printprops (all, atvalue)};

\[
\begin{align*}
\text{d} & \quad (f(\text{\texttt{1}}, \text{\texttt{2}})) = 2 + 1 \\
\text{d}\text{\texttt{1}} & \quad ! \text{\texttt{1}} = 0 \\
2 & \quad f(0, 1) = a
\end{align*}
\]

(%o3) done

(%i4) \text{diff (4*f(x,y)^2 - u(x,y)^2, x)};

\[
\begin{align*}
\frac{\text{d}}{\text{dx}} & \quad 8 f(x, y) \frac{\text{d}}{\text{dx}} (f(x, y)) - 2 u(x, y) \frac{\text{d}}{\text{dx}} (u(x, y)) \\
\end{align*}
\]

(%o4) \quad 16 a - 2 u(0, 1) \frac{\text{d}}{\text{dx}} (u(x, 1)) \\
\quad \text{\texttt{x = 0}}

\text{cartan} [\text{Function}]

The exterior calculus of differential forms is a basic tool of differential geometry developed by Elie Cartan and has important applications in the theory of partial differential equations. The \text{cartan} package implements the functions \text{ext_diff} and \text{lie_diff}, along with the operators \text{\texttt{\&}} (wedge product) and \text{\texttt{\|}} (contraction of a form with a vector.) Type \text{\texttt{demo (tensor)}} to see a brief description of these commands along with examples.

\text{cartan} was implemented by F.B. Estabrook and H.D. Wahlquist.

\text{del (x)} [\text{Function}]

\text{del (x)} represents the differential of the variable \text{x}.

\text{diff} returns an expression containing \text{del} if an independent variable is not specified. In this case, the return value is the so-called "total differential".

Examples:

(%i1) \text{diff (log (x));}

\[
\begin{align*}
\text{del(x)} & \quad \frac{\text{d}}{\text{dx}} (x) \\
\end{align*}
\]

(%o1) \quad ------- \\
\quad \text{x}

(%i2) \text{diff (exp (x*y));}

\[
\begin{align*}
\frac{\text{d}}{\text{dx}} & \quad x \ %e \ \text{del(y)} + y \ %e \ \text{del(x)} \\
\end{align*}
\]

(%o2) \quad x \ %e \ \text{del(y)} + y \ %e \ \text{del(x)}

(%i3) \text{diff (x*y*z);}

\[
\begin{align*}
\frac{\text{d}}{\text{dx}} & \quad x \ y \ \text{del(z)} + x \ z \ \text{del(y)} + y \ z \ \text{del(x)} \\
\end{align*}
\]

(%o3) \quad x \ y \ \text{del(z)} + x \ z \ \text{del(y)} + y \ z \ \text{del(x)}
delta (t) [Function]
The Dirac Delta function.
Currently only laplace knows about the delta function.
Example:

(%i1) laplace (delta (t - a) * sin(b*t), t, s);
Is a positive, negative, or zero?
p;
- a s
(%o1) sin(a b) %e

dependencies [System variable]
dependencies (f_1, ..., f_n) [Function]
The variable dependencies is the list of atoms which have functional dependencies, assigned by depends, the function dependencies, or gradef. The dependencies list is cumulative: each call to depends, dependencies, or gradef appends additional items. The default value of dependencies is [].

The function dependencies(f_1, ..., f_n) appends f_1, ..., f_n, to the dependencies list, where f_1, ..., f_n are expressions of the form f(x_1, ..., x_m), and x_1, ..., x_m are any number of arguments.
dependencies(f(x_1, ..., x_m)) is equivalent to depends(f, [x_1, ..., x_m]).
See also depends and gradef.

(%i11) dependencies;
(%o11) []
(%i12) depends (foo, [bar, baz]);
(%o12) [foo(bar, baz)]
(%i13) depends ([g, h], [a, b, c]);
(%o13) [g(a, b, c), h(a, b, c)]
(%i14) dependencies;
(%o14) [foo(bar, baz), g(a, b, c), h(a, b, c)]
(%i15) dependencies (quux (x, y), mumble (u));
(%o15) [quux(x, y), mumble(u)]
(%i16) dependencies;
(%o16) [foo(bar, baz), g(a, b, c), h(a, b, c), quux(x, y),
     mumble(u)]
(%i17) remove (quux, dependency);
(%o17) done
(%i18) dependencies;
(%o18) [foo(bar, baz), g(a, b, c), h(a, b, c), mumble(u)]

depends (f_1, x_1, ..., f_n, x_n) [Function]
Declares functional dependencies among variables for the purpose of computing derivatives. In the absence of declared dependence, diff (f, x) yields zero. If depends (f, x) is declared, diff (f, x) yields a symbolic derivative (that is, a diff noun).
Each argument \( f_1, x_1 \), etc., can be the name of a variable or array, or a list of names. Every element of \( f_i \) (perhaps just a single element) is declared to depend on every element of \( x_i \) (perhaps just a single element). If some \( f_i \) is the name of an array or contains the name of an array, all elements of the array depend on \( x_i \).

\texttt{diff} recognizes indirect dependencies established by \texttt{depends} and applies the chain rule in these cases.

\texttt{remove \( (f, \text{dependency}) \)} removes all dependencies declared for \( f \).

\texttt{depends} returns a list of the dependencies established. The dependencies are appended to the global variable \texttt{dependencies}. \texttt{depends} evaluates its arguments.

\texttt{diff} is the only Maxima command which recognizes dependencies established by \texttt{depends}. Other functions (\texttt{integrate}, \texttt{laplace}, etc.) only recognize dependencies explicitly represented by their arguments. For example, \texttt{integrate} does not recognize the dependence of \( f \) on \( x \) unless explicitly represented as \texttt{integrate \( (f(x), x) \)}.

\texttt{depends} \( (f, [x_1, \ldots, x_n]) \) is equivalent to \texttt{dependencies} \( (f(x_1, \ldots, x_n)) \).

\begin{verbatim}
(%i1) depends ([f, g], x);
(%o1) [f(x), g(x)]
(%i2) depends ([r, s], [u, v, w]);
(%o2) [r(u, v, w), s(u, v, w)]
(%i3) depends (u, t);
(%o3) [u(t)]
(%i4) dependencies;
(%o4) [f(x), g(x), r(u, v, w), s(u, v, w), u(t)]
(%i5) diff (r.s, u);
   dr    ds
   -- . s + r . --
   du    du
(%i6) diff (r.s, t);
   dr du    ds du
   -- -- . s + r . -- --
   du dt    du dt
(%i7) remove (r, dependency);
(%i8) diff (r.s, t);
   ds du
   r . -- --
   du dt
\end{verbatim}

\texttt{derivabbrev} \hspace{1cm} [Option variable]

Default value: \texttt{false}

When \texttt{derivabbrev} is \texttt{true}, symbolic derivatives (that is, \texttt{diff} nouns) are displayed as subscripts. Otherwise, derivatives are displayed in the Leibniz notation \( \frac{dy}{dx} \).

\texttt{derivdegree} \hspace{0.5cm} [Function]

Returns the highest degree of the derivative of the dependent variable \( y \) with respect to the independent variable \( x \) occurring in \( expr \).
Example:

(%i1) 'diff (y, x, 2) + 'diff (y, z, 3) + 'diff (y, x) * x^2;

(%o1) --- + --- + x --
     2 3 dx
     dz
dy

(%i2) derivdegree (%o1, y, x);

(%o2) 2

derivlist (var_1, ..., var_k) [Function]
Causes only differentiations with respect to the indicated variables, within the ev command.

derivsubst [Option variable]
Default value: false
When derivsubst is true, a non-syntactic substitution such as subst (x, 'diff (y, t), 'diff (y, t, 2)) yields 'diff (x, t).

diff [Function]

diff (expr, x_1, n_1, ..., x_m, n_m)
diff (expr, x, n)
diff (expr, x)
diff (expr)

Returns the derivative or differential of expr with respect to some or all variables in expr.

diff (expr, x, n) returns the n'th derivative of expr with respect to x.
diff (expr, x_1, n_1, ..., x_m, n_m) returns the mixed partial derivative of expr with respect to x_1, ..., x_m. It is equivalent to diff (... (diff (expr, x_m, n_m) ...), x_1, n_1).
diff (expr, x) returns the first derivative of expr with respect to the variable x.
diff (expr) returns the total differential of expr, that is, the sum of the derivatives of expr with respect to each its variables times the differential del of each variable. No further simplification of del is offered.
The noun form of diff is required in some contexts, such as stating a differential equation. In these cases, diff may be quoted (as 'diff) to yield the noun form instead of carrying out the differentiation.

When derivabbrev is true, derivatives are displayed as subscripts. Otherwise, derivatives are displayed in the Leibniz notation, dy/dx.

Examples:

(%i1) diff (exp (f(x)), x, 2);

(%o1) %e (-- (f(x))) + %e (-- (f(x)))
    dx

2

2
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(\%i2) derivabbrev: true$
(\%i3) 'integrate (f(x, y), y, g(x), h(x));
        h(x)
        /
        [\n    (\%o3) I f(x, y) dy
          ]
        / g(x)
(\%i4) diff (%o3, x);
        h(x)
        /
        [\n    (\%o4) I f(x, y) dy + f(x, h(x)) h(x) - f(x, g(x)) g(x)
          ]
        x x x x
        /
        g(x)

For the tensor package, the following modifications have been incorporated:
(1) The derivatives of any indexed objects in expr will have the variables $x_i$ appended as additional arguments. Then all the derivative indices will be sorted.
(2) The $x_i$ may be integers from 1 up to the value of the variable \texttt{dimension} [default value: 4]. This will cause the differentiation to be carried out with respect to the $x_i$'th member of the list \texttt{coordinates} which should be set to a list of the names of the coordinates, e.g., [x, y, z, t]. If \texttt{coordinates} is bound to an atomic variable, then that variable subscripted by $x_i$ will be used for the variable of differentiation. This permits an array of coordinate names or subscripted names like $X[1]$, $X[2]$, \ldots to be used. If \texttt{coordinates} has not been assigned a value, then the variables will be treated as in (1) above.

\texttt{diff} \hspace{1cm} \texttt{[Special symbol]}

When \texttt{diff} is present as an \texttt{evflag} in call to \texttt{ev}, all differentiations indicated in \texttt{expr} are carried out.

\texttt{dscalar (f)} \hspace{1cm} \texttt{[Function]}

Applies the scalar d’Alembertian to the scalar function $f$.

\texttt{load \textquoteleft \textquoteleft ctensor\textquoteright \textquoteright} loads this function.

\texttt{express (expr)} \hspace{1cm} \texttt{[Function]}

Expands differential operator nouns into expressions in terms of partial derivatives. \texttt{express} recognizes the operators \texttt{grad}, \texttt{div}, \texttt{curl}, \texttt{laplacian}. \texttt{express} also expands the cross product $\times$.

Symbolic derivatives (that is, \texttt{diff} nouns) in the return value of \texttt{express} may be evaluated by including \texttt{diff} in the \texttt{ev} function call or command line. In this context, \texttt{diff} acts as an \texttt{evfun}.

\texttt{load \textquoteleft \textquoteleft vect\textquoteright \textquoteright} loads this function.

Examples:

(\%i1) load \textquoteleft \textquoteleft vect\textquoteright \textquoteright$
(%i12) grad (x^2 + y^2 + z^2);
   2  2  2
(%o12)    grad (z + y + x)
(%i13) express (%);
   d  2  2  2  d  2  2  2  d  2  2  2
   dx dy dz
(%o13) [-- (z + y + x), -- (z + y + x), -- (z + y + x)]
   dx    dy    dz
(%i14) ev (%), diff;
(%o14) [2 x, 2 y, 2 z]
(%i15) div ([x^2, y^2, z^2]);
   2  2  2
(%o15) div [x, y, z]
(%i16) express (%);
   d  2  2  2  d  2  2  2  d  2  2  2
   dx dy dz
(%o16) [-- (z) + -- (y) + -- (x)]
   dx    dy    dz
(%i17) ev (%), diff;
(%o17) 2 z + 2 y + 2 x
(%i18) curl ([x^2, y^2, z^2]);
   2  2  2
(%o18) curl [x, y, z]
(%i19) express (%);
   d  2  2  2  d  2  2  2  d  2  2  2
   dx dy dz dz dx dy
(%o19) [- (z) - -- (y), -- (x) - -- (z), -- (y) - -- (x)]
   dy    dz    dx    dz    dx    dy
(%i20) ev (%), diff;
(%o20) [0, 0, 0]
(%i21) laplacian (x^2 * y^2 * z^2);
   2  2  2
(%o21) laplacian (x y z)
(%i22) express (%);
   2  2  2
   d  2  2  2  d  2  2  2  d  2  2  2
   dz dy dz
(%o22) [-- (x y z) + -- (x y z) + -- (x y z)]
   dx    dy    dx
(%i23) ev (%), diff;
(%o23) 2 y z + 2 x z + 2 x y
(%i24) [a, b, c] ~ [x, y, z];
(%o24) [a, b, c] ~ [x, y, z]
(%i25) express (%);
(%o25) [b z - c y, c x - a z, a y - b x]
**gradef**

\[
\text{gradef} (f(x_1, \ldots, x_n), g_1, \ldots, g_m)
\]

defines the partial derivatives (i.e., the components of the gradient) of the function \(f\) or variable \(a\).

\[
\text{gradef} (f(x_1, \ldots, x_n), g_1, \ldots, g_m)
\]
defines \(\frac{df}{dx_i}\) as \(g_i\), where \(g_i\) may be an expression; \(g_i\) may be a function call, but not the name of a function. The number of partial derivatives \(m\) may be less than the number of arguments \(n\), in which case derivatives are defined with respect to \(x_1\) through \(x_m\) only.

\[
\text{gradef} (a, x, \text{expr})
\]
defines the derivative of variable \(a\) with respect to \(x\) as \(\text{expr}\). This also establishes the dependence of \(a\) on \(x\) (via \(\text{depends} (a, x)\)).

The first argument \(f(x_1, \ldots, x_n)\) or \(a\) is quoted, but the remaining arguments \(g_1, \ldots, g_m\) are evaluated. \text{gradef} returns the function or variable for which the partial derivatives are defined.

\text{gradef} can redefine the derivatives of Maxima’s built-in functions. For example,

\[
\text{gradef} (\sin(x), \text{sqrt}(1 - \sin(x)^2))
\]

redefines the derivative of \(\sin\).

\text{gradef} cannot define partial derivatives for a subscripted function.

\text{printprops} ([\text{f}_1, \ldots, \text{f}_n], \text{gradef}) displays the partial derivatives of the functions \(\text{f}_1, \ldots, \text{f}_n\), as defined by \text{gradef}.

\text{printprops} ([\text{a}_n, \ldots, \text{a}_n], \text{atomgrad}) displays the partial derivatives of the variables \(\text{a}_n, \ldots, \text{a}_n\), as defined by \text{gradef}.

\text{grade}fs is the list of the functions for which partial derivatives have been defined by \text{gradef}. \text{grade}fs does not include any variables for which partial derivatives have been defined by \text{gradef}.

Gradients are needed when, for example, a function is not known explicitly but its first derivatives are and it is desired to obtain higher order derivatives.

\text{grade}fs

Default value: \([\]\)

\text{grade}fs is the list of the functions for which partial derivatives have been defined by \text{gradef}. \text{grade}fs does not include any variables for which partial derivatives have been defined by \text{gradef}.

\text{laplace} (\text{expr}, t, s)

Attempts to compute the Laplace transform of \text{expr} with respect to the variable \(t\) and transform parameter \(s\).

\text{laplace} recognizes in \text{expr} the functions \text{delta}, \text{exp}, \text{log}, \text{sin}, \text{cos}, \text{sinh}, \text{cosh}, and \text{erf}, as well as \text{derivative}, \text{integrate}, \text{sum}, and \text{ilt}. If \text{laplace} fails to find a transform the function \text{specint} is called. \text{specint} can find the \text{laplace} transform for expressions with special functions like the \text{bessel} functions \text{bessel_j}, \text{bessel_i}, \ldots and can handle the \text{unit} \_ \text{step} function. See also \text{specint}.

If \text{specint} cannot find a solution too, a noun \text{laplace} is returned.

\text{expr} may also be a linear, constant coefficient differential equation in which case \text{atvalue} of the dependent variable is used. The required \text{atvalue} may be supplied either before or after the transform is computed. Since the initial conditions must
be specified at zero, if one has boundary conditions imposed elsewhere he can impose these on the general solution and eliminate the constants by solving the general solution for them and substituting their values back.

`laplace` recognizes convolution integrals of the form \( \text{integrate} \left( f(x) \ast g(t-x), x, 0, t \right) \); other kinds of convolutions are not recognized.

Functional relations must be explicitly represented in `expr`; implicit relations, established by `depends`, are not recognized. That is, if \( f \) depends on \( x \) and \( y \), \( f(x, y) \) must appear in `expr`.

See also `ilt`, the inverse Laplace transform.

Examples:

\begin{verbatim}
(%i1) laplace (exp (2*t + a) * sin(t) * t, t, s);
   a
%e (2 s - 4)
   ---------------
     2    2
   (s - 4 s + 5)
(%o1)

(%i2) laplace ('diff (f (x), x), x, s);
   s laplace(f(x), x, s) - f(0)
(%o2)

(%i3) diff (diff (delta (t), t), t);
   2  
   d
   --- (delta(t))
     2
   dt
(%o3)

(%i4) laplace (%, t, s);
   !
   d ! 2
   -- (delta(t))! + s - delta(0) s
   dt !
   !t = 0
(%o4)

(%i5) assume(a>0)$
(%i6) laplace(gamma_incomplete(a,t),t,s),gamma_expand:true;
   - a - 1
   gamma(a) gamma(a) s
   ------------- - -------------
   s             1     a
     (- + 1)  
   s
(%o6)

(%i7) factor(laplace(gamma_incomplete(1/2,t),t,s));
   s + 1
   sqrt(%pi) (sqrt(s) sqrt(- + 1) - 1)
   s
   ---------------------------
     3/2  
   s + 1
   s sqrt(- + 1)
   s
(%o7)
\end{verbatim}
(%i8) assume(exp(%pi*s)>1)$
(%i9) laplace(sum((-1)^n*unit_step(t-n*pi)*sin(t),n,0,inf),t,s),
     simpsum;

\[
\frac{\pi}{s + \pi i} + \frac{\pi}{s - \pi i} = \frac{\pi^2}{(s + \pi i)(s - \pi i)(1 - e^{-\pi})}
\]

(%o9) \[
\frac{\pi^2}{(s + \pi i)(s - \pi i)(1 - e^{-\pi})}
\]
19 Integration

19.1 Introduction to Integration

Maxima has several routines for handling integration. The \texttt{integrate} function makes use of most of them. There is also the \texttt{antid} package, which handles an unspecified function (and its derivatives, of course). For numerical uses, there is a set of adaptive integrators from QUADPACK, named \texttt{quad_qag}, \texttt{quad_qags}, etc., which are described under the heading \texttt{QUADPACK}. Hypergeometric functions are being worked on, see \texttt{specint} for details. Generally speaking, Maxima only handles integrals which are integrable in terms of the "elementary functions" (rational functions, trigonometrics, logs, exponentials, radicals, etc.) and a few extensions (error function, dilogarithm). It does not handle integrals in terms of unknown functions such as \(g(x)\) and \(h(x)\).

19.2 Functions and Variables for Integration

\texttt{changevar} \((\text{expr}, f(x,y), y, x)\)

Makes the change of variable given by \(f(x,y) = 0\) in all integrals occurring in \text{expr} with integration with respect to \(x\). The new variable is \(y\).

The change of variable can also be written \(f(x) = g(y)\).

\begin{verbatim}
(%i1) assume(a > 0)$
(%i2) 'integrate (%e**sqrt(a*y), y, 0, 4);
    4
   / 
   [ sqrt(a) sqrt(y)
   ]
   /
   0
(%o2) 4

(%i3) changevar (%o2, y-z^2/a, z, y);

0

(%i3) changevar (%o3, y-z^2/a, z, y);

2
   [ abs(z)
   ]
   /
   [ z %e dz
   ]
   /
   2 sqrt(a)

(%o3) - ----------------------------
a
\end{verbatim}

An expression containing a noun form, such as the instances of \texttt{integrate} above, may be evaluated by \texttt{ev} with the \texttt{nouns} flag. For example, the expression returned by \texttt{changevar} above may be evaluated by \texttt{ev} \((%o3, \text{nouns})\).

\texttt{changevar} may also be used to changes in the indices of a sum or product. However, it must be realized that when a change is made in a sum or product, this change must be a shift, i.e., \(i = j + \ldots\), not a higher degree function. E.g.,
(%i4) sum (a[i]*x^(i-2), i, 0, inf);
   inf
   ====
   \     i - 2
   (%o4) > a x
        /      i
        ====
        i = 0

(%i5) changevar (%o4, i-2-n, n, i);
   inf
   ====
   \     n
   (%o5) > a x
        /    n + 2
        ====
        n = - 2

dblint (f, r, s, a, b)  [Function]

A double-integral routine which was written in top-level Maxima and then translated
and compiled to machine code. Use load ("dblint") to access this package. It uses
the Simpson’s rule method in both the x and y directions to calculate

\[ \int_a^b \int_{r(x)}^{s(x)} f(x, y) \, dy \, dx. \]

The function \( f \) must be a translated or compiled function of two variables, and \( r \) and
\( s \) must each be a translated or compiled function of one variable, while \( a \) and \( b \) must
be floating point numbers. The routine has two global variables which determine
the number of divisions of the x and y intervals: \( \text{dblint}_x \) and \( \text{dblint}_y \), both of
which are initially 10, and can be changed independently to other integer values (there
are \( 2 \cdot \text{dblint}_x + 1 \) points computed in the x direction, and \( 2 \cdot \text{dblint}_y + 1 \) in the y
direction). The routine subdivides the X axis and then for each value of X it first
computes \( r(x) \) and \( s(x) \); then the Y axis between \( r(x) \) and \( s(x) \) is subdivided and
the integral along the Y axis is performed using Simpson’s rule; then the integral
along the X axis is done using Simpson’s rule with the function values being the Y-
integrals. This procedure may be numerically unstable for a great variety of reasons,
but is reasonably fast: avoid using it on highly oscillatory functions and functions
with singularities (poles or branch points in the region). The Y integrals depend on
how far apart \( r(x) \) and \( s(x) \) are, so if the distance \( s(x) - r(x) \) varies rapidly with
X, there may be substantial errors arising from truncation with different step-sizes
in the various Y integrals. One can increase \( \text{dblint}_x \) and \( \text{dblint}_y \) in an effort to
improve the coverage of the region, at the expense of computation time. The function
values are not saved, so if the function is very time-consuming, you will have to wait
for re-computation if you change anything (sorry). It is required that the functions
\( f, r, \) and \( s \) be either translated or compiled prior to calling \( \text{dblint} \). This will result
in orders of magnitude speed improvement over interpreted code in many cases!

demo (dblint) executes a demonstration of \( \text{dblint} \) applied to an example problem.
defint (expr, x, a, b)  
Attempts to compute a definite integral. defint is called by integrate when limits of integration are specified, i.e., when integrate is called as integrate (expr, x, a, b). Thus from the user’s point of view, it is sufficient to call integrate.

defint returns a symbolic expression, either the computed integral or the noun form of the integral. See quad_qag and related functions for numerical approximation of definite integrals.

erfflag
Default value: true

When erfflag is false, prevents risch from introducing the erf function in the answer if there were none in the integrand to begin with.

ilt (expr, s, t)  
Computes the inverse Laplace transform of expr with respect to s and parameter t. expr must be a ratio of polynomials whose denominator has only linear and quadratic factors. By using the functions laplace and ilt together with the solve or linsolve functions the user can solve a single differential or convolution integral equation or a set of them.

(%i1) 'integrate (sinh(a*x)*f(t-x), x, 0, t) + b*f(t) = t**2;
    t
   /
   [ 2
   (%o1) I f(t - x) sinh(a x) dx + b f(t) = t
        ]
   /
   0
(%i2) laplace (%i1, t, s);
   a laplace(f(t), t, s) 2
   b laplace(f(t), t, s) +--------------------- = --
   2 2
   s - a 3
   s
(%i3) linsolve (%i2, ['laplace(f(t), t, s)]);
   2 2
   2 s - 2 a
   (%o3) [laplace(f(t), t, s) = ------------------]
      5 2 3
      b s + (a - a b) s
ilt (rhs (first (%)), s, t);
Is a \( b (a b - 1) \) positive, negative, or zero?

\[
\text{pos};
\]

\[
sqrt(a b (a b - 1)) \frac{t}{2 \cosh\left(-\frac{t}{a b}\right)} + \frac{2}{a b - 1}
\]

\[
\frac{2}{a b - 2 a b + a}
\]

\[
\frac{2}{3 2 2} + \frac{2}{a b - 2 a b + a}
\]

\text{intanalysis}

[Option variable]

Default value: \texttt{true}

When \texttt{true}, definite integration tries to find poles in the integrand in the interval of integration. If there are, then the integral is evaluated appropriately as a principal value integral. If intanalysis is \texttt{false}, this check is not performed and integration is done assuming there are no poles.

See also \texttt{ldefint}.

Examples:

Maxima can solve the following integrals, when \texttt{intanalysis} is set to \texttt{false}:

\[
\text{(%i1)} \int \frac{1}{\sqrt{x} + 1} \, dx, \text{(%o1)} \int \frac{1}{\sqrt{x} + 1} \, dx
\]

\[
\text{(%i2)} \int \frac{1}{\sqrt{x} + 1} \, dx, \text{intanalysis}: \texttt{false}; \text{(%o2)} 2 - 2 \log(2)
\]

\[
\text{(%i3)} \int \frac{\cos(a)}{\sqrt{\tan(a)^2 + 1}} \, da, \text{(%o3)} \int \frac{\cos(a)}{\sqrt{\tan(a)^2 + 1}} \, da
\]

The number 1 isn't in the domain of atanh -- an error. To debug this try: debugmode(true);

\[
\text{(%i4)} \text{intanalysis}: \texttt{false}$ \text{(%i5)} \int \frac{\cos(a)}{\sqrt{\tan(a)^2 + 1}} \, da, \text{(%o5)} \int \frac{\cos(a)}{\sqrt{\tan(a)^2 + 1}} \, da
\]

\[
\frac{\pi}{2}
\]

\[
\frac{\pi}{2}
\]
integrate  [Function]

integrate (expr, x)
integrate (expr, x, a, b)

Attempts to symbolically compute the integral of expr with respect to x. integrate(expr, x) is an indefinite integral, while integrate(expr, x, a, b) is a definite integral, with limits of integration a and b. The limits should not contain x, although integrate does not enforce this restriction. a need not be less than b. If b is equal to a, integrate returns zero.

See quad_qag and related functions for numerical approximation of definite integrals. See residue for computation of residues (complex integration). See antid for an alternative means of computing indefinite integrals.

The integral (an expression free of integrate) is returned if integrate succeeds. Otherwise the return value is the noun form of the integral (the quoted operator 'integrate) or an expression containing one or more noun forms. The noun form of integrate is displayed with an integral sign.

In some circumstances it is useful to construct a noun form by hand, by quoting integrate with a single quote, e.g., 'integrate (expr, x). For example, the integral may depend on some parameters which are not yet computed. The noun may be applied to its arguments by ev (i, nouns) where i is the noun form of interest.

integrate handles definite integrals separately from indefinite, and employs a range of heuristics to handle each case. Special cases of definite integrals include limits of integration equal to zero or infinity (inf or minf), trigonometric functions with limits of integration equal to zero and %pi or 2 %pi, rational functions, integrals related to the definitions of the beta and psi functions, and some logarithmic and trigonometric integrals. Processing rational functions may include computation of residues. If an applicable special case is not found, an attempt will be made to compute the indefinite integral and evaluate it at the limits of integration. This may include taking a limit as a limit of integration goes to infinity or negative infinity; see also ldefint.

Special cases of indefinite integrals include trigonometric functions, exponential and logarithmic functions, and rational functions. integrate may also make use of a short table of elementary integrals.

integrate may carry out a change of variable if the integrand has the form f(g(x)) * diff(g(x), x). integrate attempts to find a subexpression g(x) such that the derivative of g(x) divides the integrand. This search may make use of derivatives defined by the gradef function. See also changefor and antid.

If none of the preceding heuristics find the indefinite integral, the Risch algorithm is executed. The flag risch may be set as an evflag, in a call to ev or on the command line, e.g., ev (integrate (expr, x), risch) or integrate (expr, x), risch. If risch is present, integrate calls the risch function without attempting heuristics first. See also risch.

integrate works only with functional relations represented explicitly with the f(x) notation. integrate does not respect implicit dependencies established by the depends function.

integrate may need to know some property of a parameter in the integrand. integrate will first consult the assume database, and, if the variable of interest
is not there, \texttt{integrate} will ask the user. Depending on the question, suitable responses are \texttt{yes}; or \texttt{no}; or \texttt{pos}; \texttt{zero}; or \texttt{neg};.

\texttt{integrate} is not, by default, declared to be linear. See \texttt{declare} and \texttt{linear}.

\texttt{integrate} attempts integration by parts only in a few special cases.

Examples:

- Elementary indefinite and definite integrals.

\begin{verbatim}
(%i1) integrate (sin(x)^3, x);
   3
  3
3 cos(x) - cos(x)
(%o1)

(%i2) integrate (x/ sqrt (b^2 - x^2), x);
   2  2
2   - sqrt(b - x )
(%o2)

(%i3) integrate (cos(x)^2 * exp(x), x, 0, %pi);

  %pi
%e     3
3 ------ - -
5 5
(%o3)

(%i4) integrate (x^2 * exp(-x^2), x, minf, inf);

sqrt(%pi)
(%o4) -----------------
2
\end{verbatim}

- Use of \texttt{assume} and interactive query.

\begin{verbatim}
(%i1) assume (a > 1)$
(%i2) integrate (x**a/(x+1)**(5/2), x, 0, inf);
       2 a + 2
Is ------- an integer?
5

no;

Is 2 a - 3 positive, negative, or zero?

neg;

(%o2) beta(a + 1, - - a)
      2

\end{verbatim}

- Change of variable. There are two changes of variable in this example: one using a derivative established by \texttt{gradef}, and one using the derivation \texttt{diff(r(x))} of an unspecified function \texttt{r(x)}.

\begin{verbatim}
(%i3) gradef (q(x), sin(x**2));
(%o3) q(x)
\end{verbatim}
(%i4) diff (log (q (r (x))), x);
   2
  d (-- (r(x))) sin(r (x))
  dx

(%o4) ----------------------
      q(r(x))

(%i5) integrate (%o4, x);

(%o5) log(q(r(x)))

- Return value contains the 'integrate noun form. In this example, Maxima can extract one factor of the denominator of a rational function, but cannot factor the remainder or otherwise find its integral. grind shows the noun form 'integrate in the result. See also integrate_use_rootsof for more on integrals of rational functions.

(%i1) expand ((x-4) * (x^3+2*x+1));

(%o1) x - 4 x + 2 x - 7 x - 4

(%i2) integrate (1/%, x);
   / 2
   [ x + 4 x + 18
   I ------------ dx
   ] 3
  / x + 2 x + 1

(%o2) log(x - 4) / 73 - ('integrate((x^2+4*x+18)/(x^3+2*x+1),x))/73$

- Defining a function in terms of an integral. The body of a function is not evaluated when the function is defined. Thus the body of f_1 in this example contains the noun form of integrate. The quote-quote operator '' causes the integral to be evaluated, and the result becomes the body of f_2.

(%i1) f_1 (a) := integrate (x^3, x, 1, a);

(%o1) f_1(a) := integrate(x , x, 1, a)

(%i2) ev (f_1 (7), nouns);

(%o2) 600

(%i3) /* Note parentheses around integrate(...) here */
   f_2 (a) := ''(integrate (x^3, x, 1, a));

(%o3) f_2(a) := --
      4

(%i4) f_2 (7);

(%o4) 600

integration_constant
 Default value: %c

[System variable]
When a constant of integration is introduced by indefinite integration of an equation, the name of the constant is constructed by concatenating `integration_constant` and `integration_constant_counter`.

`integration_constant` may be assigned any symbol.

Examples:

```
(%i1) integrate (x^2 = 1, x);
   3
   x
(%o1) -- = x + %c1
   3

(%i2) integration_constant : 'k;
(%o2) k

(%i3) integrate (x^2 = 1, x);
   3
   x
(%o3) -- = x + k2
   3
```

`integration_constant_counter` [System variable]
Default value: 0

When a constant of integration is introduced by indefinite integration of an equation, the name of the constant is constructed by concatenating `integration_constant` and `integration_constant_counter`.

`integration_constant_counter` is incremented before constructing the next integration constant.

Examples:

```
(%i1) integrate (x^2 = 1, x);
   3
   x
(%o1) -- = x + %c1
   3

(%i2) integrate (x^2 = 1, x);
   3
   x
(%o2) -- = x + %c2
   3

(%i3) integrate (x^2 = 1, x);
   3
   x
(%o3) -- = x + %c3
   3

(%i4) reset (integration_constant_counter);
(%o4) [integration_constant_counter]
```
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(%i5) integrate (x^2 = 1, x);
  3
x
-- = x + %c1
  3
(%o5)

integrate_use_rootsof

[Option variable]

Default value: false

When integrate_use_rootsof is true and the denominator of a rational function cannot be factored, integrate returns the integral in a form which is a sum over the roots (not yet known) of the denominator.

For example, with integrate_use_rootsof set to false, integrate returns an unsolved integral of a rational function in noun form:

(%i1) integrate_use_rootsof: false$
(%i2) integrate (1/(1+x+x^5), x);

/ 2
[ x - 4 x + 5
I ------------ dx
  3 2
] 2 x + 1
/ x - x + 1 log(x + x + 1) sqrt(3)

(%o2) ----------------- - ------------------- + -------------------
  7 14 7 sqrt(3)

Now we set the flag to be true and the unsolved part of the integral will be expressed as a summation over the roots of the denominator of the rational function:

(%i3) integrate_use_rootsof: true$
(%i4) integrate (1/(1+x+x^5), x);

==== 2
\ (%r4 - 4 %r4 + 5) log(x - %r4)
> -------------------------------
/ 2
==== 3 %r4 - 2 %r4

3 2
%r4 in rootsof(%r4 - %r4 + 1, %r4)

(%o4) ------------------------------------------
  7

2 x + 1
5 atan(--)
log(x + x + 1) sqrt(3)

- ------------------- + -------------------
  14 7 sqrt(3)

Alternatively the user may compute the roots of the denominator separately, and then express the integrand in terms of these roots, e.g., 1/((x - a)*(x - b)*(x - c)) or 1/((x^2 - (a+b)*x + a*b)*(x - c)) if the denominator is a cubic polynomial. Sometimes this will help Maxima obtain a more useful result.
\textbf{ldefint (expr, x, a, b)} \\
Attempts to compute the definite integral of \textit{expr} by using \textit{limit} to evaluate the 
indefinite integral of \textit{expr} with respect to \textit{x} at the upper limit \textit{b} and at the lower 
limit \textit{a}. If it fails to compute the definite integral, \textit{ldefint} returns an expression 
containing limits as noun forms. 
\textit{ldefint} is not called from \textit{integrate}, so executing \textit{ldefint (expr, x, a, b)} may 
yield a different result than \textit{integrate (expr, x, a, b)}. \textit{ldefint} always uses the 
same method to evaluate the definite integral, while \textit{integrate} may employ various 
heuristics and may recognize some special cases.

\textbf{potential (givengradient)} \\
The calculation makes use of the global variable \textit{potentialzeroloc[0]} which must 
be nonlist or of the form 
\[
[indeterminate_j=expression_j, indeterminate_k=expression_k, \ldots]
\]
the former being equivalent to the nonlist expression for all right-hand sides in the lat-
ter. The indicated right-hand sides are used as the lower limit of integration. The suc-
cess of the integrations may depend upon their values and order. \textit{potentialzeroloc} 
is initially set to 0.

\textbf{residue (expr, z, z_0)} \\
Computes the residue in the complex plane of the expression \textit{expr} when the variable 
\textit{z} assumes the value \textit{z_0}. The residue is the coefficient of \((z - z_0)^{-1}\) in the Laurent 
series for \textit{expr}.

\[
\text{(%i1) residue (s/(s**2+a**2), s, a*%i);} \\
\text{1} \\
\text{2} \\
\text{3} \\
\text{a} \\
\text{6}
\]

\textbf{risch (expr, x)} \\
Integrates \textit{expr} with respect to \textit{x} using the transcendental case of the Risch algo-
rithm. (The algebraic case of the Risch algorithm has not been implemented.) This 
currently handles the cases of nested exponentials and logarithms which the main 
part of \textit{integrate} can’t do. \textit{integrate} will automatically apply \textit{risch} if given these 
cases.

\texttt{erfflag}, if \texttt{false}, prevents \textit{risch} from introducing the \texttt{erf} function in the answer 
if there were none in the integrand to begin with.

\text{(%i11) risch (x^2*erf(x), x);} \\
\text{1} \\
\text{2} \\
\text{3} \\
\text{a} \\
\text{6}

\text{(%o1)} \\
\text{3 %pi}
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\begin{verbatim}
(%i2) diff(%, x), ratsimp;
2
(%o2) x erf(x)

tldefint (expr, x, a, b) [Function]
Equivalent to ldefint with tlimswitch set to true.
\end{verbatim}

19.3 Introduction to QUADPACK

QUADPACK is a collection of functions for the numerical computation of one-dimensional definite integrals. It originated from a joint project of R. Piessens\textsuperscript{1}, E. de Doncker\textsuperscript{2}, C. Ueberhuber\textsuperscript{3}, and D. Kahaner\textsuperscript{4}.

The QUADPACK library included in Maxima is an automatic translation (via the program f2cl) of the Fortran source code of QUADPACK as it appears in the SLATEC Common Mathematical Library, Version 4.1\textsuperscript{5}. The SLATEC library is dated July 1993, but the QUADPACK functions were written some years before. There is another version of QUADPACK at Netlib\textsuperscript{6}; it is not clear how that version differs from the SLATEC version.

The QUADPACK functions included in Maxima are all automatic, in the sense that these functions attempt to compute a result to a specified accuracy, requiring an unspecified number of function evaluations. Maxima’s Lisp translation of QUADPACK also includes some non-automatic functions, but they are not exposed at the Maxima level.

Further information about QUADPACK can be found in the QUADPACK book\textsuperscript{7}.

19.3.1 Overview

quad_qag Integration of a general function over a finite interval. quad_qag implements a simple globally adaptive integrator using the strategy of Aind (Piessens, 1973). The caller may choose among 6 pairs of Gauss-Kronrod quadrature formulae for the rule evaluation component. The high-degree rules are suitable for strongly oscillating integrands.

quad_qags Integration of a general function over a finite interval. quad_qags implements globally adaptive interval subdivision with extrapolation (de Doncker, 1978) by the Epsilon algorithm (Wynn, 1956).

quad_qagi Integration of a general function over an infinite or semi-infinite interval. The interval is mapped onto a finite interval and then the same strategy as in quad_qags is applied.

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\textsuperscript{2} Applied Mathematics and Programming Division, K.U. Leuven
\textsuperscript{3} Institut für Mathematik, T.U. Wien
\textsuperscript{4} National Bureau of Standards, Washington, D.C., U.S.A
\textsuperscript{5} \url{http://www.netlib.org/slatec}
\textsuperscript{6} \url{http://www.netlib.org/quadpack}
quad_qawo
Integration of \( \cos(\omega x) \ f(x) \) or \( \sin(\omega x) \ f(x) \) over a finite interval, where \( \omega \) is a constant. The rule evaluation component is based on the modified Clenshaw-Curtis technique. \texttt{quad_qawo} applies adaptive subdivision with extrapolation, similar to \texttt{quad_qags}.

quad_qawf
Calculates a Fourier cosine or Fourier sine transform on a semi-infinite interval. The same approach as in \texttt{quad_qawo} is applied on successive finite intervals, and convergence acceleration by means of the Epsilon algorithm (Wynn, 1956) is applied to the series of the integral contributions.

quad_qaws
Integration of \( w(x) \ f(x) \) over a finite interval \([a, b]\), where \( w \) is a function of the form \((x - a)^\alpha (b - x)^\beta v(x)\) and \( v(x) \) is 1 or \( \log(x - a) \) or \( \log(b - x) \) or \( \log(x - a) \log(b - x) \), and \( \alpha > -1 \) and \( \beta > -1 \).
A globally adaptive subdivision strategy is applied, with modified Clenshaw-Curtis integration on the subintervals which contain \( a \) or \( b \).

quad_qawc
Computes the Cauchy principal value of \( f(x)/(x - c) \) over a finite interval \((a, b)\) and specified \( c \). The strategy is globally adaptive, and modified Clenshaw-Curtis integration is used on the subranges which contain the point \( x = c \).

quad_qagp
Basically the same as \texttt{quad_qags} but points of singularity or discontinuity of the integrand must be supplied. This makes it easier for the integrator to produce a good solution.

19.4 Functions and Variables for QUADPACK

quad_qag
[Funtion]

\[
\text{quad_qag} (f(x), x, a, b, key, \{\text{epsrel, epsabs, limit}\})
\]

\[
\text{quad_qag} (f, x, a, b, key, \{\text{epsrel, epsabs, limit}\})
\]
Integration of a general function over a finite interval. \texttt{quad_qag} implements a simple globally adaptive integrator using the strategy of Aind (Piessens, 1973). The caller may choose among 6 pairs of Gauss-Kronrod quadrature formulae for the rule evaluation component. The high-degree rules are suitable for strongly oscillating integrands. \texttt{quad_qag} computes the integral

\[
\int_a^b f(x) \, dx
\]

The function to be integrated is \( f(x) \), with dependent variable \( x \), and the function is to be integrated between the limits \( a \) and \( b \). \texttt{key} is the integrator to be used and should be an integer between 1 and 6, inclusive. The value of \texttt{key} selects the order of the Gauss-Kronrod integration rule. High-order rules are suitable for strongly oscillating integrands.
The integrand may be specified as the name of a Maxima or Lisp function or operator, a Maxima lambda expression, or a general Maxima expression.

The numerical integration is done adaptively by subdividing the integration region into sub-intervals until the desired accuracy is achieved.

The keyword arguments are optional and may be specified in any order. They all take the form `key=val`. The keyword arguments are:

- `epsrel` Desired relative error of approximation. Default is 1d-8.
- `epsabs` Desired absolute error of approximation. Default is 0.
- `limit` Size of internal work array. `limit` is the maximum number of subintervals to use. Default is 200.

`quad_qag` returns a list of four elements:
- an approximation to the integral,
- the estimated absolute error of the approximation,
- the number integrand evaluations,
- an error code.

The error code (fourth element of the return value) can have the values:

- 0 if no problems were encountered;
- 1 if too many sub-intervals were done;
- 2 if excessive roundoff error is detected;
- 3 if extremely bad integrand behavior occurs;
- 6 if the input is invalid.

Examples:

```maxima
(%i1) quad_qag (x^(1/2)*log(1/x), x, 0, 1, 3, 'epsrel=5d-8);
(%o1) [0.4444444444492108, 3.1700968502883E-9, 961, 0]
(%i2) integrate (x^(1/2)*log(1/x), x, 0, 1);
(%o2) 4
```

`quad_qags` implements globally adaptive interval subdivision with extrapolation (de Doncker, 1978) by the Epsilon algorithm (Wynn, 1956).

Integration of a general function over a finite interval. `quad_qags` computes the integral

\[ \int_{a}^{b} f(x) \, dx \]

The function to be integrated is \( f(x) \), with dependent variable \( x \), and the function is to be integrated between the limits \( a \) and \( b \).
The integrand may be specified as the name of a Maxima or Lisp function or operator, a Maxima lambda expression, or a general Maxima expression. The keyword arguments are optional and may be specified in any order. They all take the form `key=val`. The keyword arguments are:

- `epsrel` Desired relative error of approximation. Default is 1d-8.
- `epsabs` Desired absolute error of approximation. Default is 0.
- `limit` Size of internal work array. `limit` is the maximum number of subintervals to use. Default is 200.

`quad_qags` returns a list of four elements:
- an approximation to the integral,
- the estimated absolute error of the approximation,
- the number integrand evaluations,
- an error code.

The error code (fourth element of the return value) can have the values:

- 0 no problems were encountered;
- 1 too many sub-intervals were done;
- 2 excessive roundoff error is detected;
- 3 extremely bad integrand behavior occurs;
- 4 failed to converge
- 5 integral is probably divergent or slowly convergent
- 6 if the input is invalid.

Examples:

```plaintext
(%i1) quad_qags (x^(1/2)*log(1/x), x, 0, 1, 'epsrel=1d-10);
(%o1) [.4444444444444448, 1.11022302462516E-15, 315, 0]
```

Note that `quad_qags` is more accurate and efficient than `quad_qag` for this integrand.

`quad_qagi` is a function that integrates a general function over an infinite or semi-infinite interval. The interval is mapped onto a finite interval and then the same strategy as in `quad_qags` is applied.

`quad_qagi(f(x), x, a, b, [epsrel, epsabs, limit])`

Integration of a general function over an infinite or semi-infinite interval. The interval is mapped onto a finite interval and then the same strategy as in `quad_qags` is applied.

`quad_qagi` evaluates one of the following integrals

\[
\int_a^\infty f(x) \, dx \\
\int_0^a f(x) \, dx \\
\int_\infty^a f(x) \, dx \\
\int_{-\infty}^\infty f(x) \, dx
\]
using the Quadpack QAGI routine. The function to be integrated is \( f(x) \), with dependent variable \( x \), and the function is to be integrated over an infinite range.

The integrand may be specified as the name of a Maxima or Lisp function or operator, a Maxima lambda expression, or a general Maxima expression.

One of the limits of integration must be infinity. If not, then \texttt{quad_qagi} will just return the noun form.

The keyword arguments are optional and may be specified in any order. They all take the form \texttt{key=val}. The keyword arguments are:

- \texttt{epsrel} Desired relative error of approximation. Default is 1d-8.
- \texttt{epsabs} Desired absolute error of approximation. Default is 0.
- \texttt{limit} Size of internal work array. \texttt{limit} is the maximum number of subintervals to use. Default is 200.

\texttt{quad_qagi} returns a list of four elements:
- an approximation to the integral,
- the estimated absolute error of the approximation,
- the number integrand evaluations,
- an error code.

The error code (fourth element of the return value) can have the values:

- 0 no problems were encountered;
- 1 too many sub-intervals were done;
- 2 excessive roundoff error is detected;
- 3 extremely bad integrand behavior occurs;
- 4 failed to converge
- 5 integral is probably divergent or slowly convergent
- 6 if the input is invalid.

Examples:

\begin{verbatim}
(%i1) quad_qagi (x^2*exp(-4*x), x, 0, inf, 'epsrel=1d-8);
(%o1) [0.03125, 2.95916102995002E-11, 105, 0]
(%i2) integrate (x^2*exp(-4*x), x, 0, inf);
1
(%o2) --
32
\end{verbatim}

\texttt{quad_qawc}\hspace{1cm} [Function]

\texttt{quad_qawc (f(x), x, a, b, [epsrel, epsabs, limit])}
\texttt{quad_qawc (f, x, c, a, b, [epsrel, epsabs, limit])}

Computes the Cauchy principal value of \( f(x)/(x - c) \) over a finite interval. The strategy is globally adaptive, and modified Clenshaw-Curtis integration is used on the subranges which contain the point \( x = c \).
\texttt{quad\_qawc} computes the Cauchy principal value of

\[ \int_a^b \frac{f(x)}{x-c} \, dx \]

using the Quadpack QAWC routine. The function to be integrated is \( f(x)/(x - c) \), with dependent variable \( x \), and the function is to be integrated over the interval \( a \) to \( b \).

The integrand may be specified as the name of a Maxima or Lisp function or operator, a Maxima lambda expression, or a general Maxima expression.

The keyword arguments are optional and may be specified in any order. They all take the form \texttt{key=val}. The keyword arguments are:

- \texttt{epsrel} Desired relative error of approximation. Default is 1d-8.
- \texttt{epsabs} Desired absolute error of approximation. Default is 0.
- \texttt{limit} Size of internal work array. \texttt{limit} is the maximum number of subintervals to use. Default is 200.

\texttt{quad\_qawc} returns a list of four elements:

- an approximation to the integral,
- the estimated absolute error of the approximation,
- the number integrand evaluations,
- an error code.

The error code (fourth element of the return value) can have the values:

- 0 no problems were encountered;
- 1 too many sub-intervals were done;
- 2 excessive roundoff error is detected;
- 3 extremely bad integrand behavior occurs;
- 6 if the input is invalid.

Examples:

```
(%i1) quad_qawc (2^(-5)*((x-1)^2+4^(-5))^(-1), x, 2, 0, 5, 'epsrel=1d-7);
(%o1) [- 3.131203374159256, 1.306830140249558E-8, 495, 0]
```
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(%i2) integrate (2^(-alpha)*(((x-1)^2 + 4^(-alpha))*(x-2))^-1, x, 0, 5);
Principal Value

alpha
9 4 9
4 log(--------- + --------)
alpha alpha
64 4 64 4 + 4 + 4

(%o2) (---------------------------------)
alpha

3 alpha
-----------
2 alpha/2
2 4 atan(4 4 )
2 4 atan(4 )
alpha

(%i3) ev (%o2, alpha=5, numer);
(%o3) - 3.130120337415917

quad_qawf

quad_qawf (f(x), x, a, omega, trig, [epsabs, limit, maxp1, limlst])
quad_qawf (f, x, a, omega, trig, [epsabs, limit, maxp1, limlst])

Calculates a Fourier cosine or Fourier sine transform on a semi-infinite interval using the Quadpack QAWF function. The same approach as in quad_qawo is applied on successive finite intervals, and convergence acceleration by means of the Epsilon algorithm (Wynn, 1956) is applied to the series of the integral contributions.

quad_qawf computes the integral

\[ \int_{a}^{\infty} f(x) w(x) \, dx \]

The weight function w is selected by trig:

\[ w(x) = \cos(\text{omegax}) \]

\[ w(x) = \sin(\text{omegax}) \]

The integrand may be specified as the name of a Maxima or Lisp function or operator, a Maxima lambda expression, or a general Maxima expression.

The keyword arguments are optional and may be specified in any order. They all take the form key=val. The keyword arguments are:

epsabs Desired absolute error of approximation. Default is 1d-10.

limit Size of internal work array. \((\text{limit} - \text{limlst})/2\) is the maximum number of subintervals to use. Default is 200.
maxp1  Maximum number of Chebyshev moments. Must be greater than 0. Default is 100.

limlst  Upper bound on the number of cycles. Must be greater than or equal to 3. Default is 10.

quad_qawf returns a list of four elements:
• an approximation to the integral,
• the estimated absolute error of the approximation,
• the number integrand evaluations,
• an error code.

The error code (fourth element of the return value) can have the values:
0  no problems were encountered;
1  too many sub-intervals were done;
2  excessive roundoff error is detected;
3  extremely bad integrand behavior occurs;
6  if the input is invalid.

Examples:
(%i1) quad_qawf (exp(-x^2), x, 0, 1, 'cos, 'epsabs=1d-9);
(%o1) [.6901942235215714, 2.84846300257552E-11, 215, 0]
(%i2) integrate (exp(-x^2)*cos(x), x, 0, inf);
  - 1/4
(2) = -----------------
     %e sqrt(%pi)
(%o2) 2
(%i3) ev (%o2, numer);
(%o3) .6901942235215714

quad_qawo  [Function]
quad_qawo (f(x), x, a, b, omega, trig, [epsrel, epsabs, limit, maxp1, limlst])
quad_qawo (f, x, a, b, omega, trig, [epsrel, epsabs, limit, maxp1, limlst])

Integration of \( \cos(\omega x) f(x) \) or \( \sin(\omega x) f(x) \) over a finite interval, where \( \omega \) is a constant. The rule evaluation component is based on the modified Clenshaw-Curtis technique. quad_qawo applies adaptive subdivision with extrapolation, similar to quad_qags.

quad_qawo computes the integral using the Quadpack QAWO routine:
\[
\int_a^b f(x) w(x) \, dx
\]

The weight function \( w \) is selected by trig:
\[
\cos \quad w(x) = \cos(\omega x)
\]
\[ w(x) = \sin(\omega x) \]

The integrand may be specified as the name of a Maxima or Lisp function or operator, a Maxima lambda expression, or a general Maxima expression.

The keyword arguments are optional and may be specified in any order. They all take the form `key=val`. The keyword arguments are:

- **epsrel** Desired relative error of approximation. Default is $1d-8$.
- **epsabs** Desired absolute error of approximation. Default is 0.
- **limit** Size of internal work array. `limit/2` is the maximum number of subintervals to use. Default is 200.
- **maxp1** Maximum number of Chebyshev moments. Must be greater than 0. Default is 100.
- **limlst** Upper bound on the number of cycles. Must be greater than or equal to 3. Default is 10.

`quad_qawo` returns a list of four elements:
- an approximation to the integral,
- the estimated absolute error of the approximation,
- the number integrand evaluations,
- an error code.

The error code (fourth element of the return value) can have the values:

- **0** no problems were encountered;
- **1** too many sub-intervals were done;
- **2** excessive roundoff error is detected;
- **3** extremely bad integrand behavior occurs;
- **6** if the input is invalid.

Examples:

```maxima
(%i1) quad_qawo (x^(-1/2)*exp(-2^(-2)*x), x, 1d-8, 20*2^2, 1, cos);
(%o1) [1.376043389877692, 4.72710759424899E-11, 765, 0]
(%i2) rectform (integrate (x^(-1/2)*exp(-2^(-alpha)*x) * cos(x), x, 0, inf));

alpha/2 - 1/2 2 alpha
alpha/2 - 1/2
2 sqrt(%pi) * 2 sqrt(2 + 1) + 1
-----------------------------

d2 alpha 2 alpha
   sqrt(2 + 1)
(%o2) 1.376043390090716
(%i3) ev (%, alpha=2, numer);
(%o3) 1.376043390090716
```
quad_qaws

quad_qaws (f(x), x, a, b, alpha, beta, wfun, [epsrel, epsabs, limit])
quad_qaws (f, x, a, b, alpha, beta, wfun, [epsrel, epsabs, limit])

Integration of \(w(x)f(x)\) over a finite interval, where \(w(x)\) is a certain algebraic or logarithmic function. A globally adaptive subdivision strategy is applied, with modified Clenshaw-Curtis integration on the subintervals which contain the endpoints of the interval of integration.

quad_qaws computes the integral using the Quadpack QAWS routine:

\[
\int_{a}^{b} f(x) w(x) \, dx
\]

The weight function \(w\) is selected by \(wfun\):

1. \(w(x) = (x - a)^\alpha (b - x)^\beta\)
2. \(w(x) = (x - a)^\alpha (b - x)^\beta \log (x - a)\)
3. \(w(x) = (x - a)^\alpha (b - x)^\beta \log (b - x)\)
4. \(w(x) = (x - a)^\alpha (b - x)^\beta \log (x - a) \log (b - x)\)

The integrand may be specified as the name of a Maxima or Lisp function or operator, a Maxima lambda expression, or a general Maxima expression.

The keyword arguments are optional and may be specified in any order. They all take the form \(key=val\). The keyword arguments are:

- **epsrel** Desired relative error of approximation. Default is 1d-8.
- **epsabs** Desired absolute error of approximation. Default is 0.
- **limit** Size of internal work array. \(limit\) is the maximum number of subintervals to use. Default is 200.

quad_qaws returns a list of four elements:
- an approximation to the integral,
- the estimated absolute error of the approximation,
- the number integrand evaluations,
- an error code.

The error code (fourth element of the return value) can have the values:

- 0: no problems were encountered;
- 1: too many sub-intervals were done;
- 2: excessive roundoff error is detected;
- 3: extremely bad integrand behavior occurs;
- 6: if the input is invalid.
Examples:

(%i1) quad_qaws (1/(x+1+2^(-4)), x, -1, 1, -0.5, -0.5, 1, 'epsabs=1d-9);
(%o1) [8.750097361672832, 1.24321522715422E-10, 170, 0]

(%i2) integrate ((1-x*x)^(-1/2)/(x+1+2^(-alpha)), x, -1, 1);

alpha
Is 4 2 - 1 positive, negative, or zero?
pos;

alpha   alpha
2 %pi 2 sqrt(2 2 + 1)
----------------------------------
alpha
4 2 + 2

(%o2) 8.750097361672829

quad_qagp

quad_qagp (f(x), x, a, b, points, [epsrel, epsabs, limit])
quad_qagp (f, x, a, b, points, [epsrel, epsabs, limit])

Integration of a general function over a finite interval. quad_qagp implements globally adaptive interval subdivision with extrapolation (de Doncker, 1978) by the Epsilon algorithm (Wynn, 1956).

quad_qagp computes the integral

\[ \int_{a}^{b} f(x) \, dx \]

The function to be integrated is \( f(x) \), with dependent variable \( x \), and the function is to be integrated between the limits \( a \) and \( b \).

The integrand may be specified as the name of a Maxima or Lisp function or operator, a Maxima lambda expression, or a general Maxima expression.

To help the integrator, the user must supply a list of points where the integrand is singular or discontinuous.

The keyword arguments are optional and may be specified in any order. They all take the form \( \text{key}=\text{val} \). The keyword arguments are:

- **epsrel** Desired relative error of approximation. Default is 1d-8.
- **epsabs** Desired absolute error of approximation. Default is 0.
- **limit** Size of internal work array. \( \text{limit} \) is the maximum number of subintervals to use. Default is 200.

quad_qagp returns a list of four elements:

- an approximation to the integral,
- the estimated absolute error of the approximation,
- the number integrand evaluations,
• an error code.

The error code (fourth element of the return value) can have the values:

0  no problems were encountered;
1  too many sub-intervals were done;
2  excessive roundoff error is detected;
3  extremely bad integrand behavior occurs;
4  failed to converge
5  integral is probably divergent or slowly convergent
6  if the input is invalid.

Examples:

(%i1) quad_qagp(x^3*log(abs((x^2-1)*(x^2-2))),x,0,3,[1,sqrt(2)]);
(%o1) [52.74074838347143, 2.6247632689546663e-7, 1029, 0]
(%i2) quad_qags(x^3*log(abs((x^2-1)*(x^2-2))), x, 0, 3);
(%o2) [52.74074847951494, 4.088443219529836e-7, 1869, 0]

The integrand has singularities at 1 and sqrt(2) so we supply these points to quad_qagp. We also note that quad_qagp is more accurate and more efficient than quad_qags.

quad_control (parameter, [value]) [Function]
Control error handling for quadpack. The parameter should be one of the following symbols:

current_error
  The current error number

control
  Controls if messages are printed or not. If it is set to zero or less, messages are suppressed.

max_message
  The maximum number of times any message is to be printed.

If value is not given, then the current value of the parameter is returned. If value is given, the value of parameter is set to the given value.
20 Equations

20.1 Functions and Variables for Equations

%rnum
[System variable]
Default value: 0
%rnum is the counter for the %r variables introduced in solutions by solve and algsys.
The next %r variable is numbered %rnum+1.
See also %rnum_list.

%rnum_list
[System variable]
Default value: []
%rnum_list is the list of variables introduced in solutions by solve and algsys. %r variables
are added to %rnum_list in the order they are created. This is convenient for doing
substitutions into the solution later on.
See also %rnum.

It’s recommended to use this list rather than doing concat (’%r, j).

(%i1) solve ([x + y = 3], [x,y]);
(%o1) [[x = 3 - %r1, y = %r1]]
(%i2) %rnum_list;
(%o2) [%r1]
(%i3) sol : solve ([x + 2*y + 3*z = 4], [x,y,z]);
(%o3) [[x = - 2 %r3 - 3 %r2 + 4, y = %r3, z = %r2]]
(%i4) %rnum_list;
(%o4) [%r2, %r3]
(%i5) for i : 1 thru length (%rnum_list) do
   sol : subst (t[i], %rnum_list[i], sol)$
   (%i6) sol;
   (%o6) [[x = - 2 t - 3 t + 4, y = t , z = t ]]  
        2 1 2 1

algepsilon
[Option variable]
Default value: 10^-8
algepsilon is used by algsys.

algexact
[Option variable]
Default value: false
algexact affects the behavior of algsys as follows:
If algexact is true, algsys always calls solve and then uses realroots on solve’s failures.
If algexact is false, solve is called only if the eliminant was not univariate, or if it was
a quadratic or biquadratic.
Thus algexact: true does not guarantee only exact solutions, just that algsys will
first try as hard as it can to give exact solutions, and only yield approximations when
all else fails.
algsys

algsys ([expr_1, ..., expr_m], [x_1, ..., x_n])
algsys ([eqn_1, ..., eqn_m], [x_1, ..., x_n])

Solves the simultaneous polynomials expr_1, ..., expr_m or polynomial equations eqn_1, ..., eqn_m for the variables x_1, ..., x_n. An expression expr is equivalent to an equation expr = 0. There may be more equations than variables or vice versa.

algsys returns a list of solutions, with each solution given as a list of equations stating values of the variables x_1, ..., x_n which satisfy the system of equations. If algsys cannot find a solution, an empty list [] is returned.

The symbols %r1, %r2, ..., are introduced as needed to represent arbitrary parameters in the solution; these variables are also appended to the list %rnum_list.

The method is as follows:

1. First the equations are factored and split into subsystems.
2. For each subsystem S_i, an equation E and a variable x are selected. The variable is chosen to have lowest nonzero degree. Then the resultant of E and E_j with respect to x is computed for each of the remaining equations E_j in the subsystem S_i. This yields a new subsystem S_i' in one fewer variables, as x has been eliminated. The process now returns to (1).
3. Eventually, a subsystem consisting of a single equation is obtained. If the equation is multivariate and no approximations in the form of floating point numbers have been introduced, then solve is called to find an exact solution.

In some cases, solve is not be able to find a solution, or if it does the solution may be a very large expression.

If the equation is univariate and is either linear, quadratic, or biquadratic, then again solve is called if no approximations have been introduced. If approximations have been introduced or the equation is not univariate and neither linear, quadratic, or biquadratic, then if the switch realonly is true, the function realroots is called to find the real-valued solutions. If realonly is false, then allroots is called which looks for real and complex-valued solutions.

If algsys produces a solution which has fewer significant digits than required, the user can change the value of algepsilon to a higher value.

If algexact is set to true, solve will always be called.

4. Finally, the solutions obtained in step (3) are substituted into previous levels and the solution process returns to (1).

When algsys encounters a multivariate equation which contains floating point approximations (usually due to its failing to find exact solutions at an earlier stage), then it does not attempt to apply exact methods to such equations and instead prints the message: "algsys cannot solve - system too complicated."

Interactions with radcan can produce large or complicated expressions. In that case, it may be possible to isolate parts of the result with pickapart or reveal.

Occasionally, radcan may introduce an imaginary unit %i into a solution which is actually real-valued.

Examples:

(%i1) e1: 2*x*(1 - a1) - 2*(x - 1)*a2;
(\%o1) 2 (1 - a1) x - 2 a2 (x - 1)
(\%i2) e2: a2 - a1;
(\%o2) a2 - a1
(\%i3) e3: a1*(-y - x^2 + 1);
(\%o3) a1 (- y - x + 1)
(\%i4) e4: a2*(y - (x - 1)^2);
(\%o4) a2 (y - (x - 1) )
(\%i5) algsys ([e1, e2, e3, e4], [x, y, a1, a2]);
(\%o5) [[x = 0, y = \%r1, a1 = 0, a2 = 0],
        [x = 1, y = 0, a1 = 1, a2 = 1]]
(\%i6) e1: x^2 - y^2;
(\%o6) x^2 - y^2
(\%i7) e2: -1 - y + 2*y^2 - x + x^2;
(\%o7) 2 y - y + x - x - 1
(\%i8) algsys ([e1, e2], [x, y]);
(\%o8) [[x = - -------, y = -------],
        sqrt(3) sqrt(3)
        [x = -------, y = -------], [x = - -, y = - -], [x = 1, y = 1]]
        sqrt(3) sqrt(3) 3 3

allroots

allroots (expr)
allroots (eqn)

Computes numerical approximations of the real and complex roots of the polynomial
expr or polynomial equation eqn of one variable.

The flag polyfactor when true causes allroots to factor the polynomial over the
real numbers if the polynomial is real, or over the complex numbers, if the polynomial
is complex.

allroots may give inaccurate results in case of multiple roots. If the polynomial is
real, allroots (%i*p) may yield more accurate approximations than allroots (p),
as allroots invokes a different algorithm in that case.

allroots rejects non-polynomials. It requires that the numerator after rat’ing should
be a polynomial, and it requires that the denominator be at most a complex number.
As a result of this allroots will always return an equivalent (but factored) expression,
if polyfactor is true.

For complex polynomials an algorithm by Jenkins and Traub is used (Algorithm 419,
Comm. ACM, vol. 15, (1972), p. 97). For real polynomials the algorithm used is due
Examples:

```maxima
(%i1) eqn: (1 + 2*x)^3 = 13.5*(1 + x^5);
3 5
(2 x + 1) = 13.5 (x + 1)
(%o1)
(%i2) soln: allroots (eqn);
(%o2) [x = .8296749902129361, x = - 1.015755543828121,
x = .9659625152196369 %i - .4069597231924075,
x = - .9659625152196369 %i - .4069597231924075, x = 1.0]
(%i3) for e in soln
   do (e2: subst (e, eqn), disp (expand (lhs(e2) - rhs(e2))));
   - 3.5527136788005E-15
   - 5.32907051820075E-15
   4.44089209850063E-15 %i - 4.88498130835069E-15
   - 4.44089209850063E-15 %i - 4.88498130835069E-15
   3.5527136788005E-15
(%o3) done
(%i4) polyfactor: true$
(%i5) allroots (eqn);
(%o5) - 13.5 (x - 1.0) (x - .8296749902129361)
   2
   (x + 1.015755543828121) (x + .8139194463848151 x
   + 1.098699797110288)
```

```
bfallroots

bfallroots (expr)
bfallroots (eqn)

Computes numerical approximations of the real and complex roots of the polynomial
expr or polynomial equation eqn of one variable.
In all respects, bfallroots is identical to allroots except that bfallroots computes
the roots using bigfloats. See allroots for more information.
```

```
backsubst

Default value: true

When backsubst is false, prevents back substitution in linsolve after the equations
have been triangularized. This may be helpful in very big problems where back
substitution would cause the generation of extremely large expressions.
```

(%i1) eq1 : x + y + z = 6$
(%i2) eq2 : x - y + z = 2$
```
Chapter 20: Equations

(%i3) eq3 : x + y - z = 0$
(%i4) backsubst : false$
(%i5) linsolve ([eq1, eq2, eq3], [x,y,z]);
[ x = z - y, y = 2, z = 3 ]
(%i6) backsubst : true$
(%i7) linsolve ([eq1, eq2, eq3], [x,y,z]);
[ x = 1, y = 2, z = 3 ]

breakup

Default value: true

When breakup is true, solve expresses solutions of cubic and quartic equations in terms of common subexpressions, which are assigned to intermediate expression labels (%t1, %t2, etc.). Otherwise, common subexpressions are not identified.

breakup: true has an effect only when programmode is false.

Examples:

(%i1) programmode: false$
(%i2) breakup: true$
(%i3) solve (x^3 + x^2 - 1);

sqrt(23) 25 1/3
(sqrt(3) %i 1
--------- + --)
6 sqrt(3) 54
Solution:

sqrt(3) %i 1
--------- _ _
2 2 2 1
(%t4) x = (- --------- - -) %t3 + -------------- - - 2 2 9 %t3 3

sqrt(3) %i 1
--------- _ _
2 2 2 1
(%t5) x = (--------- - -) %t3 + ---------------- - - 2 2 9 %t3 3

1 1
(%t6) x = %t3 + ------ - - 9 %t3 3
(%o6) [%t4, %t5, %t6]

(%i6) breakup: false$
(%i7) solve (x^3 + x^2 - 1);

Solution:

sqrt(3) %i 1
--------- _ _
\( x = \frac{2}{\sqrt{23}} + \frac{1}{25} - \frac{1}{3} \) + \frac{1}{6 \sqrt{3}} - \frac{54}{9}

\( x = \frac{25}{\sqrt{23}} + \frac{1}{25} - \frac{1}{3} \) + \frac{1}{6 \sqrt{3}} - \frac{54}{9}

\( x = \frac{\sqrt{3} i}{2} - \frac{1}{2} + \frac{1}{3} \) 

\( x = \frac{\sqrt{3} i}{2} - \frac{1}{2} + \frac{1}{3} \) 

\( x = \frac{\sqrt{3} i}{2} - \frac{1}{2} + \frac{1}{3} \) 

\( x = \frac{\sqrt{3} i}{2} - \frac{1}{2} + \frac{1}{3} \)

\( x = \frac{25}{\sqrt{23}} + \frac{1}{25} - \frac{1}{3} \) + \frac{1}{6 \sqrt{3}} - \frac{54}{9}

\( x = \frac{\sqrt{3} i}{2} - \frac{1}{2} + \frac{1}{3} \) 

\( x = \frac{\sqrt{3} i}{2} - \frac{1}{2} + \frac{1}{3} \) 

\( x = \frac{\sqrt{3} i}{2} - \frac{1}{2} + \frac{1}{3} \)

\[ \text{dimension} \]
\[ \text{dimension (eqn)} \]
\[ \text{dimension (eqn_1, \ldots, eqn_n)} \]
\[ \text{dimen} \text{ is a package for dimensional analysis. load ("dimen") loads this package. demo ("dimen") displays a short demonstration.} \]

\[ \text{dispflag} \]
\[ \text{Default value: true} \]
\[ \text{If set to false within a block will inhibit the display of output generated by the solve functions called from within the block. Termination of the block with a dollar sign, $, sets dispflag to false.} \]

\[ \text{funcsolve (eqn, g(t))} \]
\[ \text{Returns [g(t) = \ldots] or [], depending on whether or not there exists a rational function g(t) satisfying eqn, which must be a first order, linear polynomial in (for this case) g(t) and g(t+1).} \]

\( \frac{(n + 1)*f(n) - (n + 3)*f(n + 1)/(n + 1) = (n - 1)/(n + 2); (n + 3) f(n + 1) n - 1} \) 

\( (n + 1) f(n) - \frac{(n + 3) f(n + 1) n - 1}{(n + 2)} = \ldots \)
\( \frac{n + 1}{n + 2} \)

(%i2) \text{funcsolve (eqn, f(n));}

Dependent equations eliminated: (4 3)

\[
\frac{n}{(n + 1)(n + 2)}
\]

(%o2) \quad f(n) = \frac{n}{(n + 1)(n + 2)}

Warning: this is a very rudimentary implementation – many safety checks and obvious generalizations are missing.

\text{globalsolve} \quad [\text{Option variable}]

Default value: false

When \text{globalsolve} is true, solved-for variables are assigned the solution values found by \text{linsolve}, and by \text{solve} when solving two or more linear equations.

When \text{globalsolve} is false, solutions found by \text{linsolve} and by \text{solve} when solving two or more linear equations are expressed as equations, and the solved-for variables are not assigned.

When solving anything other than two or more linear equations, \text{solve} ignores \text{globalsolve}. Other functions which solve equations (e.g., \text{algsys}) always ignore \text{globalsolve}.

Examples:

(%i1) \text{globalsolve: true$}

(%i2) \text{solve ([x + 3*y = 2, 2*x - y = 5], [x, y]);}

Solution

\[
\frac{17}{7}
\]

(%t2) \quad x : \frac{17}{7}

(%t3) \quad y : \frac{1}{7}

(%o3) \quad \text{[[%t2, %t3]]}

(%i3) \text{x;}

(%o3) \quad \frac{17}{7}

(%i4) \text{y;}

(%o4) \quad \frac{1}{7}

(%i5) \text{globalsolve: false$}

(%i6) \text{kill (x, y)$}

(%i7) \text{solve ([x + 3*y = 2, 2*x - y = 5], [x, y]);}

Solution
ieqn (ie, unk, tech, n, guess)

ieqn is a package for solving integral equations. load ("inteqn") loads this package.

ie is the integral equation; unk is the unknown function; tech is the technique to be tried from those given above (tech = first means: try the first technique which finds a solution; tech = all means: try all applicable techniques); n is the maximum number of terms to take for taylor, neumann, firstkindseries, or fredseries (it is also the maximum depth of recursion for the differentiation method); guess is the initial guess for neumann or firstkindseries.

Default values for the 2nd thru 5th parameters are:

unk: p(x), where p is the first function encountered in an integrand which is unknown to Maxima and x is the variable which occurs as an argument to the first occurrence of p found outside of an integral in the case of secondkind equations, or is the only other variable besides the variable of integration in firstkind equations. If the attempt to search for x fails, the user will be asked to supply the independent variable.

technique: first

n: 1

guess: none which will cause neumann and firstkindseries to use f(x) as an initial guess.

ieqnprint

Default value: true

ieqnprint governs the behavior of the result returned by the ieqn command. When ieqnprint is false, the lists returned by the ieqn function are of the form

[solution, technique used, nterms, flag]

where flag is absent if the solution is exact.

Otherwise, it is the word approximate or incomplete corresponding to an inexact or non-closed form solution, respectively. If a series method was used, nterms gives the number of terms taken (which could be less than the n given to ieqn if an error prevented generation of further terms).
**lhs** *(expr)*

Returns the left-hand side (that is, the first argument) of the expression `expr`, when the operator of `expr` is one of the relational operators `< <= = # equal notequal >= >`, one of the assignment operators `:= ::= : ::`, or a user-defined binary infix operator, as declared by `infix`.

When `expr` is an atom or its operator is something other than the ones listed above, `lhs` returns `expr`.

See also `rhs`.

Examples:

```lisp
(%i1) e: aa + bb = cc;
(%o1) bb + aa = cc
(%i2) lhs(e);
(%o2) bb + aa
(%i3) rhs(e);
(%o3) cc
(%i4) [lhs(aa < bb), lhs(aa <= bb), lhs(aa >= bb),
    lhs(aa > bb)];
(%o4) [aa, aa, aa, aa]
(%i5) [lhs(aa = bb), lhs(aa # bb), lhs(equal(aa, bb)),
    lhs(notequal(aa, bb))];
(%o5) [aa, aa, aa, aa]
(%i6) e1: '(foo(x) := 2*x);
(%o6) foo(x) := 2 x
(%i7) e2: '(bar(y) ::= 3*y);
(%o7) bar(y) ::= 3 y
(%i8) e3: '(x : y);
(%o8) x : y
(%i9) e4: '(x :: y);
(%o9) x :: y
(%i10) [lhs(e1), lhs(e2), lhs(e3), lhs(e4)];
(%o10) [foo(x), bar(y), x, x]
(%i11) infix ("["ならば"[");
(%o11) ]
(%i12) lhs(aa ][ bb);
(%o12) aa
```

**linsolve** ([`expr_1, ..., expr_m`], `[x_1, ..., x_n]`)

Solves the list of simultaneous linear equations for the list of variables. The expressions must each be polynomials in the variables and may be equations.

When `globalsolve` is true, each solved-for variable is bound to its value in the solution of the equations.

When `backsubst` is false, `linsolve` does not carry out back substitution after the equations have been triangularized. This may be necessary in very big problems where back substitution would cause the generation of extremely large expressions.

When `linsolve_params` is true, `linsolve` also generates the `%r` symbols used to represent arbitrary parameters described in the manual under `algsys`. Otherwise,
linsolve solves an under-determined system of equations with some variables expressed in terms of others.

When `programmode` is `false`, `linsolve` displays the solution with intermediate expression (`%t`) labels, and returns the list of labels.

```plaintext
(%i1) e1: x + z = y;
(%o1) z + x = y
(%i2) e2: 2*a*x - y = 2*a^2;
2
(%o2) 2 a x - y = 2 a
(%i3) e3: y - 2*z = 2;
(%o3) y - 2 z = 2
(%i4) [globalsolve: false, programmode: true];
[false, true]
(%i5) linsolve ([e1, e2, e3], [x, y, z]);
(%o5) [x = a + 1, y = 2 a, z = a - 1]
(%i6) [globalsolve: false, programmode: false];
[false, false]
(%i7) linsolve ([e1, e2, e3], [x, y, z]);
Solution
(%t7) z = a - 1
(%t8) y = 2 a
(%t9) x = a + 1
(%o9) [%t7, %t8, %t9]
(%i9) ''%;
(%o9) [z = a - 1, y = 2 a, x = a + 1]
(%i10) [globalsolve: true, programmode: false];
[true, false]
(%i10) linsolve ([e1, e2, e3], [x, y, z]);
Solution
(%t11) z : a - 1
(%t12) y : 2 a
(%t13) x : a + 1
(%o13) [%t11, %t12, %t13]
(%i13) ''%;
(%o13) [z : a - 1, y : 2 a, x : a + 1]
(%i14) [x, y, z];
(%o14) [a + 1, 2 a, a - 1]
(%i15) [globalsolve: true, programmode: true];
[true, true]
(%i15) linsolve ([e1, e2, e3], ' [x, y, z]);
```

```
Chapter 20: Equations

(%o16) [x : a + 1, y : 2 a, z : a - 1]
(%i17) [x, y, z];
(%o17) [a + 1, 2 a, a - 1]

linsolvewarn [Option variable]
Default value: true
When linsolvewarn is true, linsolve prints a message "Dependent equations eliminated".

linsolve_params [Option variable]
Default value: true
When linsolve_params is true, linsolve also generates the \%r symbols used to represent arbitrary parameters described in the manual under algsys. Otherwise, linsolve solves an under-determined system of equations with some variables expressed in terms of others.

multiplicities [System variable]
Default value: not_set_yet
multiplicities is set to a list of the multiplicities of the individual solutions returned by solve or realroots.

nroots (p, low, high) [Function]
Returns the number of real roots of the real univariate polynomial p in the half-open interval (low, high]. The endpoints of the interval may be minf or inf.
nroots uses the method of Sturm sequences.

(%i11) p: x^10 - 2*x^4 + 1/2$
(%i12) nroots (p, -6, 9.1);
(%o12) 4

nthroot (p, n) [Function]
where p is a polynomial with integer coefficients and n is a positive integer returns q, a polynomial over the integers, such that q^n = p or prints an error message indicating that p is not a perfect nth power. This routine is much faster than factor or even sqfr.

polyfactor [Option variable]
Default value: false
The option variable polyfactor when true causes allroots and bfallroots to factor the polynomial over the real numbers if the polynomial is real, or over the complex numbers, if the polynomial is complex.
See allroots for an example.

programmode [Option variable]
Default value: true
When programmode is true, solve, realroots, allroots, and linsolve return solutions as elements in a list. (Except when backsubst is set to false, in which case programmode: false is assumed.)
When programmode is false, solve, etc. create intermediate expression labels \%t1, \%t2, etc., and assign the solutions to them.
realonly

[Option variable]

Default value: false

When realonly is true, algsys returns only those solutions which are free of \%i.

realroots

[Function]

realroots (expr, bound)
realroots (eqn, bound)
realroots (expr)
realroots (eqn)

Computes rational approximations of the real roots of the polynomial expr or polynomial equation eqn of one variable, to within a tolerance of bound. Coefficients of expr or eqn must be literal numbers; symbol constants such as \%pi are rejected.

realroots assigns the multiplicities of the roots it finds to the global variable multiplicities.

realroots constructs a Sturm sequence to bracket each root, and then applies bisection to refine the approximations. All coefficients are converted to rational equivalents before searching for roots, and computations are carried out by exact rational arithmetic. Even if some coefficients are floating-point numbers, the results are rational (unless coerced to floats by the float or numer flags).

When bound is less than 1, all integer roots are found exactly. When bound is unspecified, it is assumed equal to the global variable rootsepsilon.

When the global variable programmeme is true, realroots returns a list of the form [x = x_1, x = x_2, ...]. When programmeme is false, realroots creates intermediate expression labels \%t1, \%t2, ..., assigns the results to them, and returns the list of labels.

Examples:

(%i1) realroots (-1 - x + x^5, 5e-6);
   612003
(%o1) [x = --------]
      524288

(%i2) ev (%[1], float);
(%o2) x = 1.167303085327148

(%i3) ev (-1 - x + x^5, \%);
(%o3) - 7.396496210176905E-6

(%i1) realroots (expand ((1 - x)^5 * (2 - x)^3 * (3 - x)), 1e-20);
(%o1) [x = 1, x = 2, x = 3]

(%i2) multiplicities;
(%o2) [5, 3, 1]

rhs (expr)

[Function]

Returns the right-hand side (that is, the second argument) of the expression expr, when the operator of expr is one of the relational operators < == # equal notequal >= >, one of the assignment operators := ::= ::, or a user-defined binary infix operator, as declared by infix.

When expr is an atom or its operator is something other than the ones listed above, rhs returns 0.
See also `lhs`.

Examples:

```lisp
(%i1) e: aa + bb = cc;
(%o1) bb + aa = cc
(%i2) lhs (e);
(%o2) bb + aa
(%i3) rhs (e);
(%o3) cc
(%i4) [rhs (aa < bb), rhs (aa <= bb), rhs (aa >= bb),
     rhs (aa > bb)];
(%o4) [bb, bb, bb, bb]
(%i5) [rhs (aa = bb), rhs (aa # bb), rhs (equal (aa, bb)),
     rhs (notequal (aa, bb))];
(%o5) [bb, bb, bb, bb]
(%i6) e1: '(foo(x) := 2*x);
(%o6) foo(x) := 2 x
(%i7) e2: '(bar(y) ::= 3*y);
(%o7) bar(y) ::= 3 y
(%i8) e3: '(x : y);
(%o8) x : y
(%i9) e4: '(x :: y);
(%o9) x :: y
(%i10) [rhs (e1), rhs (e2), rhs (e3), rhs (e4)];
(%o10) [2 x, 3 y, y, y]
(%i11) infix ("\[\]");
(%o11) \]
(%i12) rhs (aa \[ bb);
(%o12) bb
```

rootsconmode

Default value: `true`

`rootsconmode` governs the behavior of the `rootscontract` command. See `rootscontract` for details.

rootscontract (expr)

[Function]

Converts products of roots into roots of products. For example, `rootscontract (sqrt(x)*y^(3/2))` yields `sqrt(x*y^3)`.

When `radexpand` is `true` and `domain` is `real`, `rootscontract` converts `abs` into `sqrt`, e.g., `rootscontract (abs(x)*sqrt(y))` yields `sqrt(x^2*y)`.

There is an option `rootsconmode` affecting `rootscontract` as follows:

<table>
<thead>
<tr>
<th>Problem</th>
<th>Value of rootsconmode</th>
<th>Result of applying rootscontract</th>
</tr>
</thead>
<tbody>
<tr>
<td>x^(1/2)*y^(3/2)</td>
<td>false</td>
<td>(x*y^3)^(1/2)</td>
</tr>
<tr>
<td>x^(1/2)*y^(1/4)</td>
<td>false</td>
<td>x^(1/2)*y^(1/4)</td>
</tr>
<tr>
<td>x^(1/2)*y^(1/4)</td>
<td>true</td>
<td>(x*y^(1/2))^(1/2)</td>
</tr>
</tbody>
</table>
x^(1/2)*y^(1/3)  true  x^(1/2)*y^(1/3)
x^(1/2)*y^(1/4)  all  (x^2*y)^(1/4)
x^(1/2)*y^(1/3)  all  (x^3*y^2)^(1/6)

When rootsconmode is false, rootscontract contracts only with respect to rational number exponents whose denominators are the same. The key to the rootsconmode: true examples is simply that 2 divides into 4 but not into 3. rootsconmode: all involves taking the least common multiple of the denominators of the exponents.

rootscontract uses ratsimp in a manner similar to logcontract.

Examples:

(%i1) rootsconmode: false$
(%i2) rootscontract (x^(1/2)*y^(3/2));
  3
(%o2)  sqrt(x y )
(%i3) rootscontract (x^(1/2)*y^(1/4));
  1/4
(%o3)  sqrt(x) y
(%i4) rootsconmode: true$
(%i5) rootscontract (x^(1/2)*y^(1/4));
(%o5) sqrt(x) sqrt(y)
(%i6) rootscontract (x^(1/2)*y^(1/3));
  1/3
(%o6)  sqrt(x) y
(%i7) rootsconmode: all$
(%i8) rootscontract (x^(1/2)*y^(1/4));
  2 1/4
(%o8)  (x y)
(%i9) rootscontract (x^(1/2)*y^(1/3));
  3 2 1/6
(%o9)  (x y)
(%i10) rootsconmode: false$
(%i11) rootscontract (sqrt(sqrt(x) + sqrt(1 + x))
  *sqrt(sqrt(1 + x) - sqrt(x)));
  1
(%o11)  1
(%i12) rootsconmode: true$
(%i13) rootscontract (sqrt(5+sqrt(5)) - 5^(1/4)*sqrt(1+sqrt(5)));
  0
(%o13)  0

rootspsilson  [Option variable]

Default value: 1.0e-7

rootspsilson is the tolerance which establishes the confidence interval for the roots found by the realroots function.
solve
    solve (expr, x)
solve (expr)
solve ([eqn_1, ..., eqn_n], [x_1, ..., x_n])

Solves the algebraic equation expr for the variable x and returns a list of solution
equations in x. If expr is not an equation, the equation expr = 0 is assumed in its
place. x may be a function (e.g. f(x)), or other non-atomic expression except a sum
or product. x may be omitted if expr contains only one variable. expr may be a
rational expression, and may contain trigonometric functions, exponentials, etc.

The following method is used:
Let E be the expression and X be the variable. If E is linear in X then it is trivially
solved for X. Otherwise if E is of the form A*X^N + B then the result is (-B/A)^1/N
times the N'th roots of unity.

If E is not linear in X then the gcd of the exponents of X in E (say N) is divided
into the exponents and the multiplicity of the roots is multiplied by N. Then solve
is called again on the result. If E factors then solve is called on each of the factors.
Finally solve will use the quadratic, cubic, or quartic formulas where necessary.

In the case where E is a polynomial in some function of the variable to be solved for,
say F(X), then it is first solved for F(X) (call the result C), then the equation F(X)=C
can be solved for X provided the inverse of the function F is known.
breakup if false will cause solve to express the solutions of cubic or quartic equa-
tions as single expressions rather than as made up of several common subexpressions
which is the default.
multiplicities - will be set to a list of the multiplicities of the individual solutions
returned by solve, realroots, or allroots. Try apropos (solve) for the switches
which affect solve. describe may then by used on the individual switch names if
their purpose is not clear.
solve ([eqn_1, ..., eqn_n], [x_1, ..., x_n]) solves a system of simultaneous
(linear or non-linear) polynomial equations by calling linsolve or algsys and re-
turns a list of the solution lists in the variables. In the case of linsolve this list
would contain a single list of solutions. It takes two lists as arguments. The first list
represents the equations to be solved; the second list is a list of the unknowns to be
determined. If the total number of variables in the equations is equal to the number
of equations, the second argument-list may be omitted.

When programmode is false, solve displays solutions with intermediate expression
(%t) labels, and returns the list of labels.
When globalsolve is true and the problem is to solve two or more linear equations,
each solved-for variable is bound to its value in the solution of the equations.
Examples:

(%i1) solve (asin (cos (3*x))*(f(x) - 1), x);

solve: using arc-trig functions to get a solution.
Some solutions will be lost.
%pi
(%o1) [x = --, f(x) = 1]
6
(%i2) ev (solve (5^f(x) = 125, f(x)), solveradcan);

log(125)

(%o2) [f(x) = -------]
log(5)

(%i3) [4*x^2 - y^2 = 12, x*y - x = 2];

2 2

(%o3) [4 x - y = 12, x y - x = 2]

(%i4) solve (%[x, y]);

(%o4) [[x = 2, y = 2], [x = 0.5202594388652008 %i - 0.1331240357358706, y = 0.07678378523787788 - 3.608003221870287 %i], [x = - 0.5202594388652008 %i - 0.1331240357358706, y = 3.608003221870287 %i + 0.07678378523787788], [x = - 1.733751846381093, y = - 0.1535675710019696]]

(%i5) solve (1 + a*x + x^3, x);

3

(sqrt(3) %i 1 sqrt(4 a + 27) 1 1/3
2 2 6 sqrt(3) 2

(sqrt(3) %i 1
------------- - -) a
2 2

- -------------------------, x =
3
sqrt(4 a + 27) 1 1/3
6 sqrt(3)

3

(sqrt(3) %i 1 sqrt(4 a + 27) 1 1/3
2 2 6 sqrt(3) 2

(sqrt(3) %i 1
(- ------------- - -) a
2 2

- -------------------------, x =
3
sqrt(4 a + 27) 1 1/3
6 sqrt(3)
\[ \frac{3}{\sqrt{4a + 27}} \cdot \frac{1}{3} = \frac{a}{6 \sqrt{3}} \]

\begin{verbatim}
(%i6) solve (x^3 - 1);
   sqrt(3) %i - 1   sqrt(3) %i + 1
[x = ----------, x = ----------, x = 1]
   2               2

(%i7) solve (x^6 - 1);
   sqrt(3) %i + 1   sqrt(3) %i - 1
[x = ----------, x = ----------, x = -1,  
   2               2
   sqrt(3) %i + 1   sqrt(3) %i - 1
x = - ----------, x = - ----------, x = 1]
   2               2

(%i8) ev (x^6 - 1, %[1]);
   6
(sqrt(3) %i + 1)
64

(%i9) expand (%); 0

(%i10) x^2 - 1;
   2
x - 1

(%i11) solve (%); 

(%o11) [x = -1, x = 1]

(%i12) ev (%th(2), %[1]); 0

The symbols \%r are used to denote arbitrary constants in a solution.

(%i1) solve([x+y=1,2*x+2*y=2],[x,y]);

solve: dependent equations eliminated: (2)

(%o1) [[x = 1 - %r1, y = %r1]]

See \texttt{algsys} and \%rnum\_list for more information.

\texttt{solvedecomposes} [Option variable]

Default value: \texttt{true}

When \texttt{solvedecomposes} is \texttt{true}, \texttt{solve} calls \texttt{polydecomp} if asked to solve polynomials.
\textbf{solveexplicit} \hspace{1cm} [Option variable]

Default value: \texttt{false}

When \texttt{solveexplicit} is \texttt{true}, inhibits \texttt{solve} from returning implicit solutions, that is, solutions of the form $F(x) = 0$ where $F$ is some function.

\textbf{solvefactors} \hspace{1cm} [Option variable]

Default value: \texttt{true}

When \texttt{solvefactors} is \texttt{false}, \texttt{solve} does not try to factor the expression. The \texttt{false} setting may be desired in some cases where factoring is not necessary.

\textbf{solvenullwarn} \hspace{1cm} [Option variable]

Default value: \texttt{true}

When \texttt{solvenullwarn} is \texttt{true}, \texttt{solve} prints a warning message if called with either a null equation list or a null variable list. For example, \texttt{solve ([], [])} would print two warning messages and return \texttt{[]}.

\textbf{solveradcan} \hspace{1cm} [Option variable]

Default value: \texttt{false}

When \texttt{solveradcan} is \texttt{true}, \texttt{solve} calls \texttt{radcan} which makes \texttt{solve} slower but will allow certain problems containing exponentials and logarithms to be solved.

\textbf{solvetrigwarn} \hspace{1cm} [Option variable]

Default value: \texttt{true}

When \texttt{solvetrigwarn} is \texttt{true}, \texttt{solve} may print a message saying that it is using inverse trigonometric functions to solve the equation, and thereby losing solutions.
21 Differential Equations

21.1 Introduction to Differential Equations

This section describes the functions available in Maxima to obtain analytic solutions for some specific types of first and second-order equations. To obtain a numerical solution for a system of differential equations, see the additional package dynamics. For graphical representations in phase space, see the additional package plotdf.

21.2 Functions and Variables for Differential Equations

**bc2** (solution, xval1, yval1, xval2, yval2) [Function]

Solves a boundary value problem for a second order differential equation. Here: solution is a general solution to the equation, as found by ode2; xval1 specifies the value of the independent variable in a first point, in the form \(x = x_1\), and yval1 gives the value of the dependent variable in that point, in the form \(y = y_1\). The expressions xval2 and yval2 give the values for these variables at a second point, using the same form.

See ode2 for an example of its usage.

**desolve** [Function]

\[
\text{desolve (eqn, x)} \\
\text{desolve ([eqn_1, ..., eqn_n], [x_1, ..., x_n])}
\]

The function desolve solves systems of linear ordinary differential equations using Laplace transform. Here the eqn’s are differential equations in the dependent variables \(x_1, ..., x_n\). The functional dependence of \(x_1, ..., x_n\) on an independent variable, for instance \(x\), must be explicitly indicated in the variables and its derivatives. For example, this would not be the correct way to define two equations:

\[
\text{eqn_1: } \frac{d^2 f(x)}{dx^2} = \sin(x) + \frac{d f(x)}{dx}; \\
\text{eqn_2: } \frac{d f(x)}{dx} + x^2 - f(x) = 2\frac{d^2 g(x)}{dx^2};
\]

The correct way would be:

\[
\text{eqn_1: } \frac{d^2 f(x)}{dx^2} = \sin(x) + \frac{d f(x)}{dx}; \\
\text{eqn_2: } \frac{d f(x)}{dx} + x^2 - f(x) = 2\frac{d^2 g(x)}{dx^2};
\]

The call to the function desolve would then be

\[
\text{desolve([eqn_1, eqn_2], [f(x),g(x)]);}
\]

If initial conditions at \(x=0\) are known, they can be supplied before calling desolve by using atvalue.

\[
\text{(%i1) } \frac{d f(x)}{dx} = \frac{d f(x)}{dx} + \sin(x); \\
\text{(%o1) } \frac{--(f(x))}{dx} = \frac{--(g(x))}{dx} + \sin(x) \\
\text{(%i2) } \frac{d^2 f(x)}{dx^2} = \frac{d^2 f(x)}{dx^2} - \cos(x);
\]
\[
\frac{2}{\text{dx}} \frac{\text{d}}{\text{dx}} (g(x)) = \frac{\text{d}}{\text{dx}} (f(x)) - \cos(x)
\]

(%i3) \text{atvalue('diff(g(x),x),x=0,a);} \\
(%o3) a \\
(%i4) \text{atvalue(f(x),x=0,1);} \\
(%o4) 1 \\
(%i5) \text{desolve([%o1,%o2],[f(x),g(x)]);} \\
(%o5) \{f(x) = a \text{e}^x - a + 1, g(x) = \}
\]
\[
x \cos(x) + a \text{e}^x - a + g(0) - 1 \]

(%i6) [%o1,%o2],%o5,diff; \\
(%o6) \{a \text{e}^x = a \text{e}^x, a \text{e}^x - \cos(x) = a \text{e}^x - \cos(x)\}

If \text{desolve} cannot obtain a solution, it returns \text{false}.

\textbf{ic1 \ (solution, xval, yval)}

Solves initial value problems for first order differential equations. Here \text{solution} is a general solution to the equation, as found by \text{ode2}, \text{xval} gives an initial value for the independent variable in the form \text{x} = x_0, and \text{yval} gives the initial value for the dependent variable in the form \text{y} = y_0.

See \text{ode2} for an example of its usage.

\textbf{ic2 \ (solution, xval, yval, dval)}

Solves initial value problems for second-order differential equations. Here \text{solution} is a general solution to the equation, as found by \text{ode2}, \text{xval} gives the initial value for the independent variable in the form \text{x} = x_0, \text{yval} gives the initial value of the dependent variable in the form \text{y} = y_0, and \text{dval} gives the initial value for the first derivative of the dependent variable with respect to independent variable, in the form \text{diff(y,x)} = dy_0 (diff does not have to be quoted).

See \text{ode2} for an example of its usage.

\textbf{ode2 \ (eqn, dvar, ivar)}

The function \text{ode2} solves an ordinary differential equation (ODE) of first or second order. It takes three arguments: an ODE given by \text{eqn}, the dependent variable \text{dvar}, and the independent variable \text{ivar}. When successful, it returns either an explicit or implicit solution for the dependent variable. \%c is used to represent the integration constant in the case of first-order equations, and \%k1 and \%k2 the constants for second-order equations. The dependence of the dependent variable on the independent variable does not have to be written explicitly, as in the case of \text{desolve}, but the independent variable must always be given as the third argument.
If \texttt{ode2} cannot obtain a solution for whatever reason, it returns \texttt{false}, after perhaps printing out an error message. The methods implemented for first order equations in the order in which they are tested are: linear, separable, exact - perhaps requiring an integrating factor, homogeneous, Bernoulli’s equation, and a generalized homogeneous method. The types of second-order equations which can be solved are: constant coefficients, exact, linear homogeneous with non-constant coefficients which can be transformed to constant coefficients, the Euler or equi-dimensional equation, equations solvable by the method of variation of parameters, and equations which are free of either the independent or of the dependent variable so that they can be reduced to two first order linear equations to be solved sequentially.

In the course of solving ODE’s, several variables are set purely for informational purposes: \texttt{method} denotes the method of solution used (e.g., \texttt{linear}), \texttt{intfactor} denotes any integrating factor used, \texttt{odeindex} denotes the index for Bernoulli’s method or for the generalized homogeneous method, and \texttt{yp} denotes the particular solution for the variation of parameters technique.

In order to solve initial value problems (IVP) functions \texttt{ic1} and \texttt{ic2} are available for first and second order equations, and to solve second-order boundary value problems (BVP) the function \texttt{bc2} can be used.

Example:

\begin{verbatim}
(%i1) x^2*'diff(y,x) + 3*y*x = sin(x)/x;
   2 dy sin(x)  
x -- + 3 x y = -------
   dx x
(%i2) ode2(%o1,y,x);
   %c - cos(x)  
y = ------------
       3
      x
(%i3) ic1(%o2,x=%pi,y=0);
   cos(x) + 1  
y = - --------
            3
           x
(%i4) 'diff(y,x,2) + y*'diff(y,x)^3 = 0;
  2  
d y    dy 3
--- + y (--) = 0
 2    dx
(%i5) ode2(%o4,y,x);
 3  
y + 6 %k1 y
----------- = x + %k2
 6
(%i6) ratsimp(ic2(%o5,x=0,y=0,'diff(y,x)=2));
\end{verbatim}
\[
\frac{3}{6} - \frac{2y - 3y}{6} = x
\]

\(%i7\) bc2(%o5, x=0, y=1, x=1, y=3);
\[
\frac{3}{y - 10y}{3} = x - \frac{6}{2}
\]
22 Numerical

22.1 Introduction to fast Fourier transform

The `fft` package comprises functions for the numerical (not symbolic) computation of the fast Fourier transform.

22.2 Functions and Variables for fast Fourier transform

polartorect \((r, t)\)  
Translates complex values of the form \(r \, e^{\, i \, t}\) to the form \(a + b \, i\), where \(r\) is the magnitude and \(t\) is the phase. \(r\) and \(t\) are 1-dimensional arrays of the same size.  
The array size need not be a power of 2.  
The original values of the input arrays are replaced by the real and imaginary parts, \(a\) and \(b\), on return. The outputs are calculated as  
\[
\begin{align*}
a &= r \cos(t) \\
b &= r \sin(t)
\end{align*}
\]

polartorect is the inverse function of `recttopolar`.  
`load(fft)` loads this function. See also `fft`.

recttopolar \((a, b)\)  
Translates complex values of the form \(a + b \, i\) to the form \(r \, e^{\, i \, t}\), where \(a\) is the real part and \(b\) is the imaginary part. \(a\) and \(b\) are 1-dimensional arrays of the same size. The array size need not be a power of 2. 
The original values of the input arrays are replaced by the magnitude and angle, \(r\) and \(t\), on return. The outputs are calculated as  
\[
\begin{align*}
r &= \sqrt{a^2 + b^2} \\
t &= \arctan2(b, a)
\end{align*}
\]
The computed angle is in the range \(-\pi\) to \(\pi\). 
recttopolar is the inverse function of `polartorect`.  
`load(fft)` loads this function. See also `fft`.

inverse_fft \((y)\)  
Computes the inverse complex fast Fourier transform. \(y\) is a list or array (named or unnamed) which contains the data to transform. The number of elements must be a power of 2. The elements must be literal numbers (integers, rationals, floats, or bigfloats) or symbolic constants, or expressions \(a + b \cdot i\) where \(a\) and \(b\) are floats.  
inverse_fft returns a new object of the same type as \(y\), which is not modified. Results are always computed as floats or expressions \(a + b \cdot i\) where \(a\) and \(b\) are floats. If bigfloat precision is needed the function `bf_inverse_fft` can be used instead as a drop-in replacement of `inverse_fft` that is slower, but supports bfloats. 
The inverse discrete Fourier transform is defined as follows. Let \(x\) be the output of the inverse transform. Then for \(j\) from 0 through \(n - 1\),  
\[
x[j] = \sum_{k=0}^{n-1} y[k] \, e^{-2 \, i \, \pi \, j \, k / n}, \quad k, \, 0, \, n - 1
\]
As there are various sign and normalization conventions possible, this definition of the transform may differ from that used by other mathematical software.

load(fft) loads this function.

See also fft (forward transform), recttopolar, and polartorect.

Examples:

Real data.

```
(%i1) load ("fft") $
(%i2) fpprintprec : 4 $
(%i3) L : [1, 2, 3, 4, -1, -2, -3, -4] $
(%i4) L1 : inverse_fft (L); 
(%o4) [0.0, 14.49 %i - .8284, 0.0, 2.485 %i + 4.828, 0.0, 
                 4.828 - 2.485 %i, 0.0, - 14.49 %i - .8284]
(%i5) L2 : fft (L1); 
(%o5) [1.0, 2.0 - 2.168L-19 %i, 3.0 - 7.525L-20 %i, 
        4.0 - 4.256L-19 %i, - 1.0, 2.168L-19 %i - 2.0, 
        7.525L-20 %i - 3.0, 4.256L-19 %i - 4.0]
(%i6) lmax (abs (L2 - L)); 
(%o6) 3.545L-16
```

Complex data.

```
(%i1) load ("fft") $
(%i2) fpprintprec : 4 $
(%i3) L : [1, 1 + %i, 1 - %i, -1, -1 - %i, 1 + %i, 1] $
(%i4) L1 : inverse_fft (L); 
(%o4) [4.0, 2.711L-19 %i + 4.0, 2.0 %i - 2.0, 
       - 2.828 %i - 2.828, 0.0, 5.421L-20 %i + 4.0, - 2.0 %i - 2.0, 
       2.828 %i + 2.828]
(%i5) L2 : fft (L1); 
(%o5) [4.066E-20 %i + 1.0, 1.0 %i + 1.0, 1.0 - 1.0 %i, 
       1.55L-19 %i - 1.0, - 4.066E-20 %i - 1.0, 1.0 - 1.0 %i, 
       1.0 %i + 1.0, 1.0 - 7.368L-20 %i]
(%i6) lmax (abs (L2 - L)); 
(%o6) 6.841L-17
```

**fft (x)**

[Function]

Computes the complex fast Fourier transform. x is a list or array (named or unnamed) which contains the data to transform. The number of elements must be a power of 2. The elements must be literal numbers (integers, rationals, floats, or bigfloats) or symbolic constants, or expressions \(a + b\times\%i\) where \(a\) and \(b\) are literal numbers or symbolic constants.

**fft** returns a new object of the same type as \(x\), which is not modified. Results are always computed as floats or expressions \(a + b\times\%i\) where \(a\) and \(b\) are floats. If bigfloat precision is needed the function **bf_fft** can be used instead as a drop-in replacement of **fft** that is slower, but supports bfloats. In addition if it is known that the input consists of only real values (no imaginary parts), **real_fft** can be used which is potentially faster.
The discrete Fourier transform is defined as follows. Let \( y \) be the output of the transform. Then for \( k \) from 0 through \( n - 1 \),
\[
y[k] = \frac{1}{n} \sum_{j=0}^{n-1} x[j] \exp(+2 \pi i j k / n), \quad j, \ 0, \ n - 1
\]
As there are various sign and normalization conventions possible, this definition of the transform may differ from that used by other mathematical software.

When the data \( x \) are real, real coefficients \( a \) and \( b \) can be computed such that
\[
x[j] = \sum a[k] \cos(2\pi j k / n) + b[k] \sin(2\pi j k / n), \quad k, \ 0, \ n/2
\]
with
\[
a[0] = \text{realpart} (y[0])
\]
\[
b[0] = 0
\]
and, for \( k \) from 1 through \( n/2 - 1 \),
\[
a[k] = \text{realpart} (y[k] + y[n - k])
\]
\[
b[k] = \text{imagpart} (y[n - k] - y[k])
\]
and
\[
a[n/2] = \text{realpart} (y[n/2])
\]
\[
b[n/2] = 0
\]

load(fft) loads this function.

See also inverse_fft (inverse transform), recttopolar, and polartorect. See real_fft for FFTs of a real-valued input, and bf_fft and bf_real_fft for operations on bigfloat values.

Examples:

Real data.

```lisp
(%i1) load ("fft") $
(%i2) fpprintprec : 4 $
(%i3) L : [1, 2, 3, 4, -1, -2, -3, -4] $
(%i4) L1 : fft (L); 
(%o4) [0.0, - 1.811 %i - .1036, 0.0, .6036 - .3107 %i, 0.0, .3107 %i + .6036, 0.0, 1.811 %i - .1036]
(%i5) L2 : inverse_fft (L1); 
(%o5) [1.0, 2.168L-19 %i + 2.0, 7.525L-20 %i + 3.0, 4.256L-19 %i + 4.0, - 1.0, - 2.168L-19 %i - 2.0, - 7.525L-20 %i - 3.0, - 4.256L-19 %i - 4.0]
(%i6) lmax (abs (L2 - L)); 
(%o6) 3.545L-16
```

Complex data.

```lisp
(%i1) load ("fft") $
(%i2) fpprintprec : 4 $
(%i3) L : [1, 1 + %i, 1 - %i, -1, -1, 1 - %i, 1 + %i, 1] $
(%i4) L1 : fft (L); 
(%o4) [0.5, .3536 %i + .3536, - 0.25 %i - 0.25, 0.5 - 6.776L-21 %i, 0.0, - .3536 %i - .3536, 0.25 %i - 0.25, 0.5 - 3.388L-20 %i]
(%i5) L2 : inverse_fft (L1); 
```
(\%o5) [1.0 - 4.066E-20 %i, 1.0 %i + 1.0, 1.0 - 1.0 %i, \\
- 1.008L-19 %i - 1.0, 4.066E-20 %i - 1.0, 1.0 - 1.0 %i, \\
1.0 %i + 1.0, 1.947L-20 %i + 1.0]
(\%i6) lmax (abs (L2 - L));
(\%o6) 6.83L-17

Computation of sine and cosine coefficients.
(\%i11) load ("fft")$
(\%i12) fpprintprec : 4$
(\%i13) L : [1, 2, 3, 4, 5, 6, 7, 8]$
(\%i14) n : length (L)$
(\%i15) x : make_array (any, n)$
(\%i16) fillarray (x, L)$
(\%i17) y : fft (x)$
(\%i18) a : make_array (any, n/2 + 1)$
(\%i19) b : make_array (any, n/2 + 1)$
(\%i20) a[0] : realpart (y[0])$
(\%i21) b[0] : 0$
(\%i22) for k : 1 thru n/2 - 1 do
   (a[k] : realpart (y[k] + y[n - k]),
    b[k] : imagpart (y[n - k] - y[k]));
(\%o22) done
(\%i23) a[n/2] : y[n/2]$
(\%i24) b[n/2] : 0$
(\%i25) listarray (a);
(\%o25) [4.5, - 1.0, - 1.0, - 1.0, - 0.5]
(\%i26) listarray (b);
(\%o26) [0, - 2.414, - 1.0, - .4142, 0]
(\%i27) f(j) := sum (a[k]*cos(2*%pi*j*k/n) + b[k]*sin(2*%pi*j*k/n), \\
    k, 0, n/2) $
(\%i28) makelist (float (f (j)), j, 0, n - 1);
(\%o28) [1.0, 2.0, 3.0, 4.0, 5.0, 6.0, 7.0, 8.0]

real_fft (x) [Function]
Computes the fast Fourier transform of a real-valued sequence x. This is equivalent to performing fft(x), except that only the first N/2+1 results are returned, where N is the length of x. N must be power of two.
No check is made that x contains only real values.
The symmetry properites of the Fourier transform of real sequences to reduce he complexity. In particular the first and last output values of real_fft are purely real. For larger sequencies, real_fft may be computed more quickly than fft.
Since the output length is short, the normal inverse_fft cannot be directly used. Use inverse_real_fft to compute the inverse.

inverse_real_fft (y) [Function]
Computes the inverse Fourier transfrom of y, which must have a length of N/2+1 where N is a power of two. That is, the input x is expected to be the output of real_fft.
No check is made to ensure that the input has the correct format. (The first and last elements must be purely real.)

**bf_inverse_fft** *(y)*

Computes the inverse complex fast Fourier transform. This is the bigfloat version of `inverse_fft` that converts the input to bigfloats and returns a bigfloat result.

**bf_fft** *(y)*

Computes the forward complex fast Fourier transform. This is the bigfloat version of `fft` that converts the input to bigfloats and returns a bigfloat result.

**bf_real_fft** *(x)*

Computes the forward fast Fourier transform of a real-valued input returning a bigfloat result. This is the bigfloat version of `real_fft`.

**bf_inverse_real_fft** *(y)*

Computes the inverse fast Fourier transform with a real-valued bigfloat output. This is the bigfloat version of `inverse_real_fft`.

### 22.3 Functions for numerical solution of equations

**horner** *(Function)*

`horner (expr, x)`

`horner (expr)`

Returns a rearranged representation of `expr` as in Horner’s rule, using `x` as the main variable if it is specified. `x` may be omitted in which case the main variable of the canonical rational expression form of `expr` is used.

`horner` sometimes improves stability if `expr` is to be numerically evaluated. It is also useful if Maxima is used to generate programs to be run in Fortran. See also `stringout`.

```
(%i1) expr: 1e-155*x^2 - 5.5*x + 5.2e155;
2
(%o1) 1.e-155 x - 5.5 x + 5.2e+155

(%i2) expr2: horner (%(x), keepfloat: true);
(%o2) 1.0 ((1.e-155 x - 5.5) x + 5.2e+155)

(%i3) ev (expr, x=1e155);
Maxima encountered a Lisp error:

arithmetic error FLOATING-POINT-OVERFLOW signalled

Automatically continuing.
To enable the Lisp debugger set *debugger-hook* to nil.

(%i4) ev (expr2, x=1e155);
(%o4) 7.00000000000001e+154
```

**find_root** *(expr, x, a, b, [abserr, relerr])*  
**find_root** *(f, a, b, [abserr, relerr])*  
**bf_find_root** *(expr, x, a, b, [abserr, relerr])*

[Function]

[Function]

[Function]
bf_find_root (f, a, b, [abserr, relerr])

find_root_error
find_root_abs
find_root_rel

Finds a root of the expression expr or the function \( f \) over the closed interval \([a, b]\).

The expression expr may be an equation, in which case \( \text{find_root} \) seeks a root of \( \text{lhs(expr)} - \text{rhs(expr)} \).

Given that Maxima can evaluate expr or \( f \) over \([a, b]\) and that expr or \( f \) is continuous, \( \text{find_root} \) is guaranteed to find the root, or one of the roots if there is more than one.

\( \text{find_root} \) initially applies binary search. If the function in question appears to be smooth enough, \( \text{find_root} \) applies linear interpolation instead.

bf_find_root is a bigfloat version of \( \text{find_root} \). The function is computed using bigfloat arithmetic and a bigfloat result is returned. Otherwise, \( \text{bf_find_root} \) is identical to \( \text{find_root} \), and the following description is equally applicable to \( \text{bf_find_root} \).

The accuracy of \( \text{find_root} \) is governed by \( \text{abserr} \) and \( \text{relerr} \), which are optional keyword arguments to \( \text{find_root} \). These keyword arguments take the form \text{key}=\text{val}.

The keyword arguments are

\begin{itemize}
  \item \text{abserr} \quad \text{Desired absolute error of function value at root. Default is} \text{find_root_abs}.
  \item \text{relerr} \quad \text{Desired relative error of root. Default is} \text{find_root_rel}.
\end{itemize}

\( \text{find_root} \) stops when the function in question evaluates to something less than or equal to \( \text{abserr} \), or if successive approximants \( x_0, x_1 \) differ by no more than \( \text{relerr} \cdot \max(\text{abs}(x_0), \text{abs}(x_1)) \). The default values of \( \text{find_root_abs} \) and \( \text{find_root_rel} \) are both zero.

\( \text{find_root} \) expects the function in question to have a different sign at the endpoints of the search interval. When the function evaluates to a number at both endpoints and these numbers have the same sign, the behavior of \( \text{find_root} \) is governed by \text{find_root_error}. When \text{find_root_error} is true, \( \text{find_root} \) prints an error message. Otherwise \( \text{find_root} \) returns the value of \text{find_root_error}. The default value of \text{find_root_error} is true.

If \( f \) evaluates to something other than a number at any step in the search algorithm, \( \text{find_root} \) returns a partially-evaluated \( \text{find_root} \) expression.

The order of \( a \) and \( b \) is ignored; the region in which a root is sought is \([\min(a, b), \max(a, b)]\).

Examples:

\begin{verbatim}
(%i1) f(x) := sin(x) - x/2;

(%o1)

f(x) := sin(x) - x/2

(%i2) find_root (sin(x) - x/2, x, 0.1, %pi);

(%o2) 1.895494267033981
\end{verbatim}
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(newton (expr, x, x_0, eps))
Returns an approximate solution of \( \text{expr} = 0 \) by Newton’s method, considering \( \text{expr} \) to be a function of one variable, \( x \). The search begins with \( x = x_0 \) and proceeds until \( \text{abs(\( \text{expr} \))} < \text{eps} \) (with \( \text{expr} \) evaluated at the current value of \( x \)).

newton allows undefined variables to appear in \( \text{expr} \), so long as the termination test \( \text{abs(\( \text{expr} \))} < \text{eps} \) evaluates to true or false. Thus it is not necessary that \( \text{expr} \) evaluate to a number.

load(newton1) loads this function.

See also realroots, allroots, find_root and Chapter 72 [mnewton-pkg], page 1005.

Examples:

(%i1) load ("newton1");
(%i1) /maxima/share/numeric/newton1.mac
(%i2) newton (cos (u), u, 1, 1/100);
(%i3) ev (cos (u), u = %);
(%i4) assume (a > 0);
(%i5) newton (x^2 - a^2, x, a/2, a^2/100);
(%i6) ev (x^2 - a^2, x = %);
22.4 Introduction to numerical solution of differential equations

The Ordinary Differential Equations (ODE) solved by the functions in this section should have the form,
\[ \frac{dy}{dx} = F(x, y) \]
which is a first-order ODE. Higher order differential equations of order \( n \) must be written as a system of \( n \) first-order equations of that kind. For instance, a second-order ODE should be written as a system of two equations
\[ \frac{dx}{dt} = G(x, y, t) \quad \frac{dy}{dt} = F(x, y, t) \]

The first argument in the functions will be a list with the expressions on the right-side of the ODE’s. The variables whose derivatives are represented by those expressions should be given in a second list. In the case above those variables are \( x \) and \( y \). The independent variable, \( t \) in the examples above, might be given in a separated option. If the expressions given do not depend on that independent variable, the system is called autonomous.

22.5 Functions for numerical solution of differential equations

\texttt{plotdf}

The function \texttt{plotdf} creates a two-dimensional plot of the direction field (also called slope field) for a first-order Ordinary Differential Equation (ODE) or a system of two autonomous first-order ODE’s.

\texttt{plotdf} requires \texttt{Xmaxima}, even if its run from a \texttt{Maxima} session in a console, since the plot will be created by the Tk scripts in \texttt{Xmaxima}. If \texttt{Xmaxima} is not installed \texttt{plotdf} will not work.

\( dydx, \ dxdt \) and \( dydt \) are expressions that depend on \( x \) and \( y \). \( dvdu, \ dudt \) and \( dvdt \) are expressions that depend on \( u \) and \( v \). In addition to those two variables, the expressions can also depend on a set of parameters, with numerical values given with the \texttt{parameters} option (the option syntax is given below), or with a range of allowed values specified by a \texttt{sliders} option.

Several other options can be given within the command, or selected in the menu. Integral curves can be obtained by clicking on the plot, or with the option \texttt{trajectory.at}. The direction of the integration can be controlled with the \texttt{direction} option, which can have values of \texttt{forward}, \texttt{backward} or \texttt{both}. The number of integration steps is given by \texttt{nsteps}; at each integration step the time increment will be adjusted automatically to produce displacements much smaller than the size of the plot window. The numerical method used is 4th order Runge-Kutta with variable time steps.

\textbf{Plot window menu:}

The menu bar of the plot window has the following seven icons:
An X. Can be used to close the plot window.

A wrench and a screwdriver. Opens the configuration menu with several fields that show the ODE(s) in use and various other settings. If a pair of coordinates are entered in the field Trajectory at and the enter key is pressed, a new integral curve will be shown, in addition to the ones already shown.

Two arrows following a circle. Replots the direction field with the new settings defined in the configuration menu and replots only the last integral curve that was previously plotted.

Hard disk drive with an arrow. Used to save a copy of the plot, in Postscript format, in the file specified in a field of the box that appears when that icon is clicked.

Magnifying glass with a plus sign. Zooms in the plot.

Magnifying glass with a minus sign. Zooms out the plot. The plot can be displaced by holding down the right mouse button while the mouse is moved.

Icon of a plot. Opens another window with a plot of the two variables in terms of time, for the last integral curve that was plotted.

**Plot options:**

Options can also be given within the `plotdf` itself, each one being a list of two or more elements. The first element in each option is the name of the option, and the remainder is the value or values assigned to the option.

The options which are recognized by `plotdf` are the following:

- **nsteps** defines the number of steps that will be used for the independent variable, to compute an integral curve. The default value is 100.

- **direction** defines the direction of the independent variable that will be followed to compute an integral curve. Possible values are `forward`, to make the independent variable increase `nsteps` times, with increments `tstep`, `backward`, to make the independent variable decrease, or `both` that will lead to an integral curve that extends `nsteps` forward, and `nsteps` backward. The keywords `right` and `left` can be used as synonyms for `forward` and `backward`. The default value is `both`.

- **tinitial** defines the initial value of variable `t` used to compute integral curves. Since the differential equations are autonomous, that setting will only appear in the plot of the curves as functions of `t`. The default value is 0.

- **versus_t** is used to create a second plot window, with a plot of an integral curve, as two functions `x`, `y`, of the independent variable `t`. If `versus_t` is given any value different from 0, the second plot window will be displayed. The second plot window includes another menu, similar to the menu of the main plot window. The default value is 0.

- **trajectory_at** defines the coordinates `xinitial` and `yinitial` for the starting point of an integral curve. The option is empty by default.

- **parameters** defines a list of parameters, and their numerical values, used in the definition of the differential equations. The name and values of the parameters must be given in a string with a comma-separated sequence of pairs `name=value`.

- **sliders** defines a list of parameters that will be changed interactively using slider buttons, and the range of variation of those parameters. The names and ranges
of the parameters must be given in a string with a comma-separated sequence of elements \texttt{name=min:max}

- \texttt{xfun} defines a string with semi-colon-separated sequence of functions of \(x\) to be displayed, on top of the direction field. Those functions will be parsed by Tcl and not by Maxima.

- \(x\) should be followed by two numbers, which will set up the minimum and maximum values shown on the horizontal axis. If the variable on the horizontal axis is not \(x\), then this option should have the name of the variable on the horizontal axis. The default horizontal range is from -10 to 10.

- \(y\) should be followed by two numbers, which will set up the minimum and maximum values shown on the vertical axis. If the variable on the vertical axis is not \(y\), then this option should have the name of the variable on the vertical axis. The default vertical range is from -10 to 10.

- \texttt{xaxislabel} will be used to identify the horizontal axis. Its default value is the name of the first state variable.

- \texttt{yaxislabel} will be used to identify the vertical axis. Its default value is the name of the second state variable.

- \texttt{number_of_arrows} should be set to a square number and defines the approximate density of the arrows being drawn. The default value is 225.

Examples:

- To show the direction field of the differential equation \(y' = \exp(-x) + y\) and the solution that goes through \((2, -0.1)\):

  \[
  \texttt{(\%i1) plotdf(exp(-x)+y,[trajectory_at,2,-0.1])}
  \]

- To obtain the direction field for the equation \(\text{diff}(y, x) = x - y^2\) and the solution with initial condition \(y(-1) = 3\), we can use the command:
The graph also shows the function $y = \sqrt{x}$.

- The following example shows the direction field of a harmonic oscillator, defined by the two equations $\frac{dz}{dt} = v$ and $\frac{dv}{dt} = -kz/m$, and the integral curve through $(z,v) = (6,0)$, with a slider that will allow you to change the value of $m$ interactively ($k$ is fixed at 2):

  ```latex
  (%i1) plotdf([v,-k*z/m], [z,v], [parameters,"m=2,k=2"],
                 [sliders,"m=1:5"], [trajectory_at,6,0])$
  ```
To plot the direction field of the Duffing equation, \( mx'' + cx' + kx + bx^3 = 0 \), we introduce the variable \( y = x' \) and use:

\[
(\%i1) \text{plotdf}([y,-(k*x + c*y + b*x^3)/m],
\text{[parameters,"k=-1,m=1.0,c=0,b=1"],}
\text{[sliders,"k=-2:2,m=-1:1"],[tstep,0.1]])$
\]

The direction field for a damped pendulum, including the solution for the given initial conditions, with a slider that can be used to change the value of the mass \( m \), and with a plot of the two state variables as a function of time:
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\[
(%i1) \text{plotdf([}w,-g\sin(a)/l - b*w/m/l], [a,w],
\text{[parameters,"g=9.8,l=0.5,m=0.3,b=0.05"],}
\text{[trajectory_at,1.05,-9],[tstep,0.01],}
\text{[a,-10,2], [w,-14,14], [direction,forward],}
\text{[nsteps,300], [sliders,"m=0.1:1"], [versus_t,1]}\)
\]

\[
\text{ploteq (exp, ...options...)}
\]
Plots equipotential curves for exp, which should be an expression depending on two variables. The curves are obtained by integrating the differential equation that define the orthogonal trajectories to the solutions of the autonomous system obtained from
the gradient of the expression given. The plot can also show the integral curves for
that gradient system (option fieldlines).
This program also requires Xmaxima, even if its run from a Maxima session in a
console, since the plot will be created by the Tk scripts in Xmaxima. By default, the
plot region will be empty until the user clicks in a point (or gives its coordinate with
in the set-up menu or via the trajectory._at option).
Most options accepted by plotdf can also be used for ploteq and the plot interface is
the same that was described in plotdf.
Example:

```
(%i1) V: 900/((x+1)^2+y^2)^(1/2)-900/((x-1)^2+y^2)^(1/2)$
(%i2) ploteq(V,[x,-2,2],[y,-2,2],[fieldlines,"blue"])$
```
Clicking on a point will plot the equipotential curve that passes by that point (in red)
and the orthogonal trajectory (in blue).

**rk**

```
rk (ODE, var, initial, domain)
rk ([ODE1, ..., ODEm], [v1, ..., vm], [init1, ..., initm], domain)
```
The first form solves numerically one first-order ordinary differential equation, and the
second form solves a system of m of those equations, using the 4th order Runge-Kutta
method. var represents the dependent variable. ODE must be an expression that
depends only on the independent and dependent variables and defines the derivative
of the dependent variable with respect to the independent variable.
The independent variable is specified with domain, which must be a list of four ele-
ments as, for instance:

```
[t, 0, 10, 0.1]
```
The first element of the list identifies the independent variable, the second and third
elements are the initial and final values for that variable, and the last element sets
the increments that should be used within that interval.
If m equations are going to be solved, there should be m dependent variables v1, v2,
..., vm. The initial values for those variables will be init1, init2, ..., initm. There
will still be just one independent variable defined by domain, as in the previous case.
ODE1, ..., ODEm are the expressions that define the derivatives of each dependent
variable in terms of the independent variable. The only variables that may appear in
those expressions are the independent variable and any of the dependent variables. It
is important to give the derivatives ODE1, ..., ODEm in the list in exactly the same
order used for the dependent variables; for instance, the third element in the list will
be interpreted as the derivative of the third dependent variable.
The program will try to integrate the equations from the initial value of the inde-
pendent variable until its last value, using constant increments. If at some step one
of the dependent variables takes an absolute value too large, the integration will be
interrupted at that point. The result will be a list with as many elements as the
number of iterations made. Each element in the results list is itself another list with
m+1 elements: the value of the independent variable, followed by the values of the
dependent variables corresponding to that point.
To solve numerically the differential equation
\[
\frac{dx}{dt} = t - x^2
\]

With initial value \(x(t=0) = 1\), in the interval of \(t\) from 0 to 8 and with increments of 0.1 for \(t\), use:

\[
(\%i1) \text{results: rk}(t-x^2,x,1,[t,0,8,0.1])$
\]
\[
(\%i2) \text{plot2d ([discrete, results])}$
\]

the results will be saved in the list \texttt{results} and the plot will show the solution obtained, with \(t\) on the horizontal axis and \(x\) on the vertical axis.

To solve numerically the system:

\[
\begin{align*}
\frac{dx}{dt} &= 4 - x^2 - 4y^2 \\
\frac{dy}{dt} &= y^2 - x^2 + 1
\end{align*}
\]

for \(t\) between 0 and 4, and with values of -1.25 and 0.75 for \(x\) and \(y\) at \(t=0\):

\[
(\%i1) \text{sol: rk}([4-x^2-4*y^2,y^2-x^2+1],[x,y],[-1.25,0.75],[t,0,4,0.02])$
\]
\[
(\%i2) \text{plot2d ([discrete,makelist([p[1],p[3]],p,sol]), [xlabel,"t"],[ylabel,"y"]})$
\]

The plot will show the solution for variable \(y\) as a function of \(t\).
23 Matrices and Linear Algebra

23.1 Introduction to Matrices and Linear Algebra

23.1.1 Dot

The operator . represents noncommutative multiplication and scalar product. When the operands are 1-column or 1-row matrices $a$ and $b$, the expression $a \cdot b$ is equivalent to \( \text{sum}(a[i] \cdot b[i], i, 1, \text{length}(a)) \). If $a$ and $b$ are not complex, this is the scalar product, also called the inner product or dot product, of $a$ and $b$. The scalar product is defined as $\text{conjugate}(a) \cdot b$ when $a$ and $b$ are complex; \text{innerproduct} in the \text{eigen} package provides the complex scalar product.

When the operands are more general matrices, the product is the matrix product $a$ and $b$. The number of rows of $b$ must equal the number of columns of $a$, and the result has number of rows equal to the number of rows of $a$ and number of columns equal to the number of columns of $b$.

To distinguish . as an arithmetic operator from the decimal point in a floating point number, it may be necessary to leave spaces on either side. For example, $5 . e3$ is $5000.0$ but $5 . e3$ is $5$ times $e3$.

There are several flags which govern the simplification of expressions involving ., namely \text{dot0nscsimp}, \text{dot0simp}, \text{dot1simp}, \text{dotassoc}, \text{dotconstrules}, \text{dotdistrib}, \text{dotexptsimp}, \text{dotident}, and \text{dotscrules}.

23.1.2 Vectors

\text{vect} is a package of functions for vector analysis. \text{load ("vect")} loads this package, and \text{demo ("vect")} displays a demonstration.

The vector analysis package can combine and simplify symbolic expressions including dot products and cross products, together with the gradient, divergence, curl, and Laplacian operators. The distribution of these operators over sums or products is governed by several flags, as are various other expansions, including expansion into components in any specific orthogonal coordinate systems. There are also functions for deriving the scalar or vector potential of a field.

The \text{vect} package contains these functions: \text{vectorsimp}, \text{scalefactors}, \text{express}, \text{potential}, and \text{vectorpotential}.

By default the \text{vect} package does not declare the dot operator to be a commutative operator. To get a commutative dot operator ., the command \text{declare(".", commutative)} must be executed.

23.1.3 eigen

The package \text{eigen} contains several functions devoted to the symbolic computation of eigenvalues and eigenvectors. Maxima loads the package automatically if one of the functions \text{eigenvalues} or \text{eigenvectors} is invoked. The package may be loaded explicitly as \text{load ("eigen")}. 

demo ("eigen") displays a demonstration of the capabilities of this package. batch ("eigen") executes the same demonstration, but without the user prompt between successive computations.

The functions in the eigen package are:
innerproduct, unitvector, columnvector, gramschmidt, eigenvalues, eigenvectors, uniteigenvectors, and similaritytransform.

23.2 Functions and Variables for Matrices and Linear Algebra

addcol (M, list_1, ..., list_n) [Function]
    Appends the column(s) given by the one or more lists (or matrices) onto the matrix M.
    See also addrow and append.

addrow (M, list_1, ..., list_n) [Function]
    Appends the row(s) given by the one or more lists (or matrices) onto the matrix M.
    See also addcol and append.

adjoint (M) [Function]
    Returns the adjoint of the matrix M. The adjoint matrix is the transpose of the
    matrix of cofactors of M.

augcoefmatrix ([eqn_1, ..., eqn_m], [x_1, ..., x_n]) [Function]
    Returns the augmented coefficient matrix for the variables x_1, ..., x_n of the system
    of linear equations eqn_1, ..., eqn_m. This is the coefficient matrix with a column
    adjoined for the constant terms in each equation (i.e., those terms not dependent
    upon x_1, ..., x_n).

    (%i1) m: [2*x - (a - 1)*y = 5*b, c + b*y + a*x = 0];
    (%i2) augcoefmatrix (m, [x, y]);
    (%o2) 
    [ 2 1 - a - 5 b ]
    [ a b c ]

cauhcy_matrix [Function]
    cauchy_matrix ([x_1, x_2, ..., x_m], [y_1, y_2, ..., y_n])
    cauchy_matrix ([x_1, x_2, ..., x_n])
    Returns a n by m Cauchy matrix with the elements a[i,j] = 1/(x_i+y_i). The second
    argument of cauchy_matrix is optional. For this case the elements of the Cauchy
    matrix are a[i,j] = 1/(x_i+x_j).
    Remark: In the literature the Cauchy matrix can be found defined in two forms. A
    second definition is a[i,j] = 1/(x_i-y_i).
    Examples:

    (%i1) cauchy_matrix([[x1,x2],[y1,y2]]);
Chapter 23: Matrices and Linear Algebra

\[
\begin{bmatrix}
1 & 1 \\
\hline
y1 + x1 & y2 + x1
\end{bmatrix}
\]

(%o1)
\[
\begin{bmatrix}
1 & 1 \\
\hline
y1 + x2 & y2 + x2
\end{bmatrix}
\]

(%i2) cauchy_matrix([x1,x2]);
\[
\begin{bmatrix}
1 & 1 \\
\hline
2 x1 & x2 + x1
\end{bmatrix}
\]

(%o2)
\[
\begin{bmatrix}
1 & 1 \\
\hline
x2 + x1 & 2 x2
\end{bmatrix}
\]

ccharpoly \(M, x\)

Returns the characteristic polynomial for the matrix \(M\) with respect to variable \(x\).
That is, \(
\text{determinant}(M - \text{diagmatrix}(\text{length}(M), x))
\).

(%i1) a: matrix([3, 1], [2, 4]);
\[
\begin{bmatrix}
3 & 1 \\
\hline
2 & 4
\end{bmatrix}
\]

(%o1)
\[
\begin{bmatrix}
3 & 1 \\
\hline
2 & 4
\end{bmatrix}
\]

(%i2) expand (ccharpoly (a, lambda));
\[
2
\]

(%i3) programmode: true, solve (%);
\[
\text{lambda} = 5, \text{lambda} = 2
\]

(%i4) matrix([x1], [x2]);
\[
\begin{bmatrix}
x1 \\
\hline
x2
\end{bmatrix}
\]

(%o4)
\[
\begin{bmatrix}
x1 \\
\hline
x2
\end{bmatrix}
\]

(%i5) ev (a . % - lambda*%, %th(2)[1]);
\[
\begin{bmatrix}
x2 - 2 x1 \\
\hline
2 x1 - x2
\end{bmatrix}
\]

(%i6) %[1, 1] = 0;
(%o6) \(x2 - 2 x1 = 0\)

(%i7) x2^2 + x1^2 = 1;
\[
2
\]

(%o7) \(x2 + x1 = 1\)

(%i8) solve ([%th(2), %], [x1, x2]);
\[
\begin{align*}
(\%o8) \quad [ & x_1 = \frac{-1}{\sqrt{5}}, \quad x_2 = \frac{-2}{\sqrt{5}}, \\
& \sqrt{5} \quad \sqrt{5} \\
& [ & x_1 = \frac{1}{\sqrt{5}}, \quad x_2 = \frac{2}{\sqrt{5}}, \\
& \sqrt{5} \quad \sqrt{5} \\
\end{align*}
\]

\text{Function} \quad \text{coefmatrix} ([eqn_1, \ldots, eqn_m], [x_1, \ldots, x_n])

Returns the coefficient matrix for the variables \(x_1, \ldots, x_n\) of the system of linear equations \(eqn_1, \ldots, eqn_m\).

\[
(\%i1) \quad \text{coefmatrix}([[2*x-(a-1)*y+5*b = 0, b*y+a*x = 3]], [x,y]);
\]
\[
\begin{bmatrix}
2 & 1 & a \\
1 & b & 1 \\
\end{bmatrix}
\]

\text{col} (M, i)

Returns the \(i\)’th column of the matrix \(M\). The return value is a matrix.

\text{columnvector} (L)

\text{covect} (L)

Returns a matrix of one column and \text{length} (L) rows, containing the elements of the list \(L\).

covect is a synonym for columnvector.

load ("eigen") loads this function.

This is useful if you want to use parts of the outputs of the functions in this package in matrix calculations.

Example:

\[
(\%i1) \quad \text{load ("eigen")$}
\quad \text{Warning - you are redefining the Macsyma function eigenvalues}
\quad \text{Warning - you are redefining the Macsyma function eigenvectors}
(\%i2) \quad \text{columnvector} ([aa, bb, cc, dd]);
\]
\[
\begin{bmatrix}
aa \\
bb \\
cc \\
dd \\
\end{bmatrix}
\]

\text{copymatrix} (M)

Returns a copy of the matrix \(M\). This is the only way to make a copy aside from copying \(M\) element by element.

Note that an assignment of one matrix to another, as in \(m2: m1\), does not copy \(m1\). An assignment \(m2 [i,j] : x\) or \text{setelmx}(x, i, j, m2) also modifies \(m1 [i,j]\). Creating a copy with copymatrix and then using assignment creates a separate, modified copy.
**determinant (M)**

*Function*

Computes the determinant of $M$ by a method similar to Gaussian elimination. The form of the result depends upon the setting of the switch `ratmx`.

There is a special routine for computing sparse determinants which is called when the switches `ratmx` and `sparse` are both `true`.

**detout**

*Option variable*

Default value: `false`

When `detout` is `true`, the determinant of a matrix whose inverse is computed is factored out of the inverse.

For this switch to have an effect `doallmxops` and `doscmxops` should be `false` (see their descriptions). Alternatively this switch can be given to `ev` which causes the other two to be set correctly.

Example:

```
(%i1) m: matrix ([a, b], [c, d]);
     [ a  b ]
     [     ]
     [ c  d ]
(%i2) detout: true$
(%i3) doallmxops: false$
(%i4) doscmxops: false$
(%i5) invert (m);
     [ d  - b ]
     [        ]
     [ - c   a ]
     ------------
a d - b c
```

**diagmatrix (n, x)**

*Function*

Returns a diagonal matrix of size $n$ by $n$ with the diagonal elements all equal to $x$. `diagmatrix (n, 1)` returns an identity matrix (same as `ident (n)`).

$n$ must evaluate to an integer, otherwise `diagmatrix` complains with an error message.

$x$ can be any kind of expression, including another matrix. If $x$ is a matrix, it is not copied; all diagonal elements refer to the same instance, $x$.

**doallmxops**

*Option variable*

Default value: `true`

When `doallmxops` is `true`, all operations relating to matrices are carried out. When it is `false` then the setting of the individual dot switches govern which operations are performed.

**domxexpt**

*Option variable*

Default value: `true`

When `domxexpt` is `true`, a matrix exponential, $\exp (M)$ where $M$ is a matrix, is interpreted as a matrix with element $[i,j]$ equal to $\exp (m[i,j])$. Otherwise $\exp (M)$ evaluates to $\exp (ev(M))$. 
**domxexpt** affects all expressions of the form \( \text{base}^\text{power} \) where \( \text{base} \) is an expression assumed scalar or constant, and \( \text{power} \) is a list or matrix.

Example:

```
(%i1) m: matrix ([1, %i], [a+b, %pi]);
     [ 1   %i ]
     [     ]
     [ b + a %pi ]
(%o1) [ ]

(%i2) domxexpt: false$
(%i3) (1 - c)^m;
     [ 1   %i ]
     [     ]
     [ b + a %pi ]
(%o3) (1 - c)

(%i4) domxexpt: true$
(%i5) (1 - c)^m;
     [ %i ]
     [ 1 - c (1 - c) ]
     [     ]
     [ b + a %pi ]
     [ (1 - c) (1 - c) ]
(%o5)
```

**domxmops**

[Option variable]

Default value: true

When **domxmops** is true, all matrix-matrix or matrix-list operations are carried out (but not scalar-matrix operations); if this switch is false such operations are not carried out.

**domxncntimes**

[Option variable]

Default value: false

When **domxncntimes** is true, non-commutative products of matrices are carried out.

**dontfactor**

[Option variable]

Default value: []

dontfactor may be set to a list of variables with respect to which factoring is not to occur. (The list is initially empty.) Factoring also will not take place with respect to any variables which are less important, according the variable ordering assumed for canonical rational expression (CRE) form, than those on the dontfactor list.

**doscmxops**

[Option variable]

Default value: false

When **doscmxops** is true, scalar-matrix operations are carried out.

**doscmxplus**

[Option variable]

Default value: false

When **doscmxplus** is true, scalar-matrix operations yield a matrix result. This switch is not subsumed under doallmxops.
**dot0nscsimp**

[Option variable]

Default value: `true`

When `dot0nscsimp` is `true`, a non-commutative product of zero and a nonscalar term is simplified to a commutative product.

**dot0simp**

[Option variable]

Default value: `true`

When `dot0simp` is `true`, a non-commutative product of zero and a scalar term is simplified to a commutative product.

**dot1simp**

[Option variable]

Default value: `true`

When `dot1simp` is `true`, a non-commutative product of one and another term is simplified to a commutative product.

**dotassoc**

[Option variable]

Default value: `true`

When `dotassoc` is `true`, an expression \((A.B).C\) simplifies to \(A.(B.C)\).

**dotconstrules**

[Option variable]

Default value: `true`

When `dotconstrules` is `true`, a non-commutative product of a constant and another term is simplified to a commutative product. Turning on this flag effectively turns on `dot0simp`, `dot0nscsimp`, and `dot1simp` as well.

**dotdistrib**

[Option variable]

Default value: `false`

When `dotdistrib` is `true`, an expression \(A.(B + C)\) simplifies to \(A.B + A.C\).

**dotexptsimp**

[Option variable]

Default value: `true`

When `dotexptsimp` is `true`, an expression \(A.A\) simplifies to \(A^^2\).

**dotident**

[Option variable]

Default value: 1

`dotident` is the value returned by \(X^^0\).

**dotscrules**

[Option variable]

Default value: `false`

When `dotscrules` is `true`, an expression \(A.SC\) or \(SC.A\) simplifies to \(SC*A\) and \(A.(SC*B)\) simplifies to \(SC*(A.B)\).

**echelon \((M)\)**

[Function]

Returns the echelon form of the matrix \(M\), as produced by Gaussian elimination. The echelon form is computed from \(M\) by elementary row operations such that the first non-zero element in each row in the resulting matrix is one and the column elements under the first one in each row are all zero.
**triangularize** also carries out Gaussian elimination, but it does not normalize the leading non-zero element in each row.

**lu_factor** and **cholesky** are other functions which yield triangularized matrices.

```maxima
(%i1) M: matrix ([3, 7, aa, bb], [-1, 8, 5, 2], [9, 2, 11, 4]);
[ 3 7 aa bb ]
[ -1 8 5 2 ]
[ 9 2 11 4 ]

(%o1)

(%i2) echelon (M);
[ 1 - 8 - 5 - 2 ]
[ ]
[ 28 11 ]
[ 0 1 -- -- ]

(%o2)
[ 37 37 ]
[ ]
[ 37 bb - 119 ]
[ 0 0 1 -- ]
[ 37 aa - 313 ]
```

**eigenvalues** returns a list of two lists containing the eigenvalues of the matrix `M`. The first sublist of the return value is the list of eigenvalues of the matrix, and the second sublist is the list of the multiplicities of the eigenvalues in the corresponding order.

**eival** is a synonym for **eigenvalues**.

**eigenvalues** calls the function **solve** to find the roots of the characteristic polynomial of the matrix. Sometimes **solve** may not be able to find the roots of the polynomial; in that case some other functions in this package (except **innerproduct**, **unitvector**, **columnvector** and **gramschmidt**) will not work. Sometimes **solve** may find only a subset of the roots of the polynomial. This may happen when the factoring of the polynomial contains polynomials of degree 5 or more. In such cases a warning message is displayed and the only the roots found and their corresponding multiplicities are returned.

In some cases the eigenvalues found by **solve** may be complicated expressions. (This may happen when **solve** returns a not-so-obviously real expression for an eigenvalue which is known to be real.) It may be possible to simplify the eigenvalues using some other functions.

The package **eigen.mac** is loaded automatically when **eigenvalues** or **eigenvectors** is referenced. If **eigen.mac** is not already loaded, **load** ("eigen") loads it. After loading, all functions and variables in the package are available.

**eigenvectors** computes eigenvectors of the matrix `M`. The return value is a list of two elements. The first is a list of the eigenvalues of `M` and a list of the multiplicities of the eigen-
values. The second is a list of lists of eigenvectors. There is one list of eigenvectors for each eigenvalue. There may be one or more eigenvectors in each list.

eivects is a synonym for eigenvectors.

The package eigen.mac is loaded automatically when eigenvalues or eigenvectors is referenced. If eigen.mac is not already loaded, load ("eigen") loads it. After loading, all functions and variables in the package are available.

Note that eigenvectors internally calls eigenvalues to obtain eigenvalues. So, when eigenvalues returns a subset of all the eigenvalues, the eigenvectors returns the corresponding subset of the all the eigenvectors, with the same warning displayed as eigenvalues.

The flags that affect this function are:

nondiagonalizable is set to true or false depending on whether the matrix is nondiagonalizable or diagonalizable after eigenvectors returns.

hermitianmatrix when true, causes the degenerate eigenvectors of the Hermitian matrix to be orthogonalized using the Gram-Schmidt algorithm.

knowneigvals when true causes the eigen package to assume the eigenvalues of the matrix are known to the user and stored under the global name listeigvals. listeigvals should be set to a list similar to the output eigenvalues.

The function algsys is used here to solve for the eigenvectors. Sometimes if the eigenvalues are messy, algsys may not be able to find a solution. In some cases, it may be possible to simplify the eigenvalues by first finding them using eigenvalues command and then using other functions to reduce them to something simpler. Following simplification, eigenvectors can be called again with the knowneigvals flag set to true.

See also eigenvalues.

Examples:

A matrix which has just one eigenvector per eigenvalue.

\(\text{(%i1) M1 : matrix ([[11, -1], [1, 7]]);}\)

\(\text{(%o1)}\)

\(\text{(%i2) [vals, vecs] : eigenvectors (M1);}\)

\(\text{(%o2) [[[9 - sqrt(3), sqrt(3) + 9], [1, 1]],}\)

\(\text{[[[1, sqrt(3) + 2]], [[1, 2 - sqrt(3)]]]]}\)

\(\text{(%i3) for i thru length (vals[1]) do disp (val[i] = vals[1][i],}\)

\(\text{mult[i] = vals[2][i], vec[i] = vecs[i]);}\)

\(\text{val = 9 - sqrt(3)}\)

\(\text{1}\)

\(\text{mult = 1}\)

\(\text{1}\)

\(\text{vec = [[1, sqrt(3) + 2]]}\)

\(\text{1}\)
val = sqrt(3) + 9
2

mult = 1
2

vec = [[1, 2 - sqrt(3)]]
2

(%o3) done

A matrix which has two eigenvectors for one eigenvalue (namely 2).

(%i1) M1 : matrix ([0, 1, 0, 0], [0, 0, 0, 0], [0, 0, 2, 0], [0, 0, 0, 2]);
(%o1)
[ 0 1 0 0 ]
[ ]
[ 0 0 0 0 ]
[ ]
[ 0 0 2 0 ]
[ ]
[ 0 0 0 2 ]

(%i2) [vals, vecs] : eigenvectors (M1);
(%o2) [[[0, 2], [2, 2]], [[[1, 0, 0, 0]], [[0, 0, 1, 0], [0, 0, 0, 1]]]]

(%i3) for i thru length (vals[1]) do disp (val[i] = vals[1][i], mult[i] = vals[2][i], vec[i] = vecs[i]);

val = 0
1
mult = 2
1
vec = [[1, 0, 0, 0]]
1

val = 2
2
mult = 2
2
vec = [[0, 0, 1, 0], [0, 0, 0, 1]]
2

(%o3) done
ematrix (m, n, x, i, j)

Returns an $m$ by $n$ matrix, all elements of which are zero except for the $[i, j]$ element which is $x$.

entermatrix (m, n)

Returns an $m$ by $n$ matrix, reading the elements interactively.

If $n$ is equal to $m$, Maxima prompts for the type of the matrix (diagonal, symmetric, antisymmetric, or general) and for each element. Each response is terminated by a semicolon ; or dollar sign $\$.

If $n$ is not equal to $m$, Maxima prompts for each element.

The elements may be any expressions, which are evaluated. entermatrix evaluates its arguments.

entermatrix

(%i1) n: 3$
(%i2) m: entermatrix (n, n)$

Is the matrix 1. Diagonal 2. Symmetric 3. Antisymmetric
4. General
Answer 1, 2, 3 or 4 :
1$
Row 1 Column 1:
(a+b)^n$
Row 2 Column 2:
(a+b)^(n+1)$
Row 3 Column 3:
(a+b)^(n+2)$

Matrix entered.

(%i3) m;

(%o3) 

\[
\begin{bmatrix}
3 \\
(b + a) & 0 & 0 \\
\end{bmatrix}
\]

(%o3) 

\[
\begin{bmatrix}
4 \\
0 & (b + a) & 0 \\
\end{bmatrix}
\]

(%o3) 

\[
\begin{bmatrix}
5 \\
0 & 0 & (b + a) \\
\end{bmatrix}
\]

genmatrix

genmatrix (a, i_2, j_2, i_1, j_1)
genmatrix (a, i_2, j_2, i_1)
genmatrix (a, i_2, j_2)

Returns a matrix generated from $a$, taking element $a[i_1, j_1]$ as the upper-left element and $a[i_2, j_2]$ as the lower-right element of the matrix. Here $a$ is a declared array (created by array but not by make_array) or an undeclared array, or an array function, or a lambda expression of two arguments. (An array function is created like other functions with ::= or define, but arguments are enclosed in square brackets instead of parentheses.)
If $j_1$ is omitted, it is assumed equal to $i_1$. If both $j_1$ and $i_1$ are omitted, both are assumed equal to 1.

If a selected element $i,j$ of the array is undefined, the matrix will contain a symbolic element $a[i,j]$.

Examples:

```
(%i1) h [i, j] := 1 / (i + j - 1);
     1
h := ---------
i, j i + j - 1
(%i2) genmatrix (h, 3, 3); %o2
     [ 1 1 ]
     [ 1 - - ]
     [ 2 3 ]
     [ - - ]
     [ 1 1 1 ]
     [ - - ]
     [ 3 4 5 ]
(%o3) array (a, fixnum, 2, 2);
(%i4) a [1, 1] : %e;
(%o4) %e
(%i5) a [2, 2] : %pi;
(%o5) %pi
(%i6) genmatrix (a, 2, 2); %o6
     [ %e 0 ]
     [ 0 %pi ]
(%i7) genmatrix (lambda ([i, j], j - i), 3, 3); %o7
     [ - 1 0 1 ]
     [ - 2 - 1 0 ]
(%i8) genmatrix (B, 2, 2); %o8
     [ B B ]
     [ 1, 1 1, 2 ]
```
gramschimidt

gramschimidt (x)
gramschimidt (x, F)

Carries out the Gram-Schmidt orthogonalization algorithm on x, which is either a matrix or a list of lists. x is not modified by gramschimidt. The inner product employed by gramschimidt is F, if present, otherwise the inner product is the function innerproduct.

If x is a matrix, the algorithm is applied to the rows of x. If x is a list of lists, the algorithm is applied to the sublists, which must have equal numbers of elements. In either case, the return value is a list of lists, the sublists of which are orthogonal and span the same space as x. If the dimension of the span of x is less than the number of rows or sublists, some sublists of the return value are zero.

factor is called at each stage of the algorithm to simplify intermediate results. As a consequence, the return value may contain factored integers.

load(eigen) loads this function.

Example:
Gram-Schmidt algorithm using default inner product function.

(%i1) load ("eigen")$
(%i2) x: matrix ([1, 2, 3], [9, 18, 30], [12, 48, 60]);
[ 1 2 3 ]
[ ]
(%o2) 
[ 9 18 30 ]
[ ]
[ 12 48 60 ]
(%i3) y: gramschimidt (x);

2 2 4 3
3 3 5 2 3 2 3
(%o3) [[1, 2, 3], [- ---, - --, ---], [- ----, ----, 0]]
2 7 7 5 5

(%i4) map (innerproduct, [y[1], y[2], y[3]], [y[2], y[3], y[1]]);
(%o4) [0, 0, 0]

Gram-Schmidt algorithm using a specified inner product function.

(%i1) load ("eigen")$
(%i2) ip (f, g) := integrate (f * g, u, a, b);

(%o2) ip(f, g) := integrate(f g, u, a, b)
(%i3) y : gramschimidt([1, sin(u), cos(u)], ip), a= -%pi/2, b=%pi/2;

(%o3) [1, sin(u), 0, 0]

(%i4) map (ip, [y[1], y[2], y[3]], [y[2], y[3], y[1]]), a= -%pi/2, b=%pi/2;

(%o4) [0, 0, 0]

ident (n)

Returns an n by n identity matrix.
innerproduct (x, y)                [Function]
inprod (x, y)                     [Function]
    Returns the inner product (also called the scalar product or dot product) of x and y,
    which are lists of equal length, or both 1-column or 1-row matrices of equal length.
    The return value is conjugate (x) . y, where . is the noncommutative multiplication
    operator.
    load ("eigen") loads this function.
inprod is a synonym for innerproduct.
invert_by_adjoint (M)             [Function]
    Returns the inverse of the matrix M. The inverse is computed by the adjoint method.
invert_by_adjoint honors the ratmx and detout flags, the same as invert.
invert (M)                        [Function]
    Returns the inverse of the matrix M. The inverse is computed via the LU decom-
    position.
    When ratmx is true, elements of M are converted to canonical rational expressions
    (CRE), and the elements of the return value are also CRE.
    When ratmx is false, elements of M are not converted to a common representation.
    In particular, float and bigfloat elements are not converted to rationals.
    When detout is true, the determinant is factored out of the inverse. The global flags
doallmxops and doscmxops must be false to prevent the determinant from being
absorbed into the inverse. xthru can multiply the determinant into the inverse.
invert does not apply any simplifications to the elements of the inverse apart from
the default arithmetic simplifications. ratsimp and expand can apply additional
simplifications. In particular, when M has polynomial elements, expand(invert(M))
might be preferable.
invert(M) is equivalent to M^-1.
list_matrix_entries (M)            [Function]
    Returns a list containing the elements of the matrix M.
    Example:
    (%i1) list_matrix_entries(matrix([a,b],[c,d]));
     (%o1) [a, b, c, d]
list_matrix_entries
lmxchar                           [Option variable]
    Default value: []
lmxchar is the character displayed as the left delimiter of a matrix. See also rmxchar.
    Example:
    (%i1) lmxchar: "|"$
    (%i2) matrix ([a, b, c], [d, e, f], [g, h, i]);
          | a b c |
     (%o2) |     |    |
          | d e f |
          |     |    |
          | g h i |
**matrix (row_1, ..., row_n)**

[Function]

Returns a rectangular matrix which has the rows row_1, ..., row_n. Each row is a list of expressions. All rows must be the same length.

The operations + (addition), - (subtraction), * (multiplication), and / (division), are carried out element by element when the operands are two matrices, a scalar and a matrix, or a matrix and a scalar. The operation ^ (exponentiation, equivalently **) is carried out element by element if the operands are a scalar and a matrix or a matrix and a scalar, but not if the operands are two matrices. All operations are normally carried out in full, including . (noncommutative multiplication).

Matrix multiplication is represented by the noncommutative multiplication operator .. The corresponding noncommutative exponentiation operator is ^^. For a matrix A, A * A = A^^2 and A^^-1 is the inverse of A, if it exists. A^^-1 is equivalent to invert(A).

There are switches for controlling simplification of expressions involving dot and matrix-list operations. These are doallmxops, domxexpt, dommxops, doscmxops, and doscmxplus.

There are additional options which are related to matrices. These are: lmxchar, rmxchar, ratmx, listarith, detout, scalarmatrix and sparse.

There are a number of functions which take matrices as arguments or yield matrices as return values. See eigenvalues, eigenvectors, determinant, charpoly, genmatrix, addcol, addrow, copymatrix, transpose, echelon, and rank.

Examples:

- Construction of matrices from lists.

  ```
  (%i1) x: matrix ([17, 3], [-8, 11]);
  [ 17 3 ]
  (%o1)
  [ ]
  [ - 8 11 ]
  (%i2) y: matrix ([%pi, %e], [a, b]);
  [ %pi %e ]
  (%o2)
  [ ]
  [ a b ]
  ```

- Addition, element by element.

  ```
  (%i3) x + y;
  [ %pi + 17 %e + 3 ]
  (%o3)
  [ ]
  [ a - 8 b + 11 ]
  ```

- Subtraction, element by element.

  ```
  (%i4) x - y;
  [ 17 - %pi 3 - %e ]
  (%o4)
  [ ]
  [ - a - 8 11 - b ]
  ```

- Multiplication, element by element.

  ```
  (%i5) x * y;
  [ 17 %pi 3 %e ]
  ```
Division, element by element.

(\texttt{(%i6)} \texttt{x / y;})
\begin{verbatim}
[ 17 - 1 ]
[ --- 3 %e ]
[ %pi ]
\end{verbatim}
(\texttt{(%o6)})
\begin{verbatim}
[ 8 11 ]
[ - - -- ]
[ a b ]
\end{verbatim}

Matrix to a scalar exponent, element by element.

(\texttt{(%i7)} \texttt{x ^ 3;})
\begin{verbatim}
[ 4913 27 ]
[ ]
[ - 512 1331 ]
\end{verbatim}
(\texttt{(%o7)})

Scalar base to a matrix exponent, element by element.

(\texttt{(%i8)} \texttt{exp(y);})
\begin{verbatim}
[ %pi %e ]
[ %e %e ]
\end{verbatim}
(\texttt{(%o8)})
\begin{verbatim}
[ a b ]
[ %e %e ]
\end{verbatim}

Matrix base to a matrix exponent. This is not carried out element by element. See also \texttt{matrixexp}.

(\texttt{(%i9)} \texttt{x ^ y;})
\begin{verbatim}
[ %pi %e ]
[ ]
[ 17 3 ]
[ a b ]
\end{verbatim}
(\texttt{(%o9)})
\begin{verbatim}
[ ]
[ - 8 11 ]
\end{verbatim}

Noncommutative matrix multiplication.

(\texttt{(%i10)} \texttt{x . y;})
\begin{verbatim}
[ 3 a + 17 %pi 3 b + 17 %e ]
[ ]
[ 11 a - 8 %pi 11 b - 8 %e ]
\end{verbatim}
(\texttt{(%o10)})
\begin{verbatim}
[ ]
[ ]
\end{verbatim}

(\texttt{(%i11)} \texttt{y . x;})
\begin{verbatim}
[ 17 %pi - 8 %e 3 %pi + 11 %e ]
[ ]
[ ]
\end{verbatim}
(\texttt{(%o11)})
\begin{verbatim}
[ ]
[ ]
\end{verbatim}

Noncommutative matrix exponentiation. A scalar base \( b \) to a matrix power \( M \) is carried out element by element and so \( b^M \) is the same as \( b^m \).

(\texttt{(%i12)} \texttt{x ^ 3;})
$\begin{bmatrix}
3833 & 1719 \\
-4584 & 395
\end{bmatrix}$

(%o12)

$\begin{bmatrix}
\pi e & e \\
e & e
\end{bmatrix}$

(%i13) $e^{y};$

$\begin{bmatrix}
\pi e & e \\
e & e
\end{bmatrix}$

(%o13)

$
\begin{bmatrix}
a & b \\
e & e
\end{bmatrix}
$

- A matrix raised to a -1 exponent with noncommutative exponentiation is the matrix inverse, if it exists.

(%i14) $x^{-1};$

$\begin{bmatrix}
11 & 3 \\
--- & ---
\end{bmatrix}$

(%o14)

$\begin{bmatrix}
8 & 17 \\
--- & ---
\end{bmatrix}$

(%i15) $x . (x^{-1});$

$\begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}$

(%o15)

**matrixexp**

- matrixexp ($M$)
- matrixexp ($M$, n)
- matrixexp ($M$, $V$)

Calculates the matrix exponential $e^{MV} e^{M * V}$ . Instead of the vector $V$ a number $n$ can be specified as the second argument. If this argument is omitted matrixexp replaces it by 1.

The matrix exponential of a matrix $M$ can be expressed as a power series: $e^M = \sum_{k=0}^{\infty} \frac{M^k}{k!} e^M = sum(M^k/k!,0,inf)$

**matrixmap** ($f$, $M$)

Returns a matrix with element $i,j$ equal to $f(M[i,j])$.

See also map, fullmap, fullmapl, and apply.

**matrixp** ($expr$)

Returns true if $expr$ is a matrix, otherwise false.

**matrix_element_add**

Default value: +

matrix_element_add is the operation invoked in place of addition in a matrix multiplication. matrix_element_add can be assigned any n-ary operator (that is, a function which handles any number of arguments). The assigned value may be the
name of an operator enclosed in quote marks, the name of a function, or a lambda expression.

See also matrix_element_mult and matrix_element_transpose.

Example:

```lisp
(%i1) matrix_element_add: "*"$
(%i2) matrix_element_mult: "-"$
(%i3) aa: matrix ([a, b, c], [d, e, f]);
    [ a b c ]
    [ ]
    [ d e f ]
(%i4) bb: matrix ([u, v, w], [x, y, z]);
    [ u v w ]
    [ ]
    [ x y z ]
(%i5) aa . transpose (bb);
    [ u v w x y z ]
    [ a b c a b c ]
    [ ]
    [ u v w x y z ]
    [ d e f d e f ]
```

**matrix_element_mult**

[Option variable]

Default value: *

**matrix_element_mult** is the operation invoked in place of multiplication in a matrix multiplication. **matrix_element_mult** can be assigned any binary operator. The assigned value may be the name of an operator enclosed in quote marks, the name of a function, or a lambda expression.

The dot operator . is a useful choice in some contexts.

See also matrix_element_add and matrix_element_transpose.

Example:

```lisp
(%i1) matrix_element_add: lambda ([[x]], sqrt (apply ("+", x)))$
(%i2) matrix_element_mult: lambda ([[x, y], (x - y)^2]$ 
(%i3) [a, b, c] . [x, y, z];
    2 2 2
    sqrt((c - z) + (b - y) + (a - x))
(%i4) aa: matrix ([a, b, c], [d, e, f]);
    [ a b c ]
    [ ]
    [ d e f ]
(%i5) bb: matrix ([u, v, w], [x, y, z]);
    [ u v w ]
    [ ]
    [ x y z ]
(%i6) aa . transpose (bb);
    [ 2 2 2 ]```
Chapter 23: Matrices and Linear Algebra

\[
\begin{bmatrix}
\sqrt{(c - w) + (b - v) + (a - u)} \\
2 & 2 & 2 \\
\sqrt{(f - w) + (e - v) + (d - u)} \\
2 & 2 & 2 \\
\sqrt{(c - z) + (b - y) + (a - x)} \\
2 & 2 & 2 \\
\end{bmatrix}
\]

\(\text{Col 1} = [\]
\[
\begin{bmatrix}
2 & 2 & 2 \\
\sqrt{(f - z) + (e - y) + (d - x)} \\
2 & 2 & 2 \\
\end{bmatrix}
\]

\(\text{Col 2} = [\]
\[
\begin{bmatrix}
2 & 2 & 2 \\
\sqrt{(c - z) + (b - y) + (a - x)} \\
2 & 2 & 2 \\
\end{bmatrix}
\]

\textbf{matrix_element_transpose} [Option variable]

Default value: false

\textbf{matrix_element_transpose} is the operation applied to each element of a matrix when it is transposed. \textbf{matrix_element_mult} can be assigned any unary operator. The assigned value may be the name of an operator enclosed in quote marks, the name of a function, or a lambda expression.

When \textbf{matrix_element_transpose} equals transpose, the transpose function is applied to every element. When \textbf{matrix_element_transpose} equals nonscalars, the transpose function is applied to every nonscalar element. If some element is an atom, the nonscalars option applies transpose only if the atom is declared nonscalar, while the transpose option always applies transpose.

The default value, false, means no operation is applied.

See also \textbf{matrix_element_add} and \textbf{matrix_element_mult}.

Examples:

\(\%i1\) declare (a, nonscalar)$
\(\%i2\) transpose ([a, b]);
\[
\begin{bmatrix}
\text{transpose(a)} \\
\text{b} \\
\end{bmatrix}
\]

\(\%o2\)
\[
\begin{bmatrix}
\text{transpose(a)} \\
\text{b} \\
\end{bmatrix}
\]

\(\%i3\) matrix_element_transpose: nonscalars$
\(\%i4\) transpose ([a, b]);
\[
\begin{bmatrix}
\text{transpose(a)} \\
\text{b} \\
\end{bmatrix}
\]

\(\%o4\)
\[
\begin{bmatrix}
\text{transpose(a)} \\
\text{b} \\
\end{bmatrix}
\]

\(\%i5\) matrix_element_transpose: transpose$
\(\%i6\) transpose ([a, b]);
\[
\begin{bmatrix}
\text{transpose(a)} \\
\text{b} \\
\end{bmatrix}
\]

\(\%o6\)
\[
\begin{bmatrix}
\text{transpose(a)} \\
\text{b} \\
\end{bmatrix}
\]

\(\%i7\) matrix_element_transpose: lambda ([x], realpart(x) - %i*imagpart(x))$
\(\%i8\) m: matrix ([[1 + 5*%i, 3 - 2*%i], [7*%i, 11]]);
\[
\begin{bmatrix}
5 \%i + 1 & 3 - 2 \%i \\
7 \%i & 11 \\
\end{bmatrix}
\]

\(\%o8\)
(%i9) transpose (m);
[ 1 - 5 %i - 7 %i ]
[ ]
[ 2 %i + 3 11 ]
(%o9)

mattrace (M)  [Function]
Returns the trace (that is, the sum of the elements on the main diagonal) of the square matrix M.

mattrace is called by ncharpoly, an alternative to Maxima’s charpoly.
load ("nchrpl") loads this function.

minor (M, i, j)  [Function]
Returns the i, j minor of the matrix M. That is, M with row i and column j removed.

ncharpoly (M, x)  [Function]
Returns the characteristic polynomial of the matrix M with respect to x. This is an alternative to Maxima’s charpoly.

ncharpoly works by computing traces of powers of the given matrix, which are known to be equal to sums of powers of the roots of the characteristic polynomial. From these quantities the symmetric functions of the roots can be calculated, which are nothing more than the coefficients of the characteristic polynomial. charpoly works by forming the determinant of x * ident [n] - a. Thus ncharpoly wins, for example, in the case of large dense matrices filled with integers, since it avoids polynomial arithmetic altogether.
load ("nchrpl") loads this file.

newdet (M)  [Function]
Computes the determinant of the matrix M by the Johnson-Gentleman tree minor algorithm. newdet returns the result in CRE form.

permanent (M)  [Function]
Computes the permanent of the matrix M by the Johnson-Gentleman tree minor algorithm. A permanent is like a determinant but with no sign changes. permanent returns the result in CRE form.
See also newdet.

rank (M)  [Function]
Computes the rank of the matrix M. That is, the order of the largest non-singular subdeterminant of M.
rank may return the wrong answer if it cannot determine that a matrix element that is equivalent to zero is indeed so.

ratmx  [Option variable]
Default value: false
When ratmx is false, determinant and matrix addition, subtraction, and multiplication are performed in the representation of the matrix elements and cause the result of matrix inversion to be left in general representation.
When `ratmx` is `true`, the 4 operations mentioned above are performed in CRE form and the result of matrix inverse is in CRE form. Note that this may cause the elements to be expanded (depending on the setting of `ratfac`) which might not always be desired.

```
row (M, i)
```

Returns the $i$'th row of the matrix $M$. The return value is a matrix.

```
rmxchar
```

Default value: ]

`rmxchar` is the character drawn on the right-hand side of a matrix.

See also `lmxchar`.

```
scalarmatrixp
```

Default value: `true`

When `scalarmatrixp` is `true`, then whenever a $1 \times 1$ matrix is produced as a result of computing the dot product of matrices it is simplified to a scalar, namely the sole element of the matrix.

When `scalarmatrixp` is `all`, then all $1 \times 1$ matrices are simplified to scalars.

When `scalarmatrixp` is `false`, $1 \times 1$ matrices are not simplified to scalars.

```
scalefactors (coordinatetransform)
```

Here the argument `coordinatetransform` evaluates to the form `[[expression1, expression2, ...], indeterminate1, indeterminate2, ...]`, where the variables `indeterminate1`, `indeterminate2`, etc. are the curvilinear coordinate variables and where a set of rectangular Cartesian components is given in terms of the curvilinear coordinates by `[expression1, expression2, ...]`. `coordinates` is set to the vector `[indeterminate1, indeterminate2, ...]`, and `dimension` is set to the length of this vector. `SF[1]`, `SF[2]`, ..., `SF[dimension]` are set to the coordinate scale factors, and `sfprod` is set to the product of these scale factors. Initially, `coordinates` is `[X, Y, Z]`, `dimension` is 3, and `SF[1]=SF[2]=SF[3]=SFPROD=1`, corresponding to 3-dimensional rectangular Cartesian coordinates. To expand an expression into physical components in the current coordinate system, there is a function with usage of the form

```
setelmx (x, i, j, M)
```

Assigns $x$ to the $(i, j)$'th element of the matrix $M$, and returns the altered matrix.

$M[i, j]$: $x$ has the same effect, but returns $x$ instead of $M$.

```
similaritytransform (M)
```

`similaritytransform` computes a similarity transform of the matrix $M$. It returns a list which is the output of the `uniteigenvectors` command. In addition if the flag `nondiagonalizable` is `false` two global matrices `leftmatrix` and `rightmatrix` are computed. These matrices have the property that `leftmatrix . M . rightmatrix` is a diagonal matrix with the eigenvalues of $M$ on the diagonal. If `nondiagonalizable` is `true` the left and right matrices are not computed.
If the flag hermitianmatrix is true then leftmatrix is the complex conjugate of the transpose of rightmatrix. Otherwise leftmatrix is the inverse of rightmatrix. rightmatrix is the matrix the columns of which are the unit eigenvectors of \( M \). The other flags (see eigenvalues and eigenvectors) have the same effects since similaritytransform calls the other functions in the package in order to be able to form rightmatrix.

load ("eigen") loads this function.

simtran is a synonym for similaritytransform.

sparse  
[Option variable]
Default value: false
When sparse is true, and if ratmx is true, then determinant will use special routines for computing sparse determinants.

submatrix  
[Function]
submatrix (i_1, \ldots, i_m, M, j_1, \ldots, j_n)
submatrix (i_1, \ldots, i_m, M)
submatrix (M, j_1, \ldots, j_n)
Returns a new matrix composed of the matrix \( M \) with rows \( i_1, \ldots, i_m \) deleted, and columns \( j_1, \ldots, j_n \) deleted.

transpose (\( M \))  
[Function]
Returns the transpose of \( M \).
If \( M \) is a matrix, the return value is another matrix \( N \) such that \( N[i,j] = M[j,i] \).
If \( M \) is a list, the return value is a matrix \( N \) of length (\( m \)) rows and 1 column, such that \( N[i,1] = M[i] \).
Otherwise \( M \) is a symbol, and the return value is a noun expression \('\text{transpose} (M)\').

triangularize (\( M \))  
[Function]
Returns the upper triangular form of the matrix \( M \), as produced by Gaussian elimination. The return value is the same as echelon, except that the leading nonzero coefficient in each row is not normalized to 1.
lu_factor and cholesky are other functions which yield triangularized matrices.

(%i1) M: matrix ([3, 7, aa, bb], [-1, 8, 5, 2], [9, 2, 11, 4]);
(%o1)  
[ 3 7 aa bb ]
[ ]
[ -1 8 5 2 ]
[ ]
[ 9 2 11 4 ]
(%i2) triangularize (M);
(%o2)  
[ -1 8 5 2 ]
[ ]
[ 0 -74 -56 -22 ]
[ ]
[ 0 0 626 -74 aa 238 -74 bb ]
\textbf{uniteigenvectors} \((M)\)  \\
\textbf{ueivects} \((M)\)  \\
Computes unit eigenvectors of the matrix \(M\). The return value is a list of lists, the first sublist of which is the output of the \texttt{eigenvalues} command, and the other sublists of which are the unit eigenvectors of the matrix corresponding to those eigenvalues respectively.

The flags mentioned in the description of the \texttt{eigenvectors} command have the same effects in this one as well.

When \texttt{knowneigvects} is \texttt{true}, the \texttt{eigen} package assumes that the eigenvectors of the matrix are known to the user and are stored under the global name \texttt{listeigvects}. \texttt{listeigvects} should be set to a list similar to the output of the \texttt{eigenvectors} command.

If \texttt{knowneigvects} is set to \texttt{true} and the list of eigenvectors is given the setting of the flag \texttt{nondiagonalizable} may not be correct. If that is the case please set it to the correct value. The author assumes that the user knows what he is doing and will not try to diagonalize a matrix the eigenvectors of which do not span the vector space of the appropriate dimension.

\texttt{load ("eigen")} loads this function.

\texttt{ueivects} is a synonym for \texttt{uniteigenvectors}.

\textbf{unitvector} \((x)\)  \\
\textbf{uvect} \((x)\)  \\
Returns \(x/norm(x)\); this is a unit vector in the same direction as \(x\).

\texttt{load ("eigen")} loads this function.

\texttt{uvect} is a synonym for \texttt{unitvector}.

\textbf{vectorpotential} \((\text{givencurl})\)  \\
Returns the vector potential of a given curl vector, in the current coordinate system. \texttt{potentialzeroloc} has a similar role as for \texttt{potential}, but the order of the left-hand sides of the equations must be a cyclic permutation of the coordinate variables.

\textbf{vectorsimp} \((\text{expr})\)  \\
Applies simplifications and expansions according to the following global flags:

- \texttt{expandall, expanddot, expanddotplus, expandcross, expandcrossplus, expandcrosscross, expandgrad, expandgradplus, expandgradprod, expanddiv, expanddivplus, expanddivprod, expandcurl, expandcurlplus, expandcurlcurl, expandlaplacian, expandlaplacianplus, and expandlaplacianprod.}

All these flags have default value \texttt{false}. The \texttt{plus} suffix refers to employing additivity or distributivity. The \texttt{prod} suffix refers to the expansion for an operand that is any kind of product.

\texttt{expandcrosscross}

Simplifies \(p(qr)\) to \((p.r)*q-(p.q)*r\).

\texttt{expandcurlcurl}

Simplifies \texttt{curlcurlp} to \texttt{graddivp} + \texttt{divgradp}. 

expandlaplaciantodivgrad
   Simplifies laplacianp to divgradp.

expandcross
   Enables expandcrossplus and expandcrosscross.

expandplus
   Enables expanddotplus, expandcrossplus, expandgradplus,
   expanddivplus, expandcurlplus, and expandlaplacianplus.

expandprod
   Enables expandgradprod, expanddivprod, and expandlaplacianprod.

   These flags have all been declared evflag.

vect_cross [Option variable]
   Default value: false
   When vect_cross is true, it allows DIFF(X~Y,T) to work where ~ is defined in
   SHARE;VECT (where VECT_CROSS is set to true, anyway.)

zeromatrix (m, n) [Function]
   Returns an m by n matrix, all elements of which are zero.
24 Affine

24.1 Introduction to Affine

affine is a package to work with groups of polynomials.

24.2 Functions and Variables for Affine

**fast_linsolve** ([expr_1, ..., expr_m], [x_1, ..., x_n])

Solves the simultaneous linear equations expr_1, ..., expr_m for the variables x_1, ..., x_n. Each expr_i may be an equation or a general expression; if given as a general expression, it is treated as an equation of the form expr_i = 0.

The return value is a list of equations of the form [x_1 = a_1, ..., x_n = a_n] where a_1, ..., a_n are all free of x_1, ..., x_n.

**fast_linsolve** is faster than **linsolve** for system of equations which are sparse.

load(affine) loads this function.

**grobner_basis** ([expr_1, ..., expr_m])

Returns a Groebner basis for the equations expr_1, ..., expr_m. The function **polysimp** can then be used to simplify other functions relative to the equations.

```plaintext
grobner_basis ([3*x^2+1, y*x])
polysimp (y^2*x + x^3*9 + 2) ==> -3*x + 2
```

**polysimp** (f) yields 0 if and only if f is in the ideal generated by expr_1, ..., expr_m, that is, if and only if f is a polynomial combination of the elements of expr_1, ..., expr_m.

load(affine) loads this function.

**set_up_dot_simplifications**

set_up_dot_simplifications (eqns, check_through_degree)

set_up_dot_simplifications (eqns)

The eqns are polynomial equations in non commutative variables. The value of current_variables is the list of variables used for computing degrees. The equations must be homogeneous, in order for the procedure to terminate.

If you have checked overlapping simplifications in dot_simplifications above the degree of f, then the following is true: dotsimp (f) yields 0 if and only if f is in the ideal generated by the equations, i.e., if and only if f is a polynomial combination of the elements of the equations.

The degree is that returned by **nc_degree**. This in turn is influenced by the weights of individual variables.

load(affine) loads this function.

**declare_weights** (x_1, w_1, ..., x_n, w_n)

Assigns weights w_1, ..., w_n to x_1, ..., x_n, respectively. These are the weights used in computing **nc_degree**.

load(affine) loads this function.
nc_degree (p)  
Returns the degree of a noncommutative polynomial p. See declare_weights.
load(affine) loads this function.

dotsimp (f)  
Returns 0 if and only if f is in the ideal generated by the equations, i.e., if and only if f is a polynomial combination of the elements of the equations.
load(affine) loads this function.

fast_central_elements ([x_1, ..., x_n], n)  
If set_up_dot_simplifications has been previously done, finds the central polynomials in the variables x_1, ..., x_n in the given degree, n.
For example:

set_up_dot_simplifications ([y.x + x.y], 3);
fast_central_elements ([x, y], 2);
[y.y, x.x];
load(affine) loads this function.

check_overlaps (n, add_to_simps)  
Checks the overlaps thru degree n, making sure that you have sufficient simplification rules in each degree, for dotsimp to work correctly. This process can be speeded up if you know beforehand what the dimension of the space of monomials is. If it is of finite global dimension, then hilbert should be used. If you don’t know the monomial dimensions, do not specify a rank_function. An optional third argument reset, false says don’t bother to query about resetting things.
load(affine) loads this function.

mono ([x_1, ..., x_n], n)  
Returns the list of independent monomials relative to the current dot simplifications of degree n in the variables x_1, ..., x_n.
load(affine) loads this function.

monomial_dimensions (n)  
Compute the Hilbert series through degree n for the current algebra.
load(affine) loads this function.

extract_linear_equations ([p_1, ..., p_n], [m_1, ..., m_n])  
Makes a list of the coefficients of the noncommutative polynomials p_1, ..., p_n of the noncommutative monomials m_1, ..., m_n. The coefficients should be scalars. Use list_nc_monomials to build the list of monomials.
load(affine) loads this function.

list_nc_monomials  
list_nc_monomials ([p_1, ..., p_n])  
list_nc_monomials (p)  
Returns a list of the noncommutative monomials occurring in a polynomial p or a list of polynomials p_1, ..., p_n.
load(affine) loads this function.
all_dotsimp_denoms

[Option variable]

Default value: false

When all_dotsimp_denoms is a list, the denominators encountered by dotsimp are appended to the list. all_dotsimp_denoms may be initialized to an empty list [] before calling dotsimp.

By default, denominators are not collected by dotsimp.
25 itensor

25.1 Introduction to itensor

Maxima implements symbolic tensor manipulation of two distinct types: component tensor manipulation (ctensor package) and indicial tensor manipulation (itensor package).

Nota bene: Please see the note on 'new tensor notation' below.

Component tensor manipulation means that geometrical tensor objects are represented as arrays or matrices. Tensor operations such as contraction or covariant differentiation are carried out by actually summing over repeated (dummy) indices with do statements. That is, one explicitly performs operations on the appropriate tensor components stored in an array or matrix.

Indicial tensor manipulation is implemented by representing tensors as functions of their covariant, contravariant and derivative indices. Tensor operations such as contraction or covariant differentiation are performed by manipulating the indices themselves rather than the components to which they correspond.

These two approaches to the treatment of differential, algebraic and analytic processes in the context of Riemannian geometry have various advantages and disadvantages which reveal themselves only through the particular nature and difficulty of the user's problem. However, one should keep in mind the following characteristics of the two implementations:

The representation of tensors and tensor operations explicitly in terms of their components makes ctensor easy to use. Specification of the metric and the computation of the induced tensors and invariants is straightforward. Although all of Maxima's powerful simplification capacity is at hand, a complex metric with intricate functional and coordinate dependencies can easily lead to expressions whose size is excessive and whose structure is hidden. In addition, many calculations involve intermediate expressions which swell causing programs to terminate before completion. Through experience, a user can avoid many of these difficulties.

Because of the special way in which tensors and tensor operations are represented in terms of symbolic operations on their indices, expressions which in the component representation would be unmanageable can sometimes be greatly simplified by using the special routines for symmetrical objects in itensor. In this way the structure of a large expression may be more transparent. On the other hand, because of the special indicial representation in itensor, in some cases the user may find difficulty with the specification of the metric, function definition, and the evaluation of differentiated "indexed" objects.

The itensor package can carry out differentiation with respect to an indexed variable, which allows one to use the package when dealing with Lagrangian and Hamiltonian formalisms. As it is possible to differentiate a field Lagrangian with respect to an (indexed) field variable, one can use Maxima to derive the corresponding Euler-Lagrange equations in indicial form. These equations can be translated into component tensor (ctensor) programs using the ic_convert function, allowing us to solve the field equations in a particular coordinate representation, or to recast the equations of motion in Hamiltonian form. See einhil.dem and bradic.dem for two comprehensive examples. The first, einhil.dem, uses the Einstein-Hilbert action to derive the Einstein field tensor in
the homogeneous and isotropic case (Friedmann equations) and the spherically symmetric, static case (Schwarzschild solution.) The second, `bradic.dem`, demonstrates how to compute the Friedmann equations from the action of Brans-Dicke gravity theory, and also derives the Hamiltonian associated with the theory’s scalar field.

### 25.1.1 New tensor notation

Earlier versions of the `itensor` package in Maxima used a notation that sometimes led to incorrect index ordering. Consider the following, for instance:

```maxima
(%i2) imetric(g);
(\%) done
(%i3) ishow(g([], [j,k])*g([], [i,l])*a([i,j], [i]))$
   i  l  j  k
   g  g  a
   i  j
(%t3) g g a
   i j
(%i4) ishow(contract(%))$
   k  l
   a
(%t4) a
```

This result is incorrect unless `a` happens to be a symmetric tensor. The reason why this happens is that although `itensor` correctly maintains the order within the set of covariant and contravariant indices, once an index is raised or lowered, its position relative to the other set of indices is lost.

To avoid this problem, a new notation has been developed that remains fully compatible with the existing notation and can be used interchangeably. In this notation, contravariant indices are inserted in the appropriate positions in the covariant index list, but with a minus sign prepended. Functions like `contract_Itensor` and `ishow` are now aware of this new index notation and can process tensors appropriately.

In this new notation, the previous example yields a correct result:

```maxima
(%i5) ishow(g([-j,-k], [])*g([-i,-l], [])*a([i,j], []))$
   i  l  j  k
   g  a  g
   i  j
(%t5) g a g
   i j
(%i6) ishow(contract(%))$
   l  k
   a
(%t6) a
```

Presently, the only code that makes use of this notation is the `lc2kdt` function. Through this notation, it achieves consistent results as it applies the metric tensor to resolve Levi-Civita symbols without resorting to numeric indices.

Since this code is brand new, it probably contains bugs. While it has been tested to make sure that it doesn't break anything using the "old" tensor notation, there is a considerable chance that "new" tensors will fail to interoperate with certain functions or features. These bugs will be fixed as they are encountered... until then, caveat emptor!

### 25.1.2 Indicial tensor manipulation

The indicial tensor manipulation package may be loaded by `load(itensor)`. Demos are also available: try `demo(tensor)`. 
In *itensor* a tensor is represented as an "indexed object". This is a function of 3 groups of indices which represent the covariant, contravariant and derivative indices. The covariant indices are specified by a list as the first argument to the indexed object, and the contravariant indices by a list as the second argument. If the indexed object lacks either of these groups of indices then the empty list $[]$ is given as the corresponding argument. Thus, $g([a,b],[c])$ represents an indexed object called $g$ which has two covariant indices $(a,b)$, one contravariant index $(c)$ and no derivative indices.

The derivative indices, if they are present, are appended as additional arguments to the symbolic function representing the tensor. They can be explicitly specified by the user or be created in the process of differentiation with respect to some coordinate variable. Since ordinary differentiation is commutative, the derivative indices are sorted alphabetically, unless *iframe_flag* is set to *true*, indicating that a frame metric is being used. This canonical ordering makes it possible for Maxima to recognize that, for example, $t([a],[b],i,j)$ is the same as $t([a],[b],j,i)$. Differentiation of an indexed object with respect to some coordinate whose index does not appear as an argument to the indexed object would normally yield zero. This is because Maxima would not know that the tensor represented by the indexed object might depend implicitly on the corresponding coordinate. By modifying the existing Maxima function *diff* in *itensor*, Maxima now assumes that all indexed objects depend on any variable of differentiation unless otherwise stated. This makes it possible for the summation convention to be extended to derivative indices. It should be noted that *itensor* does not possess the capabilities of raising derivative indices, and so they are always treated as covariant.

The following functions are available in the tensor package for manipulating indexed objects. At present, with respect to the simplification routines, it is assumed that indexed objects do not by default possess symmetry properties. This can be overridden by setting the variable *allsym*[false] to *true*, which will result in treating all indexed objects completely symmetric in their lists of covariant indices and symmetric in their lists of contravariant indices.

The *itensor* package generally treats tensors as opaque objects. Tensorial equations are manipulated based on algebraic rules, specifically symmetry and contraction rules. In addition, the *itensor* package understands covariant differentiation, curvature, and torsion. Calculations can be performed relative to a metric of moving frame, depending on the setting of the *iframe_flag* variable.

A sample session below demonstrates how to load the *itensor* package, specify the name of the metric, and perform some simple calculations.

```lisp
(%i1) load(itensor);
(%o1) /share/tensor/itensor.lisp
(%i2) imetric(g);
(%o2) done
(%i3) components(g([i,j],[]),p([i,j],[])*e([],[]))$
(%i4) ishow(g([k,l],[]))$
(%t4) e p
     k l
(%i5) ishow(diff(v([i],[]),t))$
(%t5) 0
(%i6) depends(v,t);
```
(%i6) ishow(diff(v([i],[]),t))$  [v(t)]

(%t7)  
d
-- (v )
dt i

(%i8) ishow(idiff(v([i],[]),j))$

(%t8)  v
  i,j

(%i9) ishow(extdiff(v([i],[]),j))$

(%t9)  v - v
      j,i   i,j

-----------

2

(%i10) ishow(liediff(v,w([i],[])))$

(%t10)  v w + v w
      i,%3   ,i %3

(%i11) ishow(covdiff(v([i],[]),j))$

(%t11)  v - v ichr2
      i,j %4 i j

(%i12) ishow(ev(%,ichi2))$

(%t12)  v - (g v (e p + e p - e p - e p
      i,j %4   j %5,i ,i j %5   i,j %5 ,%5 i j
              + e p + e p ))/2
      i %5,j ,j i %5

(%i13) iframe_flag:true;

(%o13) true

(%i14) ishow(covdiff(v([i],[]),j))$

(%t14)  v - v icc2
      i,j %6 i j

(%i15) ishow(ev(%,iccc2))$

(%t15)  v - v icc2
      i,j %6 i j

(%i16) ishow(radcan(ev(%,icc2,icc1)))$

(%t16)  - (ifg v ifb + ifg v ifb - 2 v
      %6   j %7 i    %6   i j %7 i,j
               %6 %7
             - ifg v ifb )/2
      %6   %7 i j

(%i17) ishow(canform(s([i,j],[[]])=s([j,i])))$
(\%i18) \texttt{decsym(s,2,0,[sym(all)],[]);} \\
(\%o18) \texttt{done} \\
(\%i19) \texttt{ishow(canform(s([i,j],[])-s([j,i])))}$ \\
(\%t19) \texttt{0} \\
(\%i20) \texttt{ishow(canform(a([i,j],[])+a([j,i])))}$ \\
(\%t20) \texttt{a + a} \\
(\%i21) \texttt{decsym(a,2,0,[anti(all)],[]);} \\
(\%o21) \texttt{done} \\
(\%i22) \texttt{ishow(canform(a([i,j],[])+a([j,i])))}$ \\
(\%t22) \texttt{0}

\section*{25.2 Functions and Variables for \texttt{itensor}}

\subsection*{25.2.1 Managing indexed objects}

\texttt{dispcon} \hspace{1cm} \textbf{[Function]} \\
\texttt{dispcon (tensor\_1, tensor\_2, \ldots)} \\
\texttt{dispcon (all)} \\
Displays the contraction properties of its arguments as were given to \texttt{defcon}. \texttt{dispcon (all)} displays all the contraction properties which were defined.

\texttt{entertensor (name)} \hspace{1cm} \textbf{[Function]} \\
is a function which, by prompting, allows one to create an indexed object called \texttt{name} with any number of tensorial and derivative indices. Either a single index or a list of indices (which may be null) is acceptable input (see the example under \texttt{covdiff}).

\texttt{changename (old, new, expr)} \hspace{1cm} \textbf{[Function]} \\
will change the name of all indexed objects called \texttt{old} to \texttt{new} in \texttt{expr}. \texttt{old} may be either a symbol or a list of the form \texttt{[name, m, n]} in which case only those indexed objects called \texttt{name} with \texttt{m} covariant and \texttt{n} contravariant indices will be renamed to \texttt{new}.

\texttt{listoftens} \hspace{1cm} \textbf{[Function]} \\
Lists all tensors in a tensorial expression, complete with their indices. E.g.,

(\%i6) \texttt{ishow(a([i,j],[k])*b([u],[v])+c([x,y],[])*d([],[])*e^{k})}$ \\
(\%t6) \texttt{dec a b} \\
(\%i7) \texttt{ishow(listoftens(\%))}$ \\
(\%t7) \texttt{[a , b , c , d]} \\
(\%t8) \texttt{| i j u,v x y |}
ishow (expr)                      [Function]
displays expr with the indexed objects in it shown having their covariant indices
as subscripts and contravariant indices as superscripts. The derivative indices are
displayed as subscripts, separated from the covariant indices by a comma (see the
examples throughout this document).

indices (expr)                    [Function]
Returns a list of two elements. The first is a list of the free indices in expr (those
that occur only once). The second is the list of the dummy indices in expr (those
that occur exactly twice) as the following example demonstrates.

\begin{verbatim}
(%i1) load(itensor);
(%o1) /share/tensor/itensor.lisp
(%i2) ishow(a([i,j],[k,l],m,n)*b([k,o],[j,m,p],q,r))$
   \begin{array}{l}
   k l \\
   j m p \\
   i j, m n \ \ k o, q r
   \end{array}
(%t2) a b
   \begin{array}{l}
   i j, m n \ \ k o, q r
   \end{array}
(%i3) indices(%);
(%o3) \begin{array}{l}
[l, p, i, n, o, q, r], [k, j, m]
   \end{array}
\end{verbatim}

A tensor product containing the same index more than twice is syntactically illegal.
indices attempts to deal with these expressions in a reasonable manner; however,
when it is called to operate upon such an illegal expression, its behavior should be
considered undefined.

rename                           [Function]
rename (expr)                   [Function]
rename (expr, count)
Returns an expression equivalent to expr but with the dummy indices in each term
chosen from the set [%1, %2, ...], if the optional second argument is omitted. Oth-
wise, the dummy indices are indexed beginning at the value of count. Each dummy
index in a product will be different. For a sum, rename will operate upon each term
in the sum resetting the counter with each term. In this way rename can serve as a
tensorial simplifier. In addition, the indices will be sorted alphanumerically (if allsym
is true) with respect to covariant or contravariant indices depending upon the value
of flipflag. If flipflag is false then the indices will be renamed according to
the order of the contravariant indices. If flipflag is true the renaming will occur
according to the order of the covariant indices. It often happens that the combined
effect of the two renamings will reduce an expression more than either one by itself.

\begin{verbatim}
(%i1) load(itensor);
(%o1) /share/tensor/itensor.lisp
(%i2) allsym:true;
(%o2) true
(%i3) g([], [%4, %5])*g([], [%6, %7])*ichr2([], %1, %4, %3)*
ichr2([%2, %3], [u])*ichr2([], %5, %6, %1)*ichr2([], %7, r, %2)*
ichr2([%2, %3], [u])*ichr2([], %5, %6, %1)*ichr2([], %7, r, %2)*
g([], [%4, %5])*g([], [%6, %7])*ichr2([], %1, %2, [u])*
\end{verbatim}
ichr2([%3,%5],[%1])*ichr2([%4,%6],[%3])*ichr2([%7,r],[%2]),noeval$

(%i4) expr:ishow(%)$
   \%1\ %2\ \%3\ \%4\ \%5\ \%6\ \%7\ \%8

(%t4) g g ichr2 ichr2 ichr2 ichr2
   \%1\ %2\ %3\ %4\ %5\ %6\ %7\ %8

   %4\ %5\ %6\ %7\ u\ %1\ %2\ %3

- g g ichr2 ichr2 ichr2 ichr2
   %1\ %2\ %3\ %4\ %5\ %6\ %7\ %8

(%i5) flipflag:true;
   true

(%i6) ishow(rename(expr))$
   \%2\ %5\ %6\ %7\ %4\ u\ %1\ %3\ %2

(%t6) g g ichr2 ichr2 ichr2 ichr2
   %1\ %2\ %3\ %4\ %5\ %6\ %7\ %8

   %4\ %5\ %6\ %7\ u\ %1\ %2\ %3

- g g ichr2 ichr2 ichr2 ichr2
   %1\ %2\ %3\ %4\ %5\ %6\ %7\ %8

(%i7) flipflag:false;
   false

(%i8) rename(%th(2));
   0

(%i9) ishow(rename(expr))$
   \%1\ %2\ %3\ %4\ %5\ %6\ %7\ u

(%t9) g g ichr2 ichr2 ichr2 ichr2
   %1\ %2\ %3\ %4\ %5\ %6\ %7\ %8

   %1\ %2\ %3\ %4\ %5\ %6\ %7\ u

- g g ichr2 ichr2 ichr2 ichr2
   %1\ %2\ %3\ %4\ %5\ %6\ %7\ %8

**show (expr)**  
[Function]  
Displays expr with the indexed objects in it shown having covariant indices as subscripts, contravariant indices as superscripts. The derivative indices are displayed as subscripts, separated from the covariant indices by a comma.

**flipflag**  
[Option variable]  
Default value: false

If false then the indices will be renamed according to the order of the contravariant indices, otherwise according to the order of the covariant indices.

If flipflag is false then rename forms a list of the contravariant indices as they are encountered from left to right (if true then of the covariant indices). The first dummy index in the list is renamed to %1, the next to %2, etc. Then sorting occurs after the rename-ing (see the example under rename).
defcon

defcon (tensor_1)
defcon (tensor_1, tensor_2, tensor_3)
gives tensor_1 the property that the contraction of a product of tensor_1 and tensor_2 results in tensor_3 with the appropriate indices. If only one argument, tensor_1, is given, then the contraction of the product of tensor_1 with any indexed object having the appropriate indices (say my_tensor) will yield an indexed object with that name, i.e. my_tensor, and with a new set of indices reflecting the contractions performed. For example, if imetric: g, then defcon(g) will implement the raising and lowering of indices through contraction with the metric tensor. More than one defcon can be given for the same indexed object; the latest one given which applies in a particular contraction will be used. contractions is a list of those indexed objects which have been given contraction properties with defcon.

remcon

remcon (tensor_1, ..., tensor_n)
remcon (all)
Removes all the contraction properties from the (tensor_1, ..., tensor_n). remcon(all) removes all contraction properties from all indexed objects.

contract (expr)

Carries out the tensorial contractions in expr which may be any combination of sums and products. This function uses the information given to the defcon function. For best results, expr should be fully expanded. ratexpand is the fastest way to expand products and powers of sums if there are no variables in the denominators of the terms. The gcd switch should be false if GCD cancellations are unnecessary.

indexed_tensor (tensor)

Must be executed before assigning components to a tensor for which a built in value already exists as with ichr1, ichr2, icurvature. See the example under icurvature.

components (tensor, expr)

permits one to assign an indicial value to an expression expr giving the values of the components of tensor. These are automatically substituted for the tensor whenever it occurs with all of its indices. The tensor must be of the form t([],[]) where either list may be empty. expr can be any indexed expression involving other objects with the same free indices as tensor. When used to assign values to the metric tensor wherein the components contain dummy indices one must be careful to define these indices to avoid the generation of multiple dummy indices. Removal of this assignment is given to the function remcomps.

It is important to keep in mind that components cares only about the valence of a tensor, not about any particular index ordering. Thus assigning components to, say, x([i,-j],[]), x([-j,i],[]), or x([i],[j]) all produce the same result, namely components being assigned to a tensor named x with valence (1,1).

Components can be assigned to an indexed expression in four ways, two of which involve the use of the components command:

1) As an indexed expression. For instance:

(%i12) components(g([],[],[]),e([],[],[])*p([],[],[]))$
2) As a matrix:

(%i5) lg: ident(4)$ lg[1,1]: 1$ lg;
     [ 1 0 0 0 ]
     [        ]
     [ 0 -1 0 0 ]
     [        ]
     [ 0 0 -1 0 ]
     [        ]
     [ 0 0 0 -1 ]

(%i6) components(g([i,j],[[]]), lg);  
     done

(%i7) g([i,j],[[]]);

(%i8) g([1,1],[[]]);
    1

(%i9) g([4,4],[[]]);
    -1

3) As a function. You can use a Maxima function to specify the components of a tensor based on its indices. For instance, the following code assigns kdelta to h if h has the same number of covariant and contravariant indices and no derivative indices, and g otherwise:

(%i4) h(l1,l2,[l3]):=if length(l1)=length(l2) and length(l3)=0
   then kdelta(l1,l2) else apply(g,append([l1,l2],l3))$

(%i5) ishow(h([i],[j]));

(%i6) ishow(h([i,j],[k],[l]));

4) Using Maxima's pattern matching capabilities, specifically the defrule and applyb1 commands:

(%i1) load(itensor);
(%o1) /share/tensor/itensor.lisp
(%i2) matchdeclare(l1,listp);
(%o2) done
(%i3) defrule(r1,m(l1,[]),(i1:idummy(),
g([l1[1],l1[2]],[])\*q([i1],[])*e([],[i1])))$

(%i4) defrule(r2,m([],l1),(i1:idummy(),
w([],[l1[1],l1[2]])*e([i1],[])\*q([],[i1])))$

(%i5) ishow(m([i,n],[])*m([],[i,m]))$

(%i6) ishow(rename(applyb1(%1,%2,%3,%4,%5))$
The `showcomps` command can also display components of a tensor of rank higher than 2.

`idummy ()` [Function]

Increments `icounter` and returns as its value an index of the form `%n` where `n` is a positive integer. This guarantees that dummy indices which are needed in forming expressions will not conflict with indices already in use (see the example under `indices`).

`idummyx` [Option variable]

Default value: `%`

Is the prefix for dummy indices (see the example under `indices`).

`icounter` [Option variable]

Default value: 1

Determines the numerical suffix to be used in generating the next dummy index in the tensor package. The prefix is determined by the option `idummy` (default: `%`).

`kdelta (L1, L2)` [Function]

is the generalized Kronecker delta function defined in the `itensor` package with `L1` the list of covariant indices and `L2` the list of contravariant indices. `kdelta([i],[j])` returns the ordinary Kronecker delta. The command `ev(expr,kdelta)` causes the evaluation of an expression containing `kdelta([],[])` to the dimension of the manifold.

In what amounts to an abuse of this notation, `itensor` also allows `kdelta` to have 2 covariant and no contravariant, or 2 contravariant and no covariant indices, in effect providing a co(ntra)variant "unit matrix" capability. This is strictly considered a programming aid and not meant to imply that `kdelta([i,j],[])` is a valid tensorial object.

`kdels (L1, L2)` [Function]

Symmetrized Kronecker delta, used in some calculations. For instance:

```
(%i1) load(itensor);
(%o1) /share/tensor/itensor.lisp
(%i2) kdelta([1,2],[2,1]);
(%o2) 1
(%i3) kdels([1,2],[2,1]);
```
(%o3) 1
(%i4) ishow(kdelta([a,b],[c,d]))$
   c   d   d   c
   a   b   a   b
(%t4) kdelta kdelta - kdelta kdelta
   a   b   a   b
(%i4) ishow(kdels([a,b],[c,d]))$
   c   d   d   c
   a   b   a   b
(%t4) kdelta kdelta + kdelta kdelta
   a   b   a   b

levi_civita (L)

is the permutation (or Levi-Civita) tensor which yields 1 if the list $L$ consists of an
even permutation of integers, -1 if it consists of an odd permutation, and 0 if some
indices in $L$ are repeated.

lc2kdt (expr)

Simplifies expressions containing the Levi-Civita symbol, converting these to
Kronecker-delta expressions when possible. The main difference between this
function and simply evaluating the Levi-Civita symbol is that direct evaluation
often results in Kronecker expressions containing numerical indices. This is often
undesirable as it prevents further simplification. The lc2kdt function avoids
this problem, yielding expressions that are more easily simplified with rename or
contract.

(%i1) load(itensor);
(%o1) /share/tensor/itensor.lisp
(%i2) expr:ishow('levi_civita([],[i,j])
   *'levi_civita([k,l],[])a([j],[k]))$
   i   j   k
   j   k   l
(%t2) levi_civita a levi_civita
   j   k   l
(%i3) ishow(ev(expr,levi_civita))$
   i   j   k   1   2
   1   2   j   k   l
(%t3) kdelta a kdelta
   1   2   j   k   l
(%i4) ishow(ev(%,kdelta))$
   i   j   i   k
(%t4) (kdelta kdelta - kdelta kdelta ) a
   1   2   1   2   j
   1   2   2   1
   k   l   k   l
(%i5) ishow(lc2kdt(expr))$
   k   i   j   k   j   i
(%t5) a kdelta kdelta - a kdelta kdelta
The \texttt{lc2kdt} function sometimes makes use of the metric tensor. If the metric tensor was not defined previously with \texttt{imetric}, this results in an error.

\begin{verbatim}
(%i7) expr:ishow('levi_civita([],[i,j])*'levi_civita([],[k,l])*a([j,k],[]))$
(%t7) levi_civita levi_civita a
(%i8) ishow(lc2kdt(expr))$
Maxima encountered a Lisp error:
Error in \texttt{IMETRIC} [or a callee]:
\texttt{IMETRIC} [or a callee] requires less than two arguments.
Automatically continuing.
To reenable the Lisp debugger set \texttt{*debugger-hook*} to \texttt{nil}.
(%i10) ishow(lc2kdt(expr))$
\texttt{lc_l}
Simplification rule used for expressions containing the unevaluated Levi-Civita symbol (\texttt{levi_civita}). Along with \texttt{lc_u}, it can be used to simplify many expressions more efficiently than the evaluation of \texttt{levi_civita}. For example:
\end{verbatim}

\begin{verbatim}
(%i1) load(itensor);
(%o1) /share/tensor/itensor.lisp
(%i2) el1:ishow('levi_civita([i,j,k],[])*a([],[i])*a([],[j]))$
   i j
(%t2) a a levi_civita
(%i3) el2:ishow('levi_civita([],[i,j,k])*a([i])*a([j]))$
   i j k
\end{verbatim}
\[
\begin{align*}
\text{(\%t3)} & \quad \text{levi_civita} \quad a \quad a \quad i \quad j \\
\text{(\%t4)} & \quad \text{canform(contract(expand(applyb1(el1,lc_l,lc_u))))}; \\
\text{(\%t5)} & \quad \text{canform(contract(expand(applyb1(el2,lc_l,lc_u))))}; \\
\end{align*}
\]

\textbf{lc_u} \quad \text{[Function]}

Simplification rule used for expressions containing the unevaluated Levi-Civita symbol \texttt{(levi_civita)}. Along with \texttt{lc_u}, it can be used to simplify many expressions more efficiently than the evaluation of \texttt{levi_civita}. For details, see \texttt{lc_l}.

\textbf{canten (expr)} \quad \text{[Function]}

Simplifies \texttt{expr} by renaming (see \texttt{rename}) and permuting dummy indices. \texttt{rename} is restricted to sums of tensor products in which no derivatives are present. As such it is limited and should only be used if \texttt{canform} is not capable of carrying out the required simplification.

The \texttt{canten} function returns a mathematically correct result only if its argument is an expression that is fully symmetric in its indices. For this reason, \texttt{canten} returns an error if \texttt{allsym} is not set to \texttt{true}.

\textbf{concen (expr)} \quad \text{[Function]}

Similar to \texttt{canten} but also performs index contraction.

### 25.2.2 Tensor symmetries

\textbf{allsym} \quad \text{[Option variable]}

Default value: \texttt{false}

If \texttt{true} then all indexed objects are assumed symmetric in all of their covariant and contravariant indices. If \texttt{false} then no symmetries of any kind are assumed in these indices. Derivative indices are always taken to be symmetric unless \texttt{iframe_flag} is set to \texttt{true}.

\textbf{decsym (tensor, m, n, \texttt{[cov_1, cov_2, ...]}, \texttt{[contr_1, contr_2, ...]})} \quad \text{[Function]}

Declares symmetry properties for \texttt{tensor} of \texttt{m} covariant and \texttt{n} contravariant indices. The \texttt{cov_i} and \texttt{contr_i} are pseudofunctions expressing symmetry relations among the covariant and contravariant indices respectively. These are of the form \texttt{symoper(index_1, index_2, ...)} where \texttt{symoper} is one of \texttt{sym}, \texttt{anti} or \texttt{cyc} and the \texttt{index_i} are integers indicating the position of the index in the \texttt{tensor}. This will declare \texttt{tensor} to be symmetric, antisymmetric or cyclic respectively in the \texttt{index_i}. \texttt{symoper(all)} is also an allowable form which indicates all indices obey the symmetry condition. For example, given an object \texttt{b} with 5 covariant indices, \texttt{decsym(b,5,3,\texttt{[sym(1,2),anti(3,4)],\texttt{[cyc(all)]}})} declares \texttt{b} symmetric in its first and second and antisymmetric in its third and fourth covariant indices, and cyclic in all of its contravariant indices. Either list of symmetry declarations may
be null. The function which performs the simplifications is **canform** as the example below illustrates.

```lisp
(%i1) load(itensor);
(%o1) /share/tensor/itensor.lisp
(%i2) expr:contract( expand( a([i1, j1, k1], []) *kdels([i, j, k], [i1, j1, k1])))$
(%i3) ishow(expr)$
(%t3) a + a + a + a + a + a
    k j i k i j j k i j i k i k j i j k
(%i4) decsym(a,3,0,[sym(all)],[]);
(%o4) done
(%i5) ishow(canform(expr))$
(%t5) 6 a
    i j k
(%i6) remsym(a,3,0);
(%o6) done
(%i7) decsym(a,3,0,[anti(all)],[]);
(%o7) done
(%i8) ishow(canform(expr))$
(%t8) 0
(%i9) remsym(a,3,0);
(%o9) done
(%i10) decsym(a,3,0,[cyc(all)],[]);
(%o10) done
(%i11) ishow(canform(expr))$
(%t11) 3 a + 3 a
    i k j i j k
(%i12) dispsym(a,3,0);
(%o12) [[[cyc, [[1, 2, 3]], []]]]
```

**remsym (tensor, m, n)**

Removes all symmetry properties from tensor which has m covariant indices and n contravariant indices.

**canform**

Simplifies expr by renaming dummy indices and reordering all indices as dictated by symmetry conditions imposed on them. If **allsym** is true then all indices are assumed symmetric, otherwise symmetry information provided by **decsym** declarations will be used. The dummy indices are renamed in the same manner as in the **rename** function. When **canform** is applied to a large expression the calculation may take a considerable amount of time. This time can be shortened by calling **rename** on the expression first. Also see the example under **decsym**. Note: **canform** may not be able
to reduce an expression completely to its simplest form although it will always return a mathematically correct result.

The optional second parameter rename, if set to false, suppresses renaming.

### 25.2.3 Indicial tensor calculus

**diff (expr, v_1, [n_1, [v_2, n_2] ...])**

is the usual Maxima differentiation function which has been expanded in its abilities for `itensor`. It takes the derivative of `expr` with respect to `v_1` `n_1` times, with respect to `v_2` `n_2` times, etc. For the tensor package, the function has been modified so that the `v_i` may be integers from 1 up to the value of the variable `dim`. This will cause the differentiation to be carried out with respect to the `v_i`th member of the list `vect_coords`. If `vect_coords` is bound to an atomic variable, then that variable subscripted by `v_i` will be used for the variable of differentiation. This permits an array of coordinate names or subscripted names like `x[1]`, `x[2]`, ... to be used.

A further extension adds the ability to `diff` to compute derivatives with respect to an indexed variable. In particular, the tensor package knows how to differentiate expressions containing combinations of the metric tensor and its derivatives with respect to the metric tensor and its first and second derivatives. This capability is particularly useful when considering Lagrangian formulations of a gravitational theory, allowing one to derive the Einstein tensor and field equations from the action principle.

**idiff (expr, v_1, [n_1, [v_2, n_2] ...])**

Indicial differentiation. Unlike `diff`, which differentiates with respect to an independent variable, `idiff` can be used to differentiate with respect to a coordinate. For an indexed object, this amounts to appending the `v_i` as derivative indices. Subsequently, derivative indices will be sorted, unless `iframe_flag` is set to true.

`idiff` can also differentiate the determinant of the metric tensor. Thus, if `imetric` has been bound to `G` then `idiff(determinant(g),k)` will return $2 \ast \text{determinant}(g) \ast \text{ichr2}([%i,k],[%i])$ where the dummy index `%i` is chosen appropriately.

**liediff (v, ten)**

Computes the Lie-derivative of the tensorial expression `ten` with respect to the vector field `v`. `ten` should be any indexed tensor expression; `v` should be the name (without indices) of a vector field. For example:

```maxima
(%i1) load(itensor);
(%o1) /share/tensor/itensor.lisp
(%i2) ishow(liediff(v,a([i,j],[k])*b([],[k],l))$  
   k %2 %2 %2
   b (v a + v a + v a )
   ,l i j,%2 ,j i %2 ,i %2 j 
   %1 k %1 k %1 k
   + (v b - b v + v b ) a
   ,%1 l ,l ,%1 ,l ,%1 i j
```
**rediff** (ten)  
Evaluates all occurrences of the \texttt{idiff} command in the tensorial expression \texttt{ten}.

**undiff** (expr)  
Returns an expression equivalent to \texttt{expr} but with all derivatives of indexed objects replaced by the noun form of the \texttt{idiff} function. Its arguments would yield that indexed object if the differentiation were carried out. This is useful when it is desired to replace a differentiated indexed object with some function definition resulting in \texttt{expr} and then carry out the differentiation by saying \texttt{ev(expr, idiff)}.

**evundiff** (expr)  
Equivalent to the execution of \texttt{undiff}, followed by \texttt{ev} and \texttt{rediff}.

The point of this operation is to easily evaluate expressions that cannot be directly evaluated in derivative form. For instance, the following causes an error:

```
(%i1) load(itensor);
(%o1) /share/tensor/itensor.lisp
(%i2) icurvature([i,j,k],[l],m);
Maxima encountered a Lisp error:
Error in $ICURVATURE$ [or a callee]:
$ICURVATURE$ [or a callee] requires less than three arguments.
Automatically continuing.
To reenable the Lisp debugger set *debugger-hook* to nil.
```

However, if \texttt{icurvature} is entered in noun form, it can be evaluated using \texttt{evundiff}:

```
(%i3) ishow('icurvature([i,j,k],[l],m))$
  l
(%t3) icurvature
  i j k,m
(%i4) ishow(evundiff(%))$
  l   l  %1  l  %1
  %1
(%t4) - ichr2 - ichr2 ichr2 - ichr2 ichr2
  i k,j m  %1 j  i k,m  %1 j,m  i k
  l   l  %1  l  %1
  %1
  + ichr2 + ichr2 ichr2 + ichr2 ichr2
  i j,k m  %1 k  i j,m  %1 k,m  i j
```

Note: In earlier versions of Maxima, derivative forms of the Christoffel-symbols also could not be evaluated. This has been fixed now, so \texttt{evundiff} is no longer necessary for expressions like this:

```
(%i5) imetric(g);
(%o5) done
(%i6) ishow(ichr2([i,j],[k],1))$
  k  %3
  g (g - g + g )
  j  %3,i  l  i j,%3 1  i  %3,j  l
```
flush (expr, tensor_1, tensor_2, ...)  [Function]
    Set to zero, in expr, all occurrences of the tensor_i that have no derivative indices.

flushd (expr, tensor_1, tensor_2, ...)  [Function]
    Set to zero, in expr, all occurrences of the tensor_i that have derivative indices.

flushnd (expr, tensor, n)  [Function]
    Set to zero, in expr, all occurrences of the differentiated object tensor that have n or
    more derivative indices as the following example demonstrates.

coord (tensor_1, tensor_2, ...)  [Function]
    Gives tensor_i the coordinate differentiation property that the derivative of contravari-
    ant vector whose name is one of the tensor_i yields a Kronecker delta. For example, if
    coord(x) has been done then idiff(x([],[i]),j) gives kdelta([i],[j]). coord
    is a list of all indexed objects having this property.

remcoord  [Function]
    remcoord (tensor_1, tensor_2, ...)
    remcoord (all)
    Removes the coordinate differentiation property from the tensor_i that was estab-
    lished by the function coord. remcoord(all) removes this property from all indexed
    objects.

makebox (expr)  [Function]
    Display expr in the same manner as show; however, any tensor d’Alembertian oc-
    curring in expr will be indicated using the symbol []). For example, []p([m],[n])
    represents g([], [i,j])*p([m],[n],i,j).
**conmetderiv (expr, tensor)**

Simplifies expressions containing ordinary derivatives of both covariant and contravariant forms of the metric tensor (the current restriction). For example, `conmetderiv` can relate the derivative of the contravariant metric tensor with the Christoffel symbols as seen from the following:

```lisp
(%i1) load(itensor);
(%o1) /share/tensor/itensor.lisp
(%i2) ishow(g([],[a,b],c))$
  a b
  g , c
(%i3) ishow(conmetderiv(%),g)$
  %1 b a %1 a b
  - g ichr2 - g ichr2
  %1 c %1 c
```

**simpmetderiv**

*Simpmetderiv (expr)*

*Simpmetderiv (expr, stop)*

Simplifies expressions containing products of the derivatives of the metric tensor. Specifically, `simpmetderiv` recognizes two identities:

\[
g_{ab} g_{ab} + g_{a} g_{b} = (g_{a} g_{b}) = (k\delta_{c}^{d}) = 0
\]

\[
hence
\]

\[
g_{a} g_{b} = - g_{b} g_{a}
\]

\[
and
\]

\[
g_{a} g_{b} = g_{b} g_{a}
\]

which follows from the symmetries of the Christoffel symbols.

The `simpmetderiv` function takes one optional parameter which, when present, causes the function to stop after the first successful substitution in a product expression. The `simpmetderiv` function also makes use of the global variable `flipflag` which determines how to apply a “canonical” ordering to the product indices.

Put together, these capabilities can be used to achieve powerful simplifications that are difficult or impossible to accomplish otherwise. This is demonstrated through
the following example that explicitly uses the partial simplification features of \texttt{simpmetderiv} to obtain a contractible expression:

\begin{verbatim}
(%i1) load(itensor);
(%o1) /share/tensor/itensor.lisp
(%i2) imetric(g);
(%o2) done
(%i3) ishow(g([],[a,b])*g([],[b,c])*g([a,b],[a],d)*g([b,c],[a],e))
   a b b c
   g g g g
   a b, d b c, e
(%i4) ishow(canform(%))$
   errexp1 has improper indices
   -- an error. Quitting. To debug this try debugmode(true);
(%i5) ishow(simpmetderiv(%))$
   a b b c
   g g g g
   a b, d b c, e
(%i6) flipflag:not flipflag;
(%o6) true
(%i7) ishow(simpmetderiv(%th(2)))$
   a b b c
   g g g g
   ,d ,e a b b c
(%i8) flipflag:not flipflag;
(%o8) false
(%i9) ishow(simpmetderiv(%th(2),stop))$
   a b b c
   - g g g g
   ,e a b, d b c
(%i10) ishow(contract(%))$
   b c
   - g g
   ,e c b, d
\end{verbatim}

See also \texttt{weyl.dem} for an example that uses \texttt{simpmetderiv} and \texttt{conmetderiv} together to simplify contractions of the Weyl tensor.

**flush1deriv** \texttt{(expr, tensor)}  
Set to zero, in \texttt{expr}, all occurrences of \texttt{tensor} that have exactly one derivative index.

### 25.2.4 Tensors in curved spaces

**imetric** \texttt{(g)}  

**imetric**

[System variable]

Specifies the metric by assigning the variable `imetric: g` in addition, the contraction properties of the metric `g` are set up by executing the commands `defcon(g)`, `defcon(g, g, kdelta)`. The variable `imetric` (unbound by default), is bound to the metric, assigned by the `imetric(g)` command.

**idim (n)**

[Function]

Sets the dimensions of the metric. Also initializes the antisymmetry properties of the Levi-Civita symbols for the given dimension.

**ichr1 ([i, j, k])**

[Function]

Yields the Christoffel symbol of the first kind via the definition

\[
\frac{\left( g + g - g \right)}{2}.
\]

To evaluate the Christoffel symbols for a particular metric, the variable `imetric` must be assigned a name as in the example under `chr2`.

**ichr2 ([i, j], [k])**

[Function]

Yields the Christoffel symbol of the second kind defined by the relation

\[
ichr2([i,j],[k]) = g \frac{\left( g + g - g \right)}{2}.
\]

**icurvature ([i, j, k], [h])**

[Function]

Yields the Riemann curvature tensor in terms of the Christoffel symbols of the second kind (`ichr2`). The following notation is used:

\[
\text{icurvature} = - ichr2 - ichr2 ichr2 + ichr2 + ichr2 ichr2
\]

**covdiff (expr, v_1, v_2, ...)**

[Function]

Yields the covariant derivative of `expr` with respect to the variables `v_i` in terms of the Christoffel symbols of the second kind (`ichr2`). In order to evaluate these, one should use `ev(expr, ichr2)`.

```
(%i1) load(itensor);
(%o1) /share/tensor/itensor.lisp
(%i2) entertensor();$
Enter tensor name: a;
Enter a list of the covariant indices: [i,j];
Enter a list of the contravariant indices: [k];
Enter a list of the derivative indices: [];

(%t2) a
```
(\%i3) \text{ishow(covdiff(\%,s))}\$

\begin{align*}
(\%t3) & - a_{i j} \chi_{2} - a_{i j} \chi_{2} + a_{i j} \\
& + \chi_{2} a_{i j}
\end{align*}

(\%i4) \text{imetric:g;}

(\%o4) g

(\%i5) \text{ishow(ev(\%th(2),\chi_{2}))}\$

\begin{align*}
(\%t5) & - \frac{\text{---------------------------}}{2} \\
& - \frac{\text{---------------------------}}{2} \\
& + \frac{\text{---------------------------}}{2} + a
\end{align*}

(\%i6)

\text{lorentz\_gauge (expr)} \quad \text{[Function]}
\text{Imposes the Lorentz condition by substituting 0 for all indexed objects in expr that have a derivative index identical to a contravariant index.}

\text{igeodesic\_coords (expr, name)} \quad \text{[Function]}
\text{Causes undifferentiated Christoffel symbols and first derivatives of the metric tensor vanish in expr. The name in the igeodesic\_coords function refers to the metric name (if it appears in expr) while the connection coefficients must be called with the names ichr1 and/or ichr2. The following example demonstrates the verification of the cyclic identity satisfied by the Riemann curvature tensor using the igeodesic\_coords function.}
25.2.5 Moving frames

Maxima now has the ability to perform calculations using moving frames. These can be orthonormal frames (tetrads, vielbeins) or an arbitrary frame.

To use frames, you must first set `iframe_flag` to `true`. This causes the Christoffel-symbols, `ichr1` and `ichr2`, to be replaced by the more general frame connection coefficients `icc1` and `icc2` in calculations. Specifically, the behavior of `covdiff` and `icurvature` is changed.

The frame is defined by two tensors: the inverse frame field (`ifri`, the dual basis tetrad), and the frame metric `ifg`. The frame metric is the identity matrix for orthonormal frames, or the Lorentz metric for orthonormal frames in Minkowski spacetime. The inverse frame field defines the frame base (unit vectors). Contraction properties are defined for the frame field and the frame metric.

When `iframe_flag` is true, many `itensor` expressions use the frame metric `ifg` instead of the metric defined by `imetric` for raising and lowering indices.

IMPORTANT: Setting the variable `iframe_flag` to `true` does NOT undefine the contraction properties of a metric defined by a call to `defcon` or `imetric`. If a frame field is used, it is best to define the metric by assigning its name to the variable `imetric` and NOT invoke the `imetric` function.

Maxima uses these two tensors to define the frame coefficients (`ifc1` and `ifc2`) which form part of the connection coefficients (`icc1` and `icc2`), as the following example demonstrates:

```
(%i11) load(itensor);
```
iframe_flag: true;

%o2) true

(iframe_bracket_form: false)

block([[iframe_bracket_form: false], ishow(ifb([a,b,c]))])

iframes

Since in this version of Maxima, contraction identities for ifr and ifri are always defined, as is the frame bracket (ifb), this function does nothing.

ifb

The frame bracket. The contribution of the frame metric to the connection coefficients is expressed using the frame bracket:

\[
ifc1 = \frac{- ifb + ifb + ifb}{2}
\]

\[
abc
\]
The frame bracket itself is defined in terms of the frame field and frame metric. Two alternate methods of computation are used depending on the value of `frame_bracket_form`. If true (the default) or if the `itorsion_flag` is true:

\[
\text{ifb} = \begin{pmatrix}
d & e \\
f & d & e \\
\end{pmatrix}
\begin{pmatrix}
\text{ifr} & \text{ifr} & \text{ifr} \\
\text{ifr} & \text{ifr} & \text{ifr} \\
\text{ifr} & \text{ifr} & \text{ifr} \\
\end{pmatrix}
\begin{pmatrix}
\text{a} & \text{b} & \text{c} & \text{d} & \text{e} & \text{a} & \text{f} & \text{d} & \text{e} \\
\end{pmatrix}
\]

Otherwise:

\[
\text{ifb} = \begin{pmatrix}
d & e \\
e & d & e \\
\end{pmatrix}
\begin{pmatrix}
\text{ifr} & \text{ifr} & \text{ifr} \\
\text{ifr} & \text{ifr} & \text{ifr} \\
\end{pmatrix}
\begin{pmatrix}
\text{a} & \text{b} & \text{c} & \text{e} & \text{b} & \text{e} & \text{c} & \text{a} & \text{d} \\
\end{pmatrix}
\]

`icc1` [Variable]
Connection coefficients of the first kind. In `itensor`, defined as

\[
\text{icc1} = \begin{pmatrix}
\text{ichr1} & \text{ikt1} & \text{inmc1} \\
\text{abc} & \text{abc} & \text{abc} \\
\end{pmatrix}
\]

In this expression, if `iframe_flag` is true, the Christoffel-symbol `ichr1` is replaced with the frame connection coefficient `ifc1`. If `itorsion_flag` is false, `ikt1` will be omitted. It is also omitted if a frame base is used, as the torsion is already calculated as part of the frame bracket. Lastly, of `inonmet_flag` is false, `inmc1` will not be present.

`icc2` [Variable]
Connection coefficients of the second kind. In `itensor`, defined as

\[
\text{icc2} = \begin{pmatrix}
\text{ichr2} & \text{ikt2} & \text{inmc2} \\
\text{ab} & \text{ab} & \text{ab} \\
\end{pmatrix}
\]

In this expression, if `iframe_flag` is true, the Christoffel-symbol `ichr2` is replaced with the frame connection coefficient `ifc2`. If `itorsion_flag` is false, `ikt2` will be omitted. It is also omitted if a frame base is used, as the torsion is already calculated as part of the frame bracket. Lastly, of `inonmet_flag` is false, `inmc2` will not be present.

`ifc1` [Variable]
Frame coefficient of the first kind (also known as Ricci-rotation coefficients.) This tensor represents the contribution of the frame metric to the connection coefficient of the first kind. Defined as:
Frame coefficient of the second kind. This tensor represents the contribution of the frame metric to the connection coefficient of the second kind. Defined as a permutation of the frame bracket (ifb) with the appropriate indices raised and lowered as necessary:

\[
\text{ifc}_2 = \text{ifg} \cdot \text{ifc}_1
\]

The frame field. Contracts with the inverse frame field (ifri) to form the frame metric (ifg).

The inverse frame field. Specifies the frame base (dual basis vectors). Along with the frame metric, it forms the basis of all calculations based on frames.

The frame metric. Defaults to \text{kdelta}, but can be changed using \text{components}.

The inverse frame metric. Contracts with the frame metric (ifg) to \text{kdelta}.

Specifies how the frame bracket (ifb) is computed.

\textbf{25.2.6 Torsion and nonmetricity}

Maxima can now take into account torsion and nonmetricity. When the flag \text{itorsion\_flag} is set to \text{true}, the contribution of torsion is added to the connection coefficients. Similarly, when the flag \text{inonmet\_flag} is true, nonmetricity components are included.

The nonmetricity vector. Conformal nonmetricity is defined through the covariant derivative of the metric tensor. Normally zero, the metric tensor’s covariant derivative will evaluate to the following when \text{inonmet\_flag} is set to \text{true}:

\[
g_{ij;k} = -g_{ij} \text{inm}_{ij}^k
\]
\textbf{inmc1} \hfill [Variable]

Covariant permutation of the nonmetricity vector components. Defined as:

\[
\text{inmc1} = \frac{g_{inm} - g_{inm} g_{ab} c_{a b c} - g_{ac b}}{2 abc}
\]

(Substitute \textit{ifg} in place of \textit{g} if a frame metric is used.)

\textbf{inmc2} \hfill [Variable]

Contravariant permutation of the nonmetricity vector components. Used in the connection coefficients if \textit{inonmet\_flag} is \textbf{true}. Defined as:

\[
\text{inmc2} = \frac{-g_{inm} k_{delta} c_{a b} - k_{delta} g_{inm} + g_{inm} g_{c a b d}}{2 ab}
\]

(Substitute \textit{ifg} in place of \textit{g} if a frame metric is used.)

\textbf{ikt1} \hfill [Variable]

Covariant permutation of the torsion tensor (also known as contorsion). Defined as:

\[
\text{ikt1} = \frac{-g_{itr} d_{ad} + d_{cb} k_{delta} - g_{itr} - d_{itr} c_{a b d} = g_{itr} a b c d}{2 abc}
\]

(Substitute \textit{ifg} in place of \textit{g} if a frame metric is used.)

\textbf{ikt2} \hfill [Variable]

Contravariant permutation of the torsion tensor (also known as contorsion). Defined as:

\[
\text{ikt2} = g_{ikt1} a b c d
\]

(Substitute \textit{ifg} in place of \textit{g} if a frame metric is used.)

\textbf{itr} \hfill [Variable]

The torsion tensor. For a metric with torsion, repeated covariant differentiation on a scalar function will not commute, as demonstrated by the following example:
(\%i1) load(itensor);
(\%o1) /share/tensor/itensor.lisp
(\%i2) imetric: g;
(\%o2) g
(\%i3) covdiff( covdiff( f( [], []), i), j)
   - covdiff( covdiff( f( [], []), j), i)
(\%i4) ishow(%)$
(\%t4) f \ichr - f \ichr$
   ,\%4 j i ,\%2 i j
(\%i5) canform(%)$;
(\%o5) 0
(\%i6) itorsion_flag: true;
(\%o6) true
(\%i7) covdiff( covdiff( f( [], []), i), j)
   - covdiff( covdiff( f( [], []), j), i)
(\%i8) ishow(%)$
(\%t8) f \icc - f \icc - f + f$
   ,\%8 j i ,\%6 i j ,j i ,i j
(\%i9) ishow(canform(%)$)
(\%t9) f \icc - f \icc$
   ,\%1 j i ,\%1 i j
(\%i10) ishow(canform(ev(%,icc2)))$
(\%t10) f \ikt - f \ikt
   ,\%1 i j ,\%1 j i
(\%i11) ishow(canform(ev(%,ikt2)))$
(\%t11) f g \ikt1 - f g \ikt1
   ,\%2 i j \%1 \%1
(\%i12) ishow(factor(canform(rename(expand(ev(%,ikt1)))))$
   f g (itr - itr )
   ,\%3 \%2 \%1 j i \%1 i j
(\%t12) ------------------------------------
(\%i13) decsym(itr, 2, 1, [anti(all)], []);
(\%o13) done
(\%i14) defcon(g, g, kdelta);
(\%o14) done
(\%i15) subst(g, nounify(g), \th(3))$
(\%i16) ishow(canform(contract(%)$)
(\%t16) - f \itr
   ,\%1 i j
25.2.7 Exterior algebra

The itensor package can perform operations on totally antisymmetric covariant tensor fields. A totally antisymmetric tensor field of rank (0,L) corresponds with a differential L-form. On these objects, a multiplication operation known as the exterior product, or wedge product, is defined.

Unfortunately, not all authors agree on the definition of the wedge product. Some authors prefer a definition that corresponds with the notion of antisymmetrization: in these works, the wedge product of two vector fields, for instance, would be defined as

\[ a a - a a \]
\[ i j j i \]

a \( \wedge \) a = \-----------------
\[ i j 2 \]

More generally, the product of a p-form and a q-form would be defined as

\[ \frac{1}{i_1..i_p j_1..j_q} \]
\[ \Lambda \wedge B = \frac{1}{(p+q)!} i_1..i_p j_1..j_q k_1..k_p l_1..l_q \]

where \( D \) stands for the Kronecker-delta.

Other authors, however, prefer a “geometric” definition that corresponds with the notion of the volume element:

\[ a \wedge a = a a - a a \]
\[ i j i j j i \]

and, in the general case

\[ \frac{1}{i_1..i_p j_1..j_q} \]
\[ \Lambda \wedge B = \frac{1}{p! q!} i_1..i_p j_1..j_q k_1..k_p l_1..l_q \]

Since itensor is a tensor algebra package, the first of these two definitions appears to be the more natural one. Many applications, however, utilize the second definition. To resolve this dilemma, a flag has been implemented that controls the behavior of the wedge product: if igeowedge_flag is false (the default), the first, "tensorial" definition is used, otherwise the second, "geometric" definition will be applied.

The wedge product operator is denoted by the tilde \( \sim \). This is a binary operator. Its arguments should be expressions involving scalars, covariant tensors of rank one, or covariant tensors of rank 1 that have been declared antisymmetric in all covariant indices.

The behavior of the wedge product operator is controlled by the igeowedge_flag flag, as in the following example:

```lisp
(%i1) load(itensor);
(%o1) /share/tensor/itensor.lisp
(%i2) ishow(a([i])~b([j]))$
```

\[ a b - b a \]
i j  i j
-------------
  2
(%i3) decsym(a,2,0,[anti(all)],[]);
(%o3) done
(%i4) ishow(a([i,j])~b([k]))$
      a b + b a - a b
      i j k i j k i k j
------------
  3
(%i5) igeowedge_flag:true;
(%o5) true
(%i6) ishow(a([i])~b([j]))$
      a b - b a
      i j i j
(%i7) ishow(a([i,j])~b([k]))$
      a b + b a - a b
      i j k i j k i k j

| [Operator]

The vertical bar | denotes the "contraction with a vector" binary operation. When a totally antisymmetric covariant tensor is contracted with a contravariant vector, the result is the same regardless which index was used for the contraction. Thus, it is possible to define the contraction operation in an index-free manner.

In the itensor package, contraction with a vector is always carried out with respect to the first index in the literal sorting order. This ensures better simplification of expressions involving the | operator. For instance:

(%i1) load(itensor);
(%o1) /share/tensor/itensor.lisp
(%i2) decsym(a,2,0,[anti(all)],[]);
(%o2) done
(%i3) ishow(a([i,j],[])|v)$
%1
      v a
%1 j
(%i4) ishow(a([j,i],[])|v)$
%1
- v a
%1 j

Note that it is essential that the tensors used with the | operator be declared totally antisymmetric in their covariant indices. Otherwise, the results will be incorrect.

extdiff (expr, i) [Function]

Computes the exterior derivative of expr with respect to the index i. The exterior derivative is formally defined as the wedge product of the partial derivative operator and a differential form. As such, this operation is also controlled by the setting of igeowedge_flag. For instance:
(\%i1) load(itensor);
(\%o1) /share/tensor/itensor.lisp
(\%i2) ishow(extdiff(v([i]),j))$
\quad v - v
\quad j,i \quad i,j
(\%t2) 2
(\%i3) decsym(a,2,0,[anti(all)],[]);
(\%o3) done
(\%i4) ishow(extdiff(a([i,j]),k))$
\quad a - a + a
\quad j,k,i \quad i,k,j \quad i,j,k
(\%t4) 3
(\%i5) igeowedge_flag:true;
(\%o5) true
(\%i6) ishow(extdiff(v([i]),j))$
(\%t6) v - v
\quad j,i \quad i,j
(\%i7) ishow(extdiff(a([i,j]),k))$
(\%t7) - (a - a + a )
\quad k,j,i \quad k,i,j \quad j,i,k

hodge (expr) [Function]
Compute the Hodge-dual of expr. For instance:

(\%i1) load(itensor);
(\%o1) /share/tensor/itensor.lisp
(\%i2) imetric(g);
(\%o2) done
(\%i3) idim(4);
(\%o3) done
(\%i4) icounter:100;
(\%o4) 100
(\%i5) decsym(A,3,0,[anti(all)],[])$
(\%i6) ishow(A([i,j,k],[[]))$
(\%t6) A
\quad i,j,k
(\%i7) ishow(canform(hodge(%)))$
\quad %1 %2 %3 %4
\quad levi_civita g A %1 %102 %2 %3 %4
(\%t7) 6
(\%i8) ishow(canform(hodge(%)))$
(%t8) levi_civita levi_civita g
     g     g     g
     %1 %106 /6
     %2 %107 %3 %108 %4 %8 %5 %6 %7

(%i9) lc2kdt(%)$

(%i10) %,kdelta$

(%i11) ishow(canform(contract(expand(%))))$
(%t11)
     - A
     %106 %107 %108

igeowedge_flag

[Option variable]

Default value: false

Controls the behavior of the wedge product and exterior derivative. When set to false (the default), the notion of differential forms will correspond with that of a totally antisymmetric covariant tensor field. When set to true, differential forms will agree with the notion of the volume element.

25.2.8 Exporting TeX expressions

The itensor package provides limited support for exporting tensor expressions to TeX. Since itensor expressions appear as function calls, the regular Maxima tex command will not produce the expected output. You can try instead the tentex command, which attempts to translate tensor expressions into appropriately indexed TeX objects.

tentex(expr)

[Function]

To use the tentex function, you must first load tentex, as in the following example:

(%i1) load(itensor);
(%o1) /share/tensor/itensor.lisp
(%i2) load(tentex);
(%o2) /share/tensor/tentex.lisp
(%i3) idummyx:m;
(%o3)
(%i4) ishow(icurvature([j,k,l],[i]))$
     m1 i m1 i i
     ichr2 ichr2 - ichr2 ichr2 - ichr2
     j k m1 l j l m1 k j l,k
     + ichr2
     i
     j k,l
(%i5) tentex(%)$
$$\Gamma_{j,k}^{m_1}\Gamma_{l,m_1}^{i}-\Gamma_{j,l}^{m_1}\Gamma_{k,m_1}^{i}-\Gamma_{j,l,k}^{i}+\Gamma_{j,k,l}^{i}$$
Note the use of the `idummyx` assignment, to avoid the appearance of the percent sign in the TeX expression, which may lead to compile errors.

NB: This version of the `tentex` function is somewhat experimental.

25.2.9 Interfacing with `ctensor`

The `itensor` package has the ability to generate Maxima code that can then be executed in the context of the `ctensor` package. The function that performs this task is `ic_convert`.

**ic_convert** *(eqn)*

Converts the `itensor` equation `eqn` to a `ctensor` assignment statement. Implied sums over dummy indices are made explicit while indexed objects are transformed into arrays (the array subscripts are in the order of covariant followed by contravariant indices of the indexed objects). The derivative of an indexed object will be replaced by the noun form of `diff` taken with respect to `ct_coords` subscripted by the derivative index. The Christoffel symbols `ichr1` and `ichr2` will be translated to `lcs` and `mcs`, respectively and if `metricconvert` is `true` then all occurrences of the metric with two covariant (contravariant) indices will be renamed to `lg` (`ug`). In addition, `do` loops will be introduced summing over all free indices so that the transformed assignment statement can be evaluated by just doing `ev`. The following examples demonstrate the features of this function.

```maxima
(%i1) load(itensor);
(%o1) /share/tensor/itensor.lisp
(%i2) eqn:ishow(t([i,j],[k])=f([],[])*g([l,m],[])*a([],[m],j)*b([i],[l,k]))$
   k   m   l   k
   i   j   ,j   i   l   m
(%t2) t = f a b g
   i   j   ,j   i   l   m
(%i3) ic_convert(eqn);
(%o3) for i thru dim do (for j thru dim do (for k thru dim do
   t : f sum(sum(diff(a , ct_coords ) b
   i, j, k
   g , l, 1, dim), m, 1, dim)))
   1, m
(%i4) imetric(g);
(%o4) done
(%i5) metricconvert:true;
(%o5) true
(%i6) ic_convert(eqn);
(%o6) for i thru dim do (for j thru dim do (for k thru dim do
   t : f sum(sum(diff(a , ct_coords ) b
   i, j, k
   lg , l, 1, dim), m, 1, dim)))
   1, m
```
25.2.10 Reserved words

The following Maxima words are used by the itensor package internally and should not be redefined:

<table>
<thead>
<tr>
<th>Keyword</th>
<th>Comments</th>
</tr>
</thead>
<tbody>
<tr>
<td>indices2()</td>
<td>Internal version of indices()</td>
</tr>
<tr>
<td>conti</td>
<td>Lists contravariant indices</td>
</tr>
<tr>
<td>covi</td>
<td>Lists covariant indices of a indexed object</td>
</tr>
<tr>
<td>deri</td>
<td>Lists derivative indices of an indexed object</td>
</tr>
<tr>
<td>name</td>
<td>Returns the name of an indexed object</td>
</tr>
<tr>
<td>concan</td>
<td></td>
</tr>
<tr>
<td>irpmon</td>
<td></td>
</tr>
<tr>
<td>lc0</td>
<td></td>
</tr>
<tr>
<td>_lc2kdt0</td>
<td></td>
</tr>
<tr>
<td>_lcprod</td>
<td></td>
</tr>
<tr>
<td>_extlc</td>
<td></td>
</tr>
</tbody>
</table>
26 ctensor

26.1 Introduction to ctensor

ctensor is a component tensor manipulation package. To use the ctensor package, type load(ctensor). To begin an interactive session with ctensor, type csetup(). You are first asked to specify the dimension of the manifold. If the dimension is 2, 3 or 4 then the list of coordinates defaults to \([x,y]\), \([x,y,z]\) or \([x,y,z,t]\) respectively. These names may be changed by assigning a new list of coordinates to the variable ct_coords (described below) and the user is queried about this. Care must be taken to avoid the coordinate names conflicting with other object definitions.

Next, the user enters the metric either directly or from a file by specifying its ordinal position. The metric is stored in the matrix \(l_g\). Finally, the metric inverse is computed and stored in the matrix \(u_g\). One has the option of carrying out all calculations in a power series.

A sample protocol is begun below for the static, spherically symmetric metric (standard coordinates) which will be applied to the problem of deriving Einstein’s vacuum equations (which lead to the Schwarzschild solution) as an example. Many of the functions in ctensor will be displayed for the standard metric as examples.

(\%i1) load(ctensor);
(\%o1) /share/tensor/ctensor.mac
(\%i2) csetup();
Enter the dimension of the coordinate system:
4;
Do you wish to change the coordinate names?
n;
Do you want to
1. Enter a new metric?

2. Enter a metric from a file?

3. Approximate a metric with a Taylor series?
1;

Is the matrix 1. Diagonal 2. Symmetric 3. Antisymmetric 4. General Answer 1, 2, 3 or 4
1;
Row 1 Column 1:
a;
Row 2 Column 2:
x^{-2};
Row 3 Column 3:
x^{-2}\cdot\sin(y)^{-2};
Row 4 Column 4:
-d;
Matrix entered.
Enter functional dependencies with the DEPENDS function or 'N' if none
depends([a,d],x);
Do you wish to see the metric?
y;

\[
\begin{bmatrix}
a & 0 & 0 & 0 \\
2 & 0 & 0 & 0 \\
0 & x & 0 & 0 \\
2 & 2 & 0 & 0 \\
0 & 0 & x \sin(y) & 0 \\
0 & 0 & 0 & - d
\end{bmatrix}
\]

(%o2) done
(%i3) christof(mcs);

\[
\begin{align*}
\text{mcs} & = \frac{a}{1, 1, 1} \quad \text{when} \quad d \\
\text{mcs} & = \frac{x}{1, 2, 2} \\
\text{mcs} & = \frac{1}{1, 3, 3} \\
\text{mcs} & = \frac{d}{2, 2, 1} \\
\text{mcs} & = -\frac{\cos(y)}{2, 3, 3} \\
\text{mcs} & = -\frac{x \sin(y)}{3, 3, 1}
\end{align*}
\]
\[ mcs = -\cos(y) \sin(y) \]
\[ d \]
\[ mcs = \frac{\text{---}}{4, 4, 1 2 a} \]
\[ done \]

26.2 Functions and Variables for ctensor

26.2.1 Initialization and setup

csetup ()

A function in the ctensor (component tensor) package which initializes the package and allows the user to enter a metric interactively. See ctensor for more details.

cmetric (dis)
cmetric ()

A function in the ctensor (component tensor) package that computes the metric inverse and sets up the package for further calculations.

If cframe_flag is false, the function computes the inverse metric \( u_g \) from the (user-defined) matrix \( l_g \). The metric determinant is also computed and stored in the variable \( gdet \). Furthermore, the package determines if the metric is diagonal and sets the value of \( \text{diagmetric} \) accordingly. If the optional argument \( dis \) is present and not equal to false, the user is prompted to see the metric inverse.

If cframe_flag is true, the function expects that the values of \( fri \) (the inverse frame matrix) and \( lfg \) (the frame metric) are defined. From these, the frame matrix \( fr \) and the inverse frame metric \( ufg \) are computed.

ct_coordsys (coordinate_system, extra_arg)
ct_coordsys (coordinate_system)

Sets up a predefined coordinate system and metric. The argument coordinate_system can be one of the following symbols:

<table>
<thead>
<tr>
<th>SYMBOL</th>
<th>Dim</th>
<th>Coordinates</th>
<th>Description/comments</th>
</tr>
</thead>
<tbody>
<tr>
<td>cartesian2d</td>
<td>2</td>
<td>[x,y]</td>
<td>Cartesian 2D coordinate system</td>
</tr>
<tr>
<td>polar</td>
<td>2</td>
<td>[r,phi]</td>
<td>Polar coordinate system</td>
</tr>
<tr>
<td>elliptic</td>
<td>2</td>
<td>[r,phi]</td>
<td>Elliptic coord. system</td>
</tr>
<tr>
<td>confocalelliptic</td>
<td>2</td>
<td>[u,v]</td>
<td>Confocal elliptic coordinates</td>
</tr>
<tr>
<td>bipolar</td>
<td>2</td>
<td>[u,v]</td>
<td>Bipolar coord. system</td>
</tr>
</tbody>
</table>
coordinate_system can also be a list of transformation functions, followed by a list containing the coordinate variables. For instance, you can specify a spherical metric as follows:

(%i1) load(ctensor);
(%o1) /share/tensor/ctensor.mac
(%i2) ct_coordsys([r*cos(theta)*cos(phi),r*cos(theta)*sin(phi),
          r*sin(theta),[r,theta,phi]]);
(%o2) done
(%i3) lg:trigsimp(lg);
     [ 1  0  0 ]
[ ]
[ 2 1  0 ]
(%o3) [ 0  r  0 ]
Transformation functions can also be used when `cframe_flag` is true:

```
(\%i1) load(ctensor);
(\%o1) /share/tensor/ctensor.mac
(\%i2) cframe_flag:true;
(\%o2) true
(\%i3) ct_coordsys([r*cos(theta)*cos(phi),r*cos(theta)*sin(phi),
                   r*sin(theta),[r,theta,phi]]);
(\%o3) done
(\%i4) fri;
(\%o4)
[cos(phi)cos(theta) -cos(phi) r sin(theta) -sin(phi) r cos(theta)]
[                   ]
[sin(phi)cos(theta) -sin(phi) r sin(theta) cos(phi) r cos(theta)]
[                   ]
[  sin(theta) r cos(theta) 0     ]
(\%i5) cmetric();
(\%o5) false
(\%i6) lg:trigsimp(lg);
(\%o6)
[ 1 0 0 ]
[     ]
[  2  ]
[     ]
[ 0 r 0 ]
[     ]
[  2 2 ]
[     ]
[ 0 0 r cos (theta) ]
```

The optional argument `extra_arg` can be any one of the following:

- `cylindrical` tells `ct_coordsys` to attach an additional cylindrical coordinate.
- `minkowski` tells `ct_coordsys` to attach an additional coordinate with negative metric signature.
- `all` tells `ct_coordsys` to call `cmetric` and `christof(false)` after setting up the metric.

If the global variable `verbose` is set to `true`, `ct_coordsys` displays the values of `dim`, `ct_coords`, and either `lg` or `lfg` and `fri`, depending on the value of `cframe_flag`.
init_ctensor ()

[Function]
Initializes the ctensor package.

The init_ctensor function reinitializes the ctensor package. It removes all arrays and matrices used by ctensor, resets all flags, resets dim to 4, and resets the frame metric to the Lorentz-frame.

26.2.2 The tensors of curved space
The main purpose of the ctensor package is to compute the tensors of curved space(time), most notably the tensors used in general relativity.

When a metric base is used, ctensor can compute the following tensors:

\[
\begin{align*}
\text{lg} & \quad \text{ug} \\
\text{lcs} & \quad \text{mcs} \quad \text{ric} \quad \text{uric} \\
\text{tracer} & \quad \text{ein} \quad \text{lein} \\
\text{riem} & \quad \text{lriem} \quad \text{weyl} \\
\text{uriem} & \quad
\end{align*}
\]

ctensor can also work using moving frames. When cframe_flag is set to true, the following tensors can be calculated:

\[
\begin{align*}
\text{lfg} & \quad \text{ufg} \\
\text{fri} & \quad \text{fr} \quad \text{lcs} \quad \text{mcs} \quad \text{lriem} \quad \text{ric} \quad \text{uric} \\
\text{lg} & \quad \text{ug} \\
\text{weyl} & \quad \text{tracer} \quad \text{ein} \quad \text{lein} \\
\text{riem} & \quad
\end{align*}
\]

christof (dis)

[Function]
A function in the ctensor (component tensor) package. It computes the Christoffel symbols of both kinds. The argument dis determines which results are to be immediately displayed. The Christoffel symbols of the first and second kinds are stored in the arrays lcs[i,j,k] and mcs[i,j,k] respectively and defined to be symmetric in the first two indices. If the argument to christof is lcs or mcs then the unique non-zero values of lcs[i,j,k] or mcs[i,j,k], respectively, will be displayed. If the argument is all then the unique non-zero values of lcs[i,j,k] and mcs[i,j,k] will be displayed. If the argument is false then the display of the elements will not occur.
The array elements \( mcs[i,j,k] \) are defined in such a manner that the final index is contravariant.

**ricci (dis)**

A function in the ctensor (component tensor) package. \( \text{ricci} \) computes the covariant (symmetric) components \( \text{ric}[i,j] \) of the Ricci tensor. If the argument \( \text{dis} \) is \text{true}, then the non-zero components are displayed.

**uricci (dis)**

This function first computes the covariant components \( \text{ric}[i,j] \) of the Ricci tensor. Then the mixed Ricci tensor is computed using the contravariant metric tensor. If the value of the argument \( \text{dis} \) is \text{true}, then these mixed components, \( \text{uric}[i,j] \) (the index \( i \) is covariant and the index \( j \) is contravariant), will be displayed directly. Otherwise, \( \text{ricci}(\text{false}) \) will simply compute the entries of the array \( \text{uric}[i,j] \) without displaying the results.

**scurvature ()**

Returns the scalar curvature (obtained by contracting the Ricci tensor) of the Riemannian manifold with the given metric.

**einstein (dis)**

A function in the ctensor (component tensor) package. \( \text{einstein} \) computes the mixed Einstein tensor after the Christoffel symbols and Ricci tensor have been obtained (with the functions \( \text{christof} \) and \( \text{ricci} \)). If the argument \( \text{dis} \) is \text{true}, then the non-zero values of the mixed Einstein tensor \( \text{ein}[i,j] \) will be displayed where \( j \) is the contravariant index. The variable \( \text{rateinstein} \) will cause the rational simplification on these components. If \( \text{ratfac} \) is \text{true} then the components will also be factored.

**leinstein (dis)**

Covariant Einstein-tensor. \( \text{leinstein} \) stores the values of the covariant Einstein tensor in the array \( \text{lein} \). The covariant Einstein-tensor is computed from the mixed Einstein tensor \( \text{ein} \) by multiplying it with the metric tensor. If the argument \( \text{dis} \) is \text{true}, then the non-zero values of the covariant Einstein tensor are displayed.

**riemann (dis)**

A function in the ctensor (component tensor) package. \( \text{riemann} \) computes the Riemann curvature tensor from the given metric and the corresponding Christoffel symbols. The following index conventions are used:

\[
R[i,j,k,l] = \frac{1}{|i|} \left( \frac{1}{|j|} \left( \frac{1}{|k|} \left( \frac{1}{|l|} \right) \right) \right)
\]

This notation is consistent with the notation used by the itensor package and its \text{icurvature} function. If the optional argument \( \text{dis} \) is \text{true}, the unique non-zero components \( \text{riem}[i,j,k,l] \) will be displayed. As with the Einstein tensor, various switches set by the user control the simplification of the components of the Riemann tensor. If \( \text{ratriemann} \) is \text{true}, then rational simplification will be done. If \( \text{ratfac} \) is \text{true} then each of the components will also be factored.
If the variable `cframe_flag` is `false`, the Riemann tensor is computed directly from the Christoffel-symbols. If `cframe_flag` is `true`, the covariant Riemann-tensor is computed first from the frame field coefficients.

**lriemann (dis)**

Covariant Riemann-tensor (`lriem[]`).

Computes the covariant Riemann-tensor as the array `lriem`. If the argument `dis` is `true`, unique non-zero values are displayed.

If the variable `cframe_flag` is `true`, the covariant Riemann tensor is computed directly from the frame field coefficients. Otherwise, the (3,1) Riemann tensor is computed first.

For information on index ordering, see `riemann`.

**uriemann (dis)**

Computes the contravariant components of the Riemann curvature tensor as array elements `uriem[i,j,k,l]`. These are displayed if `dis` is `true`.

**rinvariant ()**

Forms the Kretchmann-invariant (`kinvariant`) obtained by contracting the tensors

```
lriem[i,j,k,l]*uriem[i,j,k,l].
```

This object is not automatically simplified since it can be very large.

**weyl (dis)**

Computes the Weyl conformal tensor. If the argument `dis` is `true`, the non-zero components `weyl[i,j,k,l]` will be displayed to the user. Otherwise, these components will simply be computed and stored. If the switch `ratweyl` is set to `true`, then the components will be rationally simplified; if `ratfac` is `true` then the results will be factored as well.

### 26.2.3 Taylor series expansion

The `ctensor` package has the ability to truncate results by assuming that they are Taylor-series approximations. This behavior is controlled by the `ctayswitch` variable; when set to `true`, `ctensor` makes use internally of the function `ctaylor` when simplifying results.

The `ctaylor` function is invoked by the following `ctensor` functions:

<table>
<thead>
<tr>
<th>Function</th>
<th>Comments</th>
</tr>
</thead>
<tbody>
<tr>
<td>christof()</td>
<td>For mcs only</td>
</tr>
<tr>
<td>ricci()</td>
<td></td>
</tr>
<tr>
<td>uricci()</td>
<td></td>
</tr>
<tr>
<td>einstein()</td>
<td></td>
</tr>
<tr>
<td>riemann()</td>
<td></td>
</tr>
<tr>
<td>weyl()</td>
<td></td>
</tr>
<tr>
<td>checkdiv()</td>
<td></td>
</tr>
</tbody>
</table>
ctaylor ()

The ctaylor function truncates its argument by converting it to a Taylor-series using taylor, and then calling ratdisrep. This has the combined effect of dropping terms higher order in the expansion variable ctayvar. The order of terms that should be dropped is defined by ctaypov; the point around which the series expansion is carried out is specified in ctaypt.

As an example, consider a simple metric that is a perturbation of the Minkowski metric. Without further restrictions, even a diagonal metric produces expressions for the Einstein tensor that are far too complex:

```latex
(%i1) load(ctensor);
(%o1) /share/tensor/ctensor.mac
(%i2) ratfac:true;
(%o2) true
(%i3) derivabbrev:true;
(%o3) true
(%i4) ct_coords:[t,r,theta,phi];
(%o4) [t, r, theta, phi]
(%i5) lg:matrix([-1,0,0,0],[0,1,0,0],[0,0,r^2,0],[0,0,0,r^2*sin(theta)^2]);
(%o5) 
   [ - 1 0 0 0 ]
   [           ]
   [ 0 1 0 0 ]
   [           ]
   [ 2 ]
   [ 0 0 r 0 ]
   [           ]
   [ 2 2 ]
   [ 0 0 0 r sin(theta) ]
(%i6) h:matrix([h11,0,0,0],[0,h22,0,0],[0,0,h33,0],[0,0,0,h44]);
(%o6) 
   [ h11 0 0 0 ]
   [           ]
   [ 0 h22 0 0 ]
   [           ]
   [ 0 0 h33 0 ]
   [           ]
   [ 0 0 0 h44 ]
(%i7) depends(l,r);
(%o7) [l(r)]
(%i8) lg:lg+l*h;
   [ h11 l - 1 0 0 ]
   [                ]
   [ 0 h22 l + 1 0 ]
   [                ]
   [ 2 ]
   [ 0 0 r + h33 l ]
   [                ]
   [ 0 ]
```
\[
\begin{bmatrix}
1 & 2 & 2 \\
0 & 0 & 0 \\
0 & r \sin(\theta) + h44 l
\end{bmatrix}
\]

(%i9) cmetric(false);
(%o9) done

(%i10) einstein(false);
(%o10) done

(%i11) ntermst(ein);
[[1, 1], 62]
[[1, 2], 0]
[[1, 3], 0]
[[1, 4], 0]
[[2, 1], 0]
[[2, 2], 24]
[[2, 3], 0]
[[2, 4], 0]
[[3, 1], 0]
[[3, 2], 0]
[[3, 3], 46]
[[3, 4], 0]
[[4, 1], 0]
[[4, 2], 0]
[[4, 3], 0]
[[4, 4], 46]

(%o12) done

However, if we recompute this example as an approximation that is linear in the
variable \(l\), we get much simpler expressions:

(%i14) ctayswitch:true;
(%o14) true

(%i15) ctayvar:l;
(%o15) l

(%i16) ctaypov:1;
(%o16) 1

(%i17) ctaypt:0;
(%o17) 0

(%i18) christof(false);
(%o18) done

(%i19) ricci(false);
(%o19) done

(%i20) einstein(false);
(%o20) done

(%i21) ntermst(ein);
[[1, 1], 6]
[[1, 2], 0]
This capability can be useful, for instance, when working in the weak field limit far from a gravitational source.

26.2.4 Frame fields

When the variable \texttt{cframe\_flag} is set to true, the \texttt{ctensor} package performs its calculations using a moving frame.

\textbf{frame\_bracket (fr, fri, diagframe)}

The frame bracket (fb[]).

Computes the frame bracket according to the following definition:

\[
\text{ifb} = \begin{pmatrix}
  c & c & c & d & e \\
  a & b & \text{d,e} & \text{e,d} & \text{a,b}
\end{pmatrix}
\]

26.2.5 Algebraic classification

A new feature (as of November, 2004) of \texttt{ctensor} is its ability to compute the Petrov classification of a 4-dimensional spacetime metric. For a demonstration of this capability, see the file \texttt{share/tensor/petrov.dem}.
nptetrad ()

[Function]
Computes a Newman-Penrose null tetrad (np) and its raised-index counterpart (npi).
See petrov for an example.

The null tetrad is constructed on the assumption that a four-dimensional orthonormal frame metric with metric signature (-,+,+,+) is being used. The components of the null tetrad are related to the inverse frame matrix as follows:

\[
np = \frac{(f_{ri} + f_{ri})}{\sqrt{2}}
\]
\[
np = \frac{(f_{ri} - f_{ri})}{\sqrt{2}}
\]
\[
np = \frac{(f_{ri} + \%i f_{ri})}{\sqrt{2}}
\]
\[
np = \frac{(f_{ri} - \%i f_{ri})}{\sqrt{2}}
\]

psi (dis)

[Function]
Computes the five Newman-Penrose coefficients psi[0]...psi[4]. If dis is set to true, the coefficients are displayed. See petrov for an example.

These coefficients are computed from the Weyl-tensor in a coordinate base. If a frame base is used, the Weyl-tensor is first converted to a coordinate base, which can be a computationally expensive procedure. For this reason, in some cases it may be more advantageous to use a coordinate base in the first place before the Weyl tensor is computed. Note however, that constructing a Newman-Penrose null tetrad requires a frame base. Therefore, a meaningful computation sequence may begin with a frame base, which is then used to compute lg (computed automatically by cmetric) and then ug. See petrov for an example. At this point, you can switch back to a coordinate base by setting cf\texttt{frame}\_flag to false before beginning to compute the Christoffel symbols. Changing to a frame base at a later stage could yield inconsistent results, as you may end up with a mixed bag of tensors, some computed in a frame base, some in a coordinate base, with no means to distinguish between the two.

petrov ()

[Function]
Computes the Petrov classification of the metric characterized by psi[0]...psi[4].

For example, the following demonstrates how to obtain the Petrov-classification of the Kerr metric:

\[
(\%i1) \text{load(ctensor);}\\
(\%o1) \text{/share/tensor/ctensor.mac}\\
(\%i2) \text{(cframe\_flag:}\text{true, gcd:spmod, ctrgsimp:}\text{true, ratfac:}\text{true);}\\
(\%o2) \text{true}\\
(\%i3) \text{ct_coordsys(exteriorschwarzschild, all);}\\
(\%o3) \text{done}
\]
(%i4) ug:invert(lg)$
(%i5) weyl(false);
(%o5) done
(%i6) nptetrad(true);
(%t6) np =

\[
\begin{bmatrix}
\sqrt{r - 2m} & \sqrt{r} \\
\sqrt{2} & \sqrt{r} & \sqrt{2} & \sqrt{r - 2m} \\
\sqrt{r - 2m} & \sqrt{r} \\
\sqrt{2} & \sqrt{r} & \sqrt{2} & \sqrt{r - 2m} \\
\end{bmatrix}
\]

(%t7) npi = matrix([- sqrt(r - 2m) sqrt(r) \\
                    sqrt(2) sqrt(r - 2m) sqrt(2) sqrt(r) \\
                    sqrt(r) sqrt(r - 2m) \\
                    sqrt(2) sqrt(r - 2m) sqrt(2) sqrt(r) \\
                    1 %i r sin(\theta) \\
                    0 0 --------- --------------- \\
                    sqrt(2) r sqrt(2) r sin(\theta) \\
                    sqrt(2) r sqrt(2) r sin(\theta) \\
                    1 %i \\
                    0, 0, ---------, ------------------ ],

(%o7) done
(%i7) psi(true);
(%t8) psi = 0

(%t9) psi = 0

1

m
\begin{verbatim}
(%t10) psi = --
     2 3
    r

(%t11) psi = 0

(%t12) psi = 0

(%o12)

(%i12) petrov();

(%o12) D

The Petrov classification function is based on the algorithm published in "Classifying geometries in general relativity: III Classification in practice" by Pollney, Skea, and d'Inverno, Class. Quant. Grav. 17 2885-2902 (2000). Except for some simple test cases, the implementation is untested as of December 19, 2004, and is likely to contain errors.

26.2.6 Torsion and nonmetricity

ctensor has the ability to compute and include torsion and nonmetricity coefficients in the connection coefficients.

The torsion coefficients are calculated from a user-supplied tensor \( tr \), which should be a rank (2,1) tensor. From this, the torsion coefficients \( kt \) are computed according to the following formulae:

\[
kt = \frac{-\ g_{im} \ tr_{kj} - \ g_{jm} \ tr_{ki} - \ g_{ki} \ tr_{ij} \ g_{km}}{ijkl}
\]

Note that only the mixed-index tensor is calculated and stored in the array \( kt \).

The nonmetricity coefficients are calculated from the user-supplied nonmetricity vector \( nm \). From this, the nonmetricity coefficients \( nmc \) are computed as follows:

\[
nmc = \frac{-\ nm_{ij} \ D_{km} - \ D_{ij} \ nm_{km} + \ g_{ij} \ nm_{mij}}{ijkl}
\]
\end{verbatim}
where $D$ stands for the Kronecker-delta.

When `ctorsion_flag` is set to `true`, the values of $kt$ are subtracted from the mixed-indexed connection coefficients computed by `christof` and stored in `mcs`. Similarly, if `cnonmet_flag` is set to `true`, the values of $nmc$ are subtracted from the mixed-indexed connection coefficients.

If necessary, `christof` calls the functions `contortion` and `nonmetricity` in order to compute $kt$ and $nm$.

**contortion (tr)**

Computes the $(2,1)$ contortion coefficients from the torsion tensor $tr$.

**nonmetricity (nm)**

Computes the $(2,1)$ nonmetricity coefficients from the nonmetricity vector $nm$.

### 26.2.7 Miscellaneous features

**ctransform (M)**

A function in the `ctensor` (component tensor) package which will perform a coordinate transformation upon an arbitrary square symmetric matrix $M$. The user must input the functions which define the transformation. (Formerly called `transform`.)

**findde (A, n)**

returns a list of the unique differential equations (expressions) corresponding to the elements of the $n$ dimensional square array $A$. Presently, $n$ may be 2 or 3. `deindex` is a global list containing the indices of $A$ corresponding to these unique differential equations. For the Einstein tensor ($ein$), which is a two dimensional array, if computed for the metric in the example below, `findde` gives the following independent differential equations:

```plaintext
(%i1) load(ctensor);
(%o1) /share/tensor/ctensor.mac
(%i2) derivabbrev:true;
(%o2) true
(%i3) dim:4;
(%o3) 4
(%i4) lg:matrix([a, 0, 0, 0], [ 0, x^2, 0, 0],
            [0, 0, x^2*sin(y)^2, 0], [0,0,0,-d]);
(%o4) 
  [ a  0  0  0 ]
  [ 0 x  0  0 ]
  [ 0 0 x sin(y) 0 ]
  [ 0 0 0 -d ]
```
(%i5) depends([a,d],x);
(%o5) [a(x), d(x)]

(%i6) ct_coords:[x,y,z,t];
(%o6) [x, y, z, t]

(%i7) cmetric();
(%o7) done

(%i8) einstein(false);
(%o8) done

(%i9) findde(ein,2);
(%o9) [d x - a d + d, 2 a d d x - a (d ) x - a d d x
     x       x       x       x
     2      2
     + 2 a d d - 2 a d , a x + a - a]  
     x       x       x

(%i10) deindex;
(%o10) [[1, 1], [2, 2], [4, 4]]

**cograd ()** [Function]
Computes the covariant gradient of a scalar function allowing the user to choose the corresponding vector name as the example under **contragrad** illustrates.

**contragrad ()** [Function]
Computes the contravariant gradient of a scalar function allowing the user to choose the corresponding vector name as the example below for the Schwarzschild metric illustrates:

(%i11) load(ctensor);
(%o11) /share/tensor/ctensor.mac

(%i12) derivabbrev:true;
(%o12) true

(%i13) ct_coordsys(exteriorschwarzschild,all);
(%o13) done

(%i14) depends(f,r);
(%o14) [f(r)]

(%i15) cograd(f,g1);
(%o15) done

(%i16) listarray(g1);
(%o16) [0, f , 0, 0]
      r

(%i17) contragrad(f,g2);
(%o17) done

(%i18) listarray(g2);
(%o18) [0, f r - 2 f m , 0, 0]
      r  
      r
dscalar ()  
computes the tensor d’Alembertian of the scalar function once dependencies have  
been declared upon the function. For example:

```
(%i1) load(ctensor);
(%o1) /share/tensor/ctensor.mac
(%i2) derivabbrev:true;
(%o2) true
(%i3) ct_coordsys(exteriorschwarzschild,all);
(%o3) done
(%i4) depends(p,r);
(%o4) [p(r)]
(%i5) factor(dscalar(p));
```

```
2
p r - 2 m p r + 2 p r - 2 m p
r r r r
--------------
2
---

(%o5) %r
```

checkdiv ()  
computes the covariant divergence of the mixed second rank tensor (whose first index  
must be covariant) by printing the corresponding n components of the vector field (the  
divergence) where n = dim. If the argument to the function is g then the divergence  
of the Einstein tensor will be formed and must be zero. In addition, the divergence  
(vector) is given the array name div.

cgeodesic (dis)  
A function in the ctensor (component tensor) package. cgeodesic computes the  
geodesic equations of motion for a given metric. They are stored in the array geod[i].  
If the argument dis is true then these equations are displayed.

bdvac (f)  
generates the covariant components of the vacuum field equations of the Brans-  
Dicke gravitational theory. The scalar field is specified by the argument f, which should be  
a (quoted) function name with functional dependencies, e.g., 'p(x).

The components of the second rank covariant field tensor are represented by the array  
bd.

invariant1 ()  
generates the mixed Euler- Lagrange tensor (field equations) for the invariant density  
of R^2. The field equations are the components of an array named inv1.

invariant2 ()  
*** NOT YET IMPLEMENTED ***
generates the mixed Euler- Lagrange tensor (field equations) for the invariant density  
of ric[i,j]*uriem[i,j]. The field equations are the components of an array named  
inv2.
bimetric ()

*** NOT YET IMPLEMENTED ***

generates the field equations of Rosen’s bimetric theory. The field equations are the components of an array named rosen.

26.2.8 Utility functions

diagmatrixp (M, n)

Returns true if the first n rows and n columns of M form a diagonal matrix or (2D) array.

symmetricp (M, n)

Returns true if M is a n by n symmetric matrix or two-dimensional array, otherwise false.

If n is less than the size of M, symmetricp considers only the n by n submatrix (respectively, subarray) comprising rows 1 through n and columns 1 through n.

ntermst (f)

gives the user a quick picture of the "size" of the doubly subscripted tensor (array) f. It prints two element lists where the second element corresponds to NTERMS of the components specified by the first elements. In this way, it is possible to quickly find the non-zero expressions and attempt simplification.

cdisplay (ten)

displays all the elements of the tensor ten, as represented by a multidimensional array. Tensors of rank 0 and 1, as well as other types of variables, are displayed as with ldisplay. Tensors of rank 2 are displayed as 2-dimensional matrices, while tensors of higher rank are displayed as a list of 2-dimensional matrices. For instance, the Riemann-tensor of the Schwarzschild metric can be viewed as:

\[
(\text{i1}) \text{load(ctensor);} \\
(\text{o1}) /\text{share/tensor/ctensor.mac} \\
(\text{i2}) \text{ratfac:}\text{true;} \\
(\text{i3}) \text{ct coordsys(exteriorschwarzschild,all);} \\
(\text{o3}) \text{done} \\
(\text{i4}) \text{riemann(false);} \\
(\text{o4}) \text{done} \\
(\text{i5}) \text{cdisplay(riem);} \\
\begin{bmatrix}
0 & 0 & 0 & 0 \\
[2] & \\
[3 m (r - 2 m) & m & 2 m] \\
[0 & \text{---} & \text{---} & 0 & 0] \\
[4 & 3 & 4] \\
[r & r & r] \\
\end{bmatrix}
\]

\text{riem} = [0 \\
1, 1 \begin{bmatrix}
\text{m (r - 2 m)} \\
0 & \text{---} & 0] \\
\end{bmatrix}
\[
\begin{array}{ccc}
\left[
\begin{array}{cccc}
4 & \\
\alpha & \\
\beta & \\
\gamma & \\
\end{array}
\right]
\end{array}
\]

\[
\begin{array}{ccc}
\left[
\begin{array}{cccc}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{array}
\right]
\end{array}
\]

\[
\begin{array}{ccc}
\text{riem} = [ \\
1, 2 \\
\left[
\begin{array}{cccc}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{array}
\right]
\end{array}
\]

\[
\begin{array}{ccc}
\left[
\begin{array}{cccc}
2 \alpha & \beta & \gamma & \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{array}
\right]
\end{array}
\]

\[
\begin{array}{ccc}
\text{riem} = [ \\
1, 3 \\
\left[
\begin{array}{cccc}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{array}
\right]
\end{array}
\]

\[
\begin{array}{ccc}
\left[
\begin{array}{cccc}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{array}
\right]
\end{array}
\]

\[
\begin{array}{ccc}
\text{riem} = [ \\
1, 4 \\
\left[
\begin{array}{cccc}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{array}
\right]
\end{array}
\]

\[
\begin{array}{ccc}
\left[
\begin{array}{cccc}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{array}
\right]
\end{array}
\]

\[
\begin{array}{ccc}
\text{riem} = [ \\
2, 1 \\
\left[
\begin{array}{cccc}
2 & \\
0 & \\
0 & \\
0 & \\
\end{array}
\right]
\end{array}
\]

\[
\begin{array}{ccc}
\left[
\begin{array}{cccc}
r (r - 2 \alpha) & \\
0 & \\
0 & \\
0 & \\
\end{array}
\right]
\end{array}
\]
\[
\text{riem} = \begin{bmatrix}
2m \\
\frac{-r(r - 2m)}{2} \\
\frac{m}{2} \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
m & 0 & 0 & 0 \\
\end{bmatrix}
\]

\[
\text{riem} = \begin{bmatrix}
2 \\
r(r - 2m) \\
m \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

\[
\text{riem} = \begin{bmatrix}
2 \\
r(r - 2m) \\
m \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

\[
\text{riem} = \begin{bmatrix}
2 \\
r(r - 2m) \\
m \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{bmatrix}
\]
\[
\text{riem} = \begin{bmatrix}
    m \\
    -0 & 0 & 0 \\
    r \\
    0 & 0 & 0 \\
    0 & 0 & 0 \\
\end{bmatrix}
\]

\[
\text{riem} = \begin{bmatrix}
    m \\
    0 & 0 & 0 \\
    r \\
    0 & 0 & 0 \\
    0 & 0 & 0 \\
\end{bmatrix}
\]

\[
\text{riem} = \begin{bmatrix}
    m \\
    -0 & 0 & 0 \\
    r \\
    0 & 0 & 0 \\
    0 & 0 & 0 \\
\end{bmatrix}
\]

\[
\text{riem} = \begin{bmatrix}
    r \\
    0 & 0 & 0 \\
    0 & 0 & 0 \\
    0 & 0 & 0 \\
    2m - r \\
\end{bmatrix}
\]

\[
\text{riem} = \begin{bmatrix}
    2m \\
    0 & 0 & 0 \\
    r \\
    0 & 0 & 0 \\
    0 & 0 & 0 \\
\end{bmatrix}
\]

\[
\text{riem} = \begin{bmatrix}
    0 & 0 & 0 & 0 \\
\end{bmatrix}
\]
\[
\text{riem} = \begin{bmatrix}
0 & 0 & 0 & 0 \\
\frac{2 \sin(\theta)}{r} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

\[
\text{riem} = \begin{bmatrix}
0 & 0 & 0 & 0 \\
\frac{2 \sin(\theta)}{r} & \frac{2 \sin(\theta)}{r} & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

\[
\text{riem} = \begin{bmatrix}
0 & 0 & 0 & 0 \\
\frac{2 \sin(\theta)}{r} & \frac{2 \sin(\theta)}{r} & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

\[
\text{riem} = \begin{bmatrix}
0 & 0 & 0 & 0 \\
\frac{2 \sin(\theta)}{r} & \frac{2 \sin(\theta)}{r} & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

(\%o5) done
deleten \((L, n)\) \[
\text{Returns a new list consisting of } L \text{ with the } n\text{'th element deleted.}
\]

### 26.2.9 Variables used by ctensor

- **dim** \[Option variable\]
  - Default value: 4
  - An option in the ctensor (component tensor) package. dim is the dimension of the manifold with the default 4. The command dim: n will reset the dimension to any other value n.

- **diagmetric** \[Option variable\]
  - Default value: false
  - An option in the ctensor (component tensor) package. If diagmetric is true special routines compute all geometrical objects (which contain the metric tensor explicitly) by taking into consideration the diagonality of the metric. Reduced run times will, of course, result. Note: this option is set automatically by csetup if a diagonal metric is specified.

- **ctrsgsimp** \[Option variable\]
  - Causes trigonometric simplifications to be used when tensors are computed. Presently, ctrsgsimp affects only computations involving a moving frame.

- **cframe_flag** \[Option variable\]
  - Causes computations to be performed relative to a moving frame as opposed to a holonomic metric. The frame is defined by the inverse frame array fri and the frame metric lfg. For computations using a Cartesian frame, lfg should be the unit matrix of the appropriate dimension; for computations in a Lorentz frame, lfg should have the appropriate signature.

- **ctorsion_flag** \[Option variable\]
  - Causes the contortion tensor to be included in the computation of the connection coefficients. The contortion tensor itself is computed by contortion from the user-supplied tensor tr.

- **cnonmet_flag** \[Option variable\]
  - Causes the nonmetricity coefficients to be included in the computation of the connection coefficients. The nonmetricity coefficients are computed from the user-supplied nonmetricity vector nm by the function nonmetricity.

- **ctayswitch** \[Option variable\]
  - If set to true, causes some ctensor computations to be carried out using Taylor-series expansions. Presently, christof, ricci, uricci, einstein, and weyl take into account this setting.

- **ctayvar** \[Option variable\]
  - Variable used for Taylor-series expansion if ctayswitch is set to true.

- **ctaypov** \[Option variable\]
  - Maximum power used in Taylor-series expansion when ctayswitch is set to true.
ctaypt  [Option variable]
Point around which Taylor-series expansion is carried out when ctayswitch is set to true.

gdet  [System variable]
The determinant of the metric tensor lg. Computed by cmetric when cframe_flag is set to false.

ratchristof  [Option variable]
Causes rational simplification to be applied by christof.

rateinstein  [Option variable]
Default value: true
If true rational simplification will be performed on the non-zero components of Einstein tensors; if ratfac is true then the components will also be factored.

ratriemann  [Option variable]
Default value: true
One of the switches which controls simplification of Riemann tensors; if true, then rational simplification will be done; if ratfac is true then each of the components will also be factored.

ratweyl  [Option variable]
Default value: true
If true, this switch causes the weyl function to apply rational simplification to the values of the Weyl tensor. If ratfac is true, then the components will also be factored.

lfg  [Variable]
The covariant frame metric. By default, it is initialized to the 4-dimensional Lorentz frame with signature (+,+,+,-). Used when cframe_flag is true.

ufg  [Variable]
The inverse frame metric. Computed from lfg when cmetric is called while cframe_flag is set to true.

riem  [Variable]
The (3,1) Riemann tensor. Computed when the function riemann is invoked. For information about index ordering, see the description of riemann.
If cframe_flag is true, riem is computed from the covariant Riemann-tensor lriem.

lriem  [Variable]
The covariant Riemann tensor. Computed by lriemann.

uriem  [Variable]
The contravariant Riemann tensor. Computed by uriemann.

ric  [Variable]
The mixed Ricci-tensor. Computed by ricci.
uric
The contravariant Ricci-tensor. Computed by uricci.

lg
The metric tensor. This tensor must be specified (as a \texttt{dim} by \texttt{dim} matrix) before other computations can be performed.

ug
The inverse of the metric tensor. Computed by cmetric.

weyl
The Weyl tensor. Computed by weyl.

fb
Frame bracket coefficients, as computed by frame_bracket.

tkinvariant
The Kretchmann invariant. Computed by rinvariant.

np
A Newman-Penrose null tetrad. Computed by nptetrad.

npi
The raised-index Newman-Penrose null tetrad. Computed by nptetrad. Defined as \texttt{ug.np}. The product \texttt{np.transpose(npi)} is constant:

\begin{verbatim}
(%i39) trigsimp(np.transpose(npi));
[ 0 - 1 0 0 ]
[   ]
[ - 1 0 0 0 ]
[   ]
[ 0 0 0 1 ]
[   ]
[ 0 0 1 0 ]
\end{verbatim}

tr
User-supplied rank-3 tensor representing torsion. Used by contortion.

kt
The contortion tensor, computed from \texttt{tr} by contortion.

nm
User-supplied nonmetricity vector. Used by nonmetricity.

nmc
The nonmetricity coefficients, computed from \texttt{nm} by nonmetricity.

tensorkill
Variable indicating if the tensor package has been initialized. Set and used by csetup, reset by init_ctensor.
ct_coords

Default value: []

An option in the ctensor (component tensor) package. ct_coords contains a list of coordinates. While normally defined when the function csetup is called, one may redefine the coordinates with the assignment ct_coords: [j1, j2, ..., jn] where the j’s are the new coordinate names. See also csetup.

26.2.10 Reserved names

The following names are used internally by the ctensor package and should not be redefined:

<table>
<thead>
<tr>
<th>Name</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>_lg()</td>
<td>Evaluates to lfg if frame metric used, lg otherwise</td>
</tr>
<tr>
<td>_ug()</td>
<td>Evaluates to ufg if frame metric used, ug otherwise</td>
</tr>
<tr>
<td>cleanup()</td>
<td>Removes items from the deindex list</td>
</tr>
<tr>
<td>contract4()</td>
<td>Used by psi()</td>
</tr>
<tr>
<td>filemet()</td>
<td>Used by csetup() when reading the metric from a file</td>
</tr>
<tr>
<td>findde1()</td>
<td>Used by findde()</td>
</tr>
<tr>
<td>findde2()</td>
<td>Used by findde()</td>
</tr>
<tr>
<td>findde3()</td>
<td>Used by findde()</td>
</tr>
<tr>
<td>kdelta()</td>
<td>Kronecker-delta (not generalized)</td>
</tr>
<tr>
<td>newmet()</td>
<td>Used by csetup() for setting up a metric interactively</td>
</tr>
<tr>
<td>setflags()</td>
<td>Used by init_ctensor()</td>
</tr>
<tr>
<td>readvalue()</td>
<td></td>
</tr>
<tr>
<td>resimp()</td>
<td></td>
</tr>
<tr>
<td>sermet()</td>
<td>Used by csetup() for entering a metric as Taylor-series</td>
</tr>
<tr>
<td>txyzsum()</td>
<td></td>
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<tr>
<td>tmetric()</td>
<td>Frame metric, used by cmetric() when cframe_flag:true</td>
</tr>
<tr>
<td>triemman()</td>
<td>Riemann-tensor in frame base, used when cframe_flag:true</td>
</tr>
<tr>
<td>tricci()</td>
<td>Ricci-tensor in frame base, used when cframe_flag:true</td>
</tr>
<tr>
<td>trrc()</td>
<td>Ricci rotation coefficients, used by christof()</td>
</tr>
<tr>
<td>yesp()</td>
<td></td>
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</table>

26.2.11 Changes

In November, 2004, the ctensor package was extensively rewritten. Many functions and variables have been renamed in order to make the package compatible with the commercial version of Macsyma.

<table>
<thead>
<tr>
<th>New Name</th>
<th>Old Name</th>
<th>Description</th>
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</thead>
<tbody>
<tr>
<td>ctaylor()</td>
<td>DLGTAYLOR()</td>
<td>Taylor-series expansion of an expression</td>
</tr>
<tr>
<td>lgeod[]</td>
<td>EM</td>
<td>Geodesic equations</td>
</tr>
<tr>
<td>ein[]</td>
<td>G[]</td>
<td>Mixed Einstein-tensor</td>
</tr>
<tr>
<td>ric[]</td>
<td>LR[]</td>
<td>Mixed Ricci-tensor</td>
</tr>
<tr>
<td>ricci()</td>
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<td>ctaypov</td>
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<td>cgeodesic()</td>
<td>MOTION</td>
<td>Compute geodesic equations</td>
</tr>
<tr>
<td>ct_coords</td>
<td>OMEGA</td>
<td>Metric coordinates</td>
</tr>
</tbody>
</table>
ctayvar PARAM Taylor-series expansion variable
lriem[] R[] Covariant Riemann-tensor
uriemann() RAISERIEMANN() Compute the contravariant Riemann-tensor
ratriemann RATRIEMAN Rational simplif. of the Riemann-tensor
uric[] RICCI[] Contravariant Ricci-tensor
uricci() RICCICOM() Compute the contravariant Ricci-tensor
cmetric() SETMETRIC() Set up the metric
c Taypt TAYPT Point for Taylor-series expansion
c TAYSWITCH Taylor-series setting switch
c csetup() TSETUP() Start interactive setup session
ctransform() TTRANSFORM() Interactive coordinate transformation
uriem[] UR[] Contravariant Riemann-tensor
weyl[] W[] (3,1) Weyl-tensor
27 atensor

27.1 Introduction to atensor

atensor is an algebraic tensor manipulation package. To use atensor, type `load(atensor)`, followed by a call to the `init_atensor` function.

The essence of atensor is a set of simplification rules for the noncommutative (dot) product operator ("."). atensor recognizes several algebra types; the corresponding simplification rules are put into effect when the `init_atensor` function is called.

The capabilities of atensor can be demonstrated by defining the algebra of quaternions as a Clifford-algebra Cl(0,2) with two basis vectors. The three quaternionic imaginary units are then the two basis vectors and their product, i.e.:

\[ i = v \quad j = v \quad k = v \cdot v \]

1 2 1 2

Although the atensor package has a built-in definition for the quaternion algebra, it is not used in this example, in which we endeavour to build the quaternion multiplication table as a matrix:

```
(%i1) load(atensor);
(%o1) /share/tensor/atensor.mac
(%i2) init_atensor(clifford,0,0,2);
(%o2) done
(%i3) atensimp(v[1].v[1]);
(%o3) -1
(%i4) atensimp((v[1].v[2]).(v[1].v[2]));
(%o4) -1
(%i5) q:zeromatrix(4,4);  
[ 0 0 0 0 ]
[ ]
[ 0 0 0 0 ]
[ ]
[ 0 0 0 0 ]
[ ]
[ 0 0 0 0 ]
(%o5) 
(%i6) q[1,1]:1;
(%o6) 1
(%i7) for i thru adim do q[1,i+1]:q[i+1,1]:v[i];
(%o7) done
(%i8) q[1,4]:q[4,1]:v[1].v[2];
(%o8) v . v
1 2
(%i9) for i from 2 thru 4 do for j from 2 thru 4 do
   q[i,j]:atensimp(q[i,1].q[1,j]);
(%o9) done
(%i10) q;
```
atensor recognizes as base vectors indexed symbols, where the symbol is that stored in asymbol and the index runs between 1 and adim. For indexed symbols, and indexed symbols only, the bilinear forms sf, af, and av are evaluated. The evaluation substitutes the value of aform[i,j] in place of fun(v[i],v[j]) where v represents the value of asymbol and fun is either af or sf; or, it substitutes v[aform[i,j]] in place of av(v[i],v[j]).

Needless to say, the functions sf, af and av can be redefined.

When the atensor package is loaded, the following flags are set:

```
dotscrules: true;

dotdistrib: true;

dotexptsimp: false;
```

If you wish to experiment with a nonassociative algebra, you may also consider setting dotassoc to false. In this case, however, atensimp will not always be able to obtain the desired simplifications.

### 27.2 Functions and Variables for atensor

**init_atensor**

```
init_atensor (alg_type, opt_dims)
init_atensor (alg_type)
```

Initializes the atensor package with the specified algebra type. alg_type can be one of the following:

- **universal**: The universal algebra has no commutation rules.
- **grassmann**: The Grassman algebra is defined by the commutation relation \( u.v + v.u = 0 \).
- **clifford**: The Clifford algebra is defined by the commutation relation \( u.v + v.u = -2*sf(u,v) \) where \( sf \) is a symmetric scalar-valued function. For this algebra, opt_dims can be up to three nonnegative integers, representing the number of positive, degenerate, and negative dimensions of the algebra, respectively. If any opt_dims values are supplied, atensor will configure the values of adim and aform appropriately. Otherwise, adim will default to 0 and aform will not be defined.
- **symmetric**: The symmetric algebra is defined by the commutation relation \( u.v - v.u = 0 \).
- **symplectic**: The symplectic algebra is defined by the commutation relation \( u.v - v.u = 2*af(u,v) \) where \( af \) is an antisymmetric scalar-valued function. For the
symplectic algebra, \( \text{opt\_dims} \) can be up to two nonnegative integers, representing the nondegenerate and degenerate dimensions, respectively. If any \( \text{opt\_dims} \) values are supplied, \( \text{atensor} \) will configure the values of \( \text{adim} \) and \( \text{aform} \) appropriately. Otherwise, \( \text{adim} \) will default to 0 and \( \text{aform} \) will not be defined.

\text{lie\_envelop}: \) The algebra of the Lie envelope is defined by the commutation relation \( u.v-v.u=2*\text{av}(u,v) \) where \( \text{av} \) is an antisymmetric function.

The \( \text{init\_atensor} \) function also recognizes several predefined algebra types:

- \( \text{complex} \) implements the algebra of complex numbers as the Clifford algebra \( \text{Cl}(0,1) \).
- The call \( \text{init\_atensor}(\text{complex}) \) is equivalent to \( \text{init\_atensor}(\text{clifford},0,0,1) \).
- \( \text{quaternion} \) implements the algebra of quaternions. The call \( \text{init\_atensor}(\text{quaternion}) \) is equivalent to \( \text{init\_atensor}(\text{clifford},0,0,2) \).
- \( \text{pauli} \) implements the algebra of Pauli-spinors as the Clifford-algebra \( \text{Cl}(3,0) \). A call to \( \text{init\_atensor}(\text{pauli}) \) is equivalent to \( \text{init\_atensor}(\text{clifford},3) \).
- \( \text{dirac} \) implements the algebra of Dirac-spinors as the Clifford-algebra \( \text{Cl}(3,1) \). A call to \( \text{init\_atensor}(\text{dirac}) \) is equivalent to \( \text{init\_atensor}(\text{clifford},3,0,1) \).

\[ \text{atensimp} \left( \text{expr} \right) \] [Function]

Simplifies an algebraic tensor expression \( \text{expr} \) according to the rules configured by a call to \( \text{init\_atensor} \). Simplification includes recursive application of commutation relations and resolving calls to \( \text{sf}, \text{af}, \) and \( \text{av} \) where applicable. A safeguard is used to ensure that the function always terminates, even for complex expressions.

\[ \text{alg\_type} \] [Function]

The algebra type. Valid values are \text{universal}, \text{grassmann}, \text{clifford}, \text{symmetric}, \text{symplectic} and \text{lie\_envelop}.

\[ \text{adim} \] [Variable]

Default value: 0

The dimensionality of the algebra. \( \text{atensor} \) uses the value of \( \text{adim} \) to determine if an indexed object is a valid base vector. See \( \text{abasep} \).

\[ \text{aform} \] [Variable]

Default value: \( \text{ident}(3) \)

Default values for the bilinear forms \( \text{sf}, \text{af}, \) and \( \text{av} \). The default is the identity matrix \( \text{ident}(3) \).

\[ \text{asymbol} \] [Variable]

Default value: \( v \)

The symbol for base vectors.

\[ \text{sf} \left( u, v \right) \] [Function]

A symmetric scalar function that is used in commutation relations. The default implementation checks if both arguments are base vectors using \( \text{abasep} \) and if that is the case, substitutes the corresponding value from the matrix \( \text{aform} \).

\[ \text{af} \left( u, v \right) \] [Function]

An antisymmetric scalar function that is used in commutation relations. The default implementation checks if both arguments are base vectors using \( \text{abasep} \) and if that is the case, substitutes the corresponding value from the matrix \( \text{aform} \).
av (u, v)  [Function]
An antisymmetric function that is used in commutation relations. The default implementation checks if both arguments are base vectors using abasep and if that is the case, substitutes the corresponding value from the matrix aform.

For instance:

(%i1) load(atensor);
(%o1) /share/tensor/atensor.mac
(%i2) adim:3;
(%o2) 3
(%i3) aform:matrix([0,3,-2],[-3,0,1],[2,-1,0]);
   [ 0  3  -2 ]
   [           ]
   [ -3  0  1 ]
   [           ]
   [ 2  -1  0 ]
(%o3)                     
(%i4) asymbol:x;
(%o4) x
(%i5) av(x[1],x[2]);
(%o5) x

abasep (v)  [Function]
Checks if its argument is an atensor base vector. That is, if it is an indexed symbol, with the symbol being the same as the value of asymbol, and the index having a numeric value between 1 and adim.
28 Sums, Products, and Series

28.1 Functions and Variables for Sums and Products

bashindices (expr) [Function]
Transforms the expression expr by giving each summation and product a unique index. This gives changevar greater precision when it is working with summations or products. The form of the unique index is jnumber. The quantity number is determined by referring to gensumnum, which can be changed by the user. For example, gensumnum:0$ resets it.

lsum (expr, x, L) [Function]
Represents the sum of expr for each element x in L. A noun form 'lsum is returned if the argument L does not evaluate to a list.

Examples:

(%i1) lsum (x^i, i, [1, 2, 7]);
    7    2
(%o1) x + x + x

(%i2) lsum (i^2, i, rootsof (x^3 - 1, x));
   ==
   2
(%o2) > i
 / 
   ==
     3
 i in rootsof(x - 1, x)

intosum (expr) [Function]
Moves multiplicative factors outside a summation to inside. If the index is used in the outside expression, then the function tries to find a reasonable index, the same as it does for sumcontract. This is essentially the reverse idea of the outative property of summations, but note that it does not remove this property, it only bypasses it.

In some cases, a scanmap (multthru, expr) may be necessary before the intosum.

simpproduct [Option variable]
Default value: false

When simpproduct is true, the result of a product is simplified. This simplification may sometimes be able to produce a closed form. If simpproduct is false or if the quoted form 'product is used, the value is a product noun form which is a representation of the pi notation used in mathematics.

product (expr, i, i_0, i_1) [Function]
Represents a product of the values of expr as the index i varies from i_0 to i_1. The noun form 'product is displayed as an uppercase letter pi.

product evaluates expr and lower and upper limits i_0 and i_1, product quotes (does not evaluate) the index i.
If the upper and lower limits differ by an integer, \( expr \) is evaluated for each value of the index \( i \), and the result is an explicit product.

Otherwise, the range of the index is indefinite. Some rules are applied to simplify the product. When the global variable \( \textbf{simpproduct} \) is \texttt{true}, additional rules are applied. In some cases, simplification yields a result which is not a product; otherwise, the result is a noun form 'product.

See also \texttt{nouns} and \texttt{evflag}.

Examples:

\begin{verbatim}
(%i1) product (x + i*(i+1)/2, i, 1, 4);
(%o1) (x + 1) (x + 3) (x + 6) (x + 10)

(%i2) product (i^2, i, 1, 7);
(%o2) 25401600

(%i3) product (a[i], i, 1, 7);
(%o3)  a  a  a  a  a  a  a
     1  2  3  4  5  6  7

(%i4) product (a(i), i, 1, 7);
(%o4) a(1) a(2) a(3) a(4) a(5) a(6) a(7)

(%i5) product (a(i), i, 1, n);
(%o5) \texttt{n}
    \texttt{/===}
    \texttt{!  a(i)}
    \texttt{!  i = 1}

(%i6) product (k, k, 1, n);
(%o6) \texttt{n}
    \texttt{/===}
    \texttt{!  k}
    \texttt{!  k = 1}

(%i7) product (k, k, 1, n), simpproduct;
(%o7) \texttt{n!}

(%i8) product (integrate (x^k, x, 0, 1), k, 1, n);
(%o8) \texttt{n}
    \texttt{/===}
    \texttt{!  1}
    \texttt{!  ----}
    \texttt{!  k + 1}
    \texttt{k = 1}

(%i9) product (if k <= 5 then a^k else b^k, k, 1, 10);
(%o9) a  b
\end{verbatim}

\texttt{simpsum} \hspace{1cm} \texttt{[Option variable]}

Default value: \texttt{false}
When \texttt{simpsum} is \texttt{true}, the result of a \texttt{sum} is simplified. This simplification may sometimes be able to produce a closed form. If \texttt{simpsum} is \texttt{false} or if the quoted form \texttt{'}sum\texttt{'} is used, the value is a sum noun form which is a representation of the sigma notation used in mathematics.

\begin{verbatim}
sum (expr, i, i_0, i_1) [Function]
Represents a summation of the values of \texttt{expr} as the index \texttt{i} varies from \texttt{i_0} to \texttt{i_1}. The noun form \texttt{'}sum\texttt{'} is displayed as an uppercase letter \texttt{sigma}.
sum evaluates its summand \texttt{expr} and lower and upper limits \texttt{i_0} and \texttt{i_1}, \texttt{sum} quotes (does not evaluate) the index \texttt{i}.
If the upper and lower limits differ by an integer, the summand \texttt{expr} is evaluated for each value of the summation index \texttt{i}, and the result is an explicit sum.
Otherwise, the range of the index is indefinite. Some rules are applied to simplify the summation. When the global variable \texttt{simpsum} is \texttt{true}, additional rules are applied. In some cases, simplification yields a result which is not a summation; otherwise, the result is a noun form \texttt{'}sum\texttt{'}.

When the \texttt{evflag} (evaluation flag) \texttt{cauchysum} is \texttt{true}, a product of summations is expressed as a Cauchy product, in which the index of the inner summation is a function of the index of the outer one, rather than varying independently.

The global variable \texttt{genindex} is the alphabetic prefix used to generate the next index of summation, when an automatically generated index is needed.

\texttt{gensumnum} is the numeric suffix used to generate the next index of summation, when an automatically generated index is needed. When \texttt{gensumnum} is \texttt{false}, an automatically-generated index is only \texttt{genindex} with no numeric suffix.

See also \texttt{lsum}, \texttt{sumcontract}, \texttt{intosum}, \texttt{bashindices}, \texttt{niceindices}, \texttt{nouns}, \texttt{evflag}, and Chapter 87 [zeilberger-pkg], page 1129.

Examples:
\begin{verbatim}
(%i1) sum (i^2, i, 1, 7); 140
(%o1) 140

(%i2) sum (a[i], i, 1, 7);
    a + a + a + a + a + a
    7 6 5 4 3 2 1

(%o2) a(7) + a(6) + a(5) + a(4) + a(3) + a(2) + a(1)

(%i3) sum (a(i), i, 1, n); n
  =========
\( \backslash \)
  a(i)
\  
  \  =========
  i = 1

(%i4) sum (2^i + i^2, i, 0, n); n
  =========
\( \backslash \)
  a(i)
\  
  \  =========
  i = 1

(%i5) sum (2^i + i^2, i, 0, n); n
  =========
\( \backslash \)
  a(i)
\  
  \  =========
  i = 1
\end{verbatim}
\end{verbatim}
\( \frac{i^2}{2} \)

\[
\text{(o5)} \quad \frac{2 + i}{\text{====}} \\
\quad i = 0
\]

\[
\text{(i6)} \quad \text{sum}(2^i + i^2, i, 0, n), \text{simpsum}; \\
\frac{3}{2} \quad \frac{n + 1}{2} \quad \frac{2n + 3n + n}{6} \\
\text{(o6)} \quad \frac{2}{6} + \frac{1}{6} - 1
\]

\[
\text{(i7)} \quad \text{sum}(1/3^i, i, 1, \text{inf}); \\
\frac{1}{3} \quad \text{====} \quad 1 \\
\text{(o7)} \quad \frac{1}{i} \\
\text{====} \quad 3 \\
\quad i = 1
\]

\[
\text{(i8)} \quad \text{sum}(1/3^i, i, 1, \text{inf}), \text{simpsum}; \\
\frac{1}{3} \quad - \\
\text{(o8)} \quad \frac{2}{3}
\]

\[
\text{(i9)} \quad \text{sum}(i^2, i, 1, 4) * \text{sum}(1/i^2, i, 1, \text{inf}); \\
\frac{\text{inf}}{\text{====}} \\
\frac{1}{\text{====} \quad 1} \\
\text{(o9)} \quad \frac{30}{\text{====} \quad 2} \\
\quad \text{====} \quad i \\
\quad i = 1
\]

\[
\text{(i10)} \quad \text{sum}(i^2, i, 1, 4) * \text{sum}(1/i^2, i, 1, \text{inf}), \text{simpsum}; \\
\frac{2}{\text{====} \quad 2} \\
\text{(o10)} \quad 5 \frac{\%pi}{\text{====} \quad 1} \\
\text{(i11)} \quad \text{integrate}(x^k, x, 0, 1), k, 1, n); \\
\frac{\text{inf}}{\text{====}} \\
\frac{1}{\text{====} \quad 1} \\
\text{(o11)} \quad \frac{\text{====} \quad k + 1} \\
\quad \frac{\text{====}}{\text{====} \quad k + 1} \\
\quad \text{====} \\
\quad \text{====} \\
\quad \text{====} \\
\quad \text{====} \\
\quad \text{====} \\
\quad \text{====} \\
\quad \text{====} \\
\quad \text{====} \\
\quad \text{====} \\
\text{(i12)} \quad \text{sum}(\text{if } k \leq 5 \text{ then } a^k \text{ else } b^k, k, 1, 10); \\
\frac{10}{\text{====}} \\
\frac{9}{\text{====}} \\
\frac{8}{\text{====}} \\
\frac{7}{\text{====}} \\
\frac{6}{\text{====}} \\
\frac{5}{\text{====}} \\
\frac{4}{\text{====}} \\
\frac{3}{\text{====}} \\
\frac{2}{\text{====}} \\
\text{(o12)} \quad \frac{b}{b} + \frac{b}{b} + \frac{b}{b} + \frac{a}{a} + \frac{a}{a} + \frac{a}{a} + \frac{a}{a}
\]
sumcontract (expr)  
Combines all sums of an addition that have upper and lower bounds that differ by constants. The result is an expression containing one summation for each set of such summations added to all appropriate extra terms that had to be extracted to form this sum. sumcontract combines all compatible sums and uses one of the indices from one of the sums if it can, and then try to form a reasonable index if it cannot use any supplied.

It may be necessary to do an intosum (expr) before the sumcontract.

sumexpand  
[Option variable]
Default value: false
When sumexpand is true, products of sums and exponentiated sums simplify to nested sums.

See also cauchysum.

Examples:

(%i1) sumexpand: true$
(%i2) sum (f (i), i, 0, m) * sum (g (j), j, 0, n);
   m      n
(\%i2) > > f(i1) g(i2)
   \ \   \ \ 
   i1 = 0 i2 = 0
(%o2) > > f(i1) g(i2)
   /   / 
   \ \ i1 = 0 i2 = 0
(%i3) sum (f (i), i, 0, m)^2;
   m      m
(\%i3) > > f(i3) f(i4)
   \ \   \ \ 
   i3 = 0 i4 = 0
(%o3) > > f(i3) f(i4)
   /   / 
   \ \ i3 = 0 i4 = 0

28.2 Introduction to Series
Maxima contains functions taylor and powerseries for finding the series of differentiable functions. It also has tools such as nusum capable of finding the closed form of some series. Operations such as addition and multiplication work as usual on series. This section presents the global variables which control the expansion.

28.3 Functions and Variables for Series
cauchysum  
[Option variable]
Default value: false
When multiplying together sums with inf as their upper limit, if sumexpand is true and cauchysum is true then the Cauchy product will be used rather than the usual
product. In the Cauchy product the index of the inner summation is a function of the index of the outer one rather than varying independently.

Example:

```maxima
(%i1) sumexpand: false$
(%i2) cauchysum: false$
(%i3) s: sum (f(i), i, 0, inf) * sum (g(j), j, 0, inf);
inf inf
==== ====
\ / \ \\
(%o3) ( > f(i)) > g(j)
/ / \\
==== ====
i = 0 j = 0
(%i4) sumexpand: true$
(%i5) cauchysum: true$
(%i6) expand(s,0,0);
inf i1
==== ====
\ / \ \\
(%o6) > > g(i1 - i2) f(i2)
/ / \\
==== ==== i1 = 0 i2 = 0
```

deftaylor (f_1(x_1), expr_1, ..., f_n(x_n), expr_n)

For each function \( f_i \) of one variable \( x_i \), \texttt{deftaylor} defines \( expr_i \) as the Taylor series about zero. \( expr_i \) is typically a polynomial in \( x_i \) or a summation; more general expressions are accepted by \texttt{deftaylor} without complaint.

\texttt{powerseries} \((f_i(x_i), x_i, 0)\) returns the series defined by \texttt{deftaylor}.

\texttt{deftaylor} returns a list of the functions \( f_1, \ldots, f_n \). \texttt{deftaylor} evaluates its arguments.

Example:

```maxima
(%i1) deftaylor (f(x), x^2 + sum(x^i/(2^i*i!^2), i, 4, inf));
(%o1) \%f
(%i2) powerseries (f(x), x, 0);
inf
==== i1
\ / x \\
(%o2) > > ------- + x
/ / i1 2
==== 2 i1!
 i1 = 4
(%i3) taylor (exp (sqrt (f(x))), x, 0, 4);
2 3 4
 x 3073 x 12817 x
(%o3)/T/ 1 + x + -- + ------- + -------- + ... 
```

maxtayorder

Default value: true

When maxtayorder is true, then during algebraic manipulation of (truncated) Taylor series, taylor tries to retain as many terms as are known to be correct.

niceindices (expr)

Renames the indices of sums and products in expr. niceindices attempts to rename each index to the value of niceindicespref[1], unless that name appears in the summand or multiplicand, in which case niceindices tries the succeeding elements of niceindicespref in turn, until an unused variable is found. If the entire list is exhausted, additional indices are constructed by appending integers to the value of niceindicespref[1], e.g., i0, i1, i2, ...

niceindices returns an expression. niceindices evaluates its argument.

Example:

(%i1) niceindicespref;
(%o1) [i, j, k, l, m, n]

(%i2) product (sum (f (foo + i*j*bar), foo, 1, inf), bar, 1, inf);

inf inf
/===

! ! > f(bar i j + foo)
! ! 

bar = 1 ====

foo = 1

(%i3) niceindices (%);

inf inf
/===\ =====

! ! \ 

(%o3) ! ! > f(i j l + k)
! ! /

l = 1 =====

k = 1

niceindicespref

Default value: [i, j, k, l, m, n]

niceindicespref is the list from which niceindices takes the names of indices for sums and products.

The elements of niceindicespref are typically names of variables, although that is not enforced by niceindices.

Example:

(%i1) niceindicespref: [p, q, r, s, t, u]$ 

(%i2) product (sum (f (foo + i*j*bar), foo, 1, inf), bar, 1, inf);

inf inf
/===\ =====
\begin{verbatim}
(%o2) > f(bar i j + foo)
       \bar = 1 ====
       foo = 1

(%i3) niceindices (%);
inf  inf
/>===\  ====
   \   
(%o3) > f(i j q + p)
       \ q = 1 ====
       p = 1

nusum (expr, x, i_0, i_1)

[Function]

Carries out indefinite hypergeometric summation of expr with respect to x using a
decision procedure due to R.W. Gosper. expr and the result must be expressible as
products of integer powers, factorials, binomials, and rational functions.

The terms "definite" and "indefinite summation" are used analogously to "definite"
and "indefinite integration". To sum indefinitely means to give a symbolic result for
the sum over intervals of variable length, not just e.g. 0 to inf. Thus, since there is
no formula for the general partial sum of the binomial series, \texttt{nusum} can't do it.

\texttt{nusum} and \texttt{unsum} know a little about sums and differences of finite products. See also
\texttt{unsum}.

Examples:

(%i1) nusum (n*n!, n, 0, n);

Dependent equations eliminated: (1)

(%o1) (n + 1)! - 1

(%i2) nusum (n^4*4^n/binomial(2*n,n), n, 0, n);

\begin{verbatim}
   4 3 2 n
2 (n + 1) (63 n + 112 n + 18 n - 22 n + 3) 4 2
----------------------------------------------- - -------
693 binomial(2 n, n) 3 11 7
\end{verbatim}

(%i3) unsum (%/=%, n);

\begin{verbatim}
4  n
n 4
-------------------
binomial(2 n, n)
\end{verbatim}

(%i4) unsum (prod (i^2, i, 1, n), n);

\begin{verbatim}
\( \Rightarrow \)
2
\end{verbatim}

(%o4) (i + 1) (n - 1) (n + 1)

i = 1
\end{verbatim}

(%i5) nusum (% , n, 1, n);

Dependent equations eliminated: (2 3)

\[
\begin{align*}
\text{n} & \\
\text{/===} & \\
\text{2} & \\
\text{i - 1} & \\
\text{i = 1}
\end{align*}
\]

pade (taylor_series, numer_deg_bound, denom_deg_bound)  [Function]
Returns a list of all rational functions which have the given Taylor series expansion
where the sum of the degrees of the numerator and the denominator is less than or
equal to the truncation level of the power series, i.e. are "best" approximants, and
which additionally satisfy the specified degree bounds.

taylor_series is a univariate Taylor series. numer_deg_bound and denom_deg_bound
are positive integers specifying degree bounds on the numerator and denominator.

taylor_series can also be a Laurent series, and the degree bounds can be inf which
causes all rational functions whose total degree is less than or equal to the length
of the power series to be returned. Total degree is defined as numer_deg_bound +
denom_deg_bound. Length of a power series is defined as "truncation level" + 1 -
min(0, "order of series").

(%i11) taylor (1 + x + x^2 + x^3, x, 0, 3);
2 3
\[
1 + x + x + x + \ldots 
\]
(%i12) pade (% , 1, 1);
1
\[
\text{[- -----]} \\
\text{x - 1}
\]
(%i13) t: taylor(-\(83787\times^{10} - 45552\times^{9} - 187296\times^{8}
+ 387072\times^{7} + 86016\times^{6} - 1507328\times^{5}
+ 1966080\times^{4} + 4194304\times^{3} - 25165824\times^{2}
+ 67108864\times - 134217728\))
/134217728, x, 0, 10);
2 3 4 5 6 7
\[
x^{3} x \ x \ 15 \ x \ 23 \ x \ 21 \ x \ \text{189} \ x
\]
(%o3)/T/ 1 - - + ----- - -- - ----- + ----- - ----- - -------
2 16 32 1024 2048 32768 65536
8 9 10
\[
5853 \ x \ 2847 \ x \ 83787 \ x
+ \text{---------} + \text{---------} - \text{---------} + \ldots
4194304 \ 8388608 \ 134217728
\]
(%i4) pade (t, 4, 4);
(%o4) []
There is no rational function of degree 4 numerator/denominator, with this power series expansion. You must in general have degree of the numerator and degree of the denominator adding up to at least the degree of the power series, in order to have enough unknown coefficients to solve.

(%i5) pade (t, 5, 5);

5 4 3
- (520256329 x - 96719020632 x - 489651410240 x
2
- 1619100813312 x - 2176885157888 x - 2386516803584)

5 4 3
/ (47041365435 x + 381702613848 x + 1360678489152 x
2
+ 2856700692480 x + 3370143559680 x + 2386516803584)]

powerseries (expr, x, a) [Function]
Returns the general form of the power series expansion for expr in the variable x about the point a (which may be inf for infinity):

\[
\inf \sum_{n=0}^{\infty} b_n (x - a)^n
\]

If powerseries is unable to expand expr, taylor may give the first several terms of the series.

When verbose is true, powerseries prints progress messages.

(%i1) verbose: true$
(%i2) powerseries (log(sin(x)/x), x, 0);
can't expand
log(sin(x))
so we'll try again after applying the rule:
\[
\frac{d}{dx} \log(sin(x)) = \frac{i \cot(x)}{sin(x)}
\]
in the first simplification we have returned:

\[
[ i \cot(x) \ dx - \log(x)
\]
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\[
\frac{\prod_{i_1=1}^{\infty} \left( -1 \right)^{i_1} \frac{\text{bern}(2 i_1) x}{(2 i_1)!}}{\frac{i_1 (2 i_1)!}{2}} = 1
\]

\([\text{Option variable}]\)

\text{psexpand}

Default value: \text{false}

When \text{psexpand} is \text{true}, an extended rational function expression is displayed fully expanded. The switch \text{ratexpand} has the same effect.

When \text{psexpand} is \text{false}, a multivariate expression is displayed just as in the rational function package.

When \text{psexpand} is \text{multi}, then terms with the same total degree in the variables are grouped together.

\text{revert (expr, x)} \quad [\text{Function}]

\text{revert2 (expr, x, n)} \quad [\text{Function}]

These functions return the reversion of \text{expr}, a Taylor series about zero in the variable \text{x}. \text{revert} returns a polynomial of degree equal to the highest power in \text{expr}. \text{revert2} returns a polynomial of degree \text{n}, which may be greater than, equal to, or less than the degree of \text{expr}.

\text{load ("revert")} loads these functions.

Examples:

\[(\%i1)\] \text{load ("revert")}$
\[(\%i2)\] \text{t: taylor (exp(x) - 1, x, 0, 6)};
\[
\begin{array}{cccccc}
2 & 3 & 4 & 5 & 6 \\
x & x & x & x & x
\end{array}
\]
\[(\%o2)/T/\]
\[
\begin{array}{cccccc}
2 & 6 & 24 & 120 & 720 \\
x + \text{---} + \text{---} + \text{---} + \text{---} + \text{---} + \ldots
\end{array}
\]
\[(\%i3)\] \text{revert (t, x)};
\[
\begin{array}{cccccc}
6 & 5 & 4 & 3 & 2 \\
10 x - 12 x + 15 x - 20 x + 30 x - 60 x
\end{array}
\]
\[(\%o3)/R/ - \frac{\text{---} \text{---} \text{---} \text{---} \text{---}}{60}
\]
\[(\%i4)\] \text{ratexpand (\%)};
\[
\begin{array}{cccccc}
6 & 5 & 4 & 3 & 2 \\
x & x & x & x & x
\end{array}
\]
\[(\%o4)\]
\[
\text{---} + \text{---} + \text{---} + \text{---} + \text{---} + \text{x}
\]
\[(\%i5)\] \text{taylor (log(x+1), x, 0, 6)};
\[
\begin{array}{cccccc}
2 & 3 & 4 & 5 & 6
\end{array}
\]
\[
\frac{x}{T} = \frac{x}{2} - \frac{x}{3} + \frac{x}{4} - \frac{x}{5} + \frac{x}{6} + \ldots
\]

\[%i6\) ratsimp (revert (t, x) - taylor (log(x+1), x, 0, 6));

\%

\[%i7\) revert2 (t, x, 4);

\]

\begin{verbatim}
taylor

taylor (expr, x, a, n)
taylor (expr, [x_1, x_2, ...], a, n)
taylor (expr, [x, a, n, 'asymp])
taylor (expr, [x_1, x_2, ...], [a_1, a_2, ...], [n_1, n_2, ...])
taylor (expr, [x_1, a_1, n_1], [x_2, a_2, n_2], ...)

\end{verbatim}

taylor (expr, x, a, n) expands the expression expr in a truncated Taylor or \textit{Laurent} series in the variable x around the point a, containing terms through \((x - a)^n\).

If expr is of the form \(f(x)/g(x)\) and \(g(x)\) has no terms up to degree \(n\) then \texttt{taylor} attempts to expand \(g(x)\) up to degree \(2n\). If there are still no nonzero terms, \texttt{taylor} doubles the degree of the expansion of \(g(x)\) so long as the degree of the expansion is less than or equal to \(n 2^{\text{taylordepth}}\).

taylor (expr, [x_1, x_2, ...], a, n) returns a truncated power series of degree \(n\) in all variables \(x_1, x_2, \ldots\) about the point \((a, a, \ldots)\).

taylor (expr, [x_1, a_1, n_1], [x_2, a_2, n_2], ...) returns a truncated power series in the variables \(x_1, x_2, \ldots\) about the point \((a_1, a_2, \ldots)\), truncated at \(n_1, n_2, \ldots\)

taylor (expr, [x_1, x_2, ...], [a_1, a_2, ...], [n_1, n_2, ...]) returns a truncated power series in the variables \(x_1, x_2, \ldots\) about the point \((a_1, a_2, \ldots)\), truncated at \(n_1, n_2, \ldots\)

taylor (expr, [x, a, n, 'asymp]) returns an expansion of \(x - a\). The highest order term is \((x - a)^{-n}\). The switch \texttt{maxtaylororder} is true, then during algebraic manipulation of (truncated) Taylor series, \texttt{taylor} tries to retain as many terms as are known to be correct.

When \texttt{psexpand} is true, an extended rational function expression is displayed fully expanded. The switch \texttt{ratexpand} has the same effect. When \texttt{psexpand} is false, a multivariate expression is displayed just as in the rational function package. When \texttt{psexpand} is \texttt{multi}, then terms with the same total degree in the variables are grouped together.

See also the \texttt{taylor_logexpand} switch for controlling expansion.

Examples:

\[%i11\) taylor (sqrt (sin(x) + a*x + 1), x, 0, 3);

\[
\frac{2}{2} (a + 1) x + (a + 2 a + 1) x
\]
(%o1)/T/ \(1 + \frac{\frac{3}{2}}{8} - \frac{\frac{3}{(3a + 9a + 9a - 1)x}}{48} + \cdots\)

(%i2) (%o2)/T/ \(1 + (a + 1)x - \frac{3}{6} + \cdots\)

(%i3) taylor(sqrt(x + 1), x, 0, 5);
\[\frac{x}{2} + \frac{x^3}{8} + \frac{5x^5}{16} + \cdots\]

(%o3)/T/ \(1 + \frac{x}{2} - \frac{x^3}{8} + \frac{x^5}{16} + \cdots\)

(%i4) (%o4)/T/ \(1 + x + \cdots\)

(%i5) product((1 + x^i)^2.5, i, 1, inf)/(1 + x^2);
\[\sum_{i=1}^{\text{inf}} \frac{(x + 1)^{2.5i}}{2x + 1}\]

(%i6) ev(taylor(%), x, 0, 3), keepfloat;
\[1 + 2.5x + 3.375x^2 + 6.5625x^3 + \cdots\]

(%i7) taylor(1/log(x + 1), x, 0, 3);
\[-\frac{1}{x} - \frac{1}{12}x - \frac{1}{24}x^2 - \frac{1}{720}x^3 + \cdots\]

(%i8) taylor((cos(x) - sec(x))^3, x, 0, 5);
\[0 + \cdots\]

(%i9) taylor(1/(cos(x) - sec(x))^3, x, 0, 5);
\[1 + 1 + \frac{11}{347} + \frac{347}{6767}x + \frac{6767}{15377}x + \cdots\]
(%o10) \( - \frac{6}{4} + \frac{2}{6} + \frac{15120}{604800} + \frac{7983360}{x^2} + \ldots \)

(%i11) taylor (sqrt (1 - k^2*sin(x)^2), x, 0, 6);
2 4 2 4
k x (3 k - 4 k ) x
(%o11) \( \frac{1}{2} - \frac{6}{24} x + \frac{45 k - 60 k + 16 k}{720} x + \ldots \)

(%i12) taylor ((x + 1)^n, x, 0, 4);
2 2 3 2 3
(n - n) x (n - 3 n + 2 n) x
(%o12) \( 1 + n x + \frac{n}{2} - \frac{n}{6} x + \frac{n - 6 n + 11 n - 6 n}{24} x + \ldots \)

(%i13) taylor (sin (y + x), x, 0, 3, y, 0, 3);
3
y
(%o13) \( y - \frac{1}{6} + \frac{1}{2} + \left( \frac{1}{12} - \frac{1}{2} + \ldots \right) x + \ldots \)

(%i14) taylor (sin (y + x), [x, y], 0, 3);
3 2 3
x + 3 y x + 3 y x + y
(%o14) \( y + x - \frac{1}{6} + \ldots + \frac{1}{3} x + \frac{1}{2} y + \ldots \)

(%i15) taylor (1/sin (y + x), x, 0, 3, y, 0, 3);
1 1 1 2
y 1 1 2
(%o15) \( - \frac{1}{y} + \frac{1}{y} x + \left( - \frac{1}{y} - \frac{1}{y} + \ldots \right) x + \frac{1}{y} x + \ldots \)
\[ \frac{1}{\sin(y + x)} \]

\[ \frac{1}{x + y} + \frac{x + y}{6} + \frac{7 x + 21 y x + 21 y x + 7 y}{360} + \ldots \]

\( \text{taylordepth} \)

Default value: 3

If there are still no nonzero terms, \texttt{taylor} doubles the degree of the expansion of \( g(x) \) so long as the degree of the expansion is less than or equal to \( n \cdot 2^\text{taylordepth} \).

\( \text{taylorinfo} (\text{expr}) \)

Returns information about the Taylor series \( \text{expr} \). The return value is a list of lists. Each list comprises the name of a variable, the point of expansion, and the degree of the expansion.

\text{taylorinfo} returns \texttt{false} if \( \text{expr} \) is not a Taylor series.

Example:

\[ \frac{(1 - y^2)/(1 - x), x, 0, 3, [y, a, \infty]}{2} \]

\[ \frac{2}{(y - a) - 2 a (y - a) + (1 - a )} \]

\[ + (1 - a - 2 a (y - a) - (y - a)) x \]

\[ + (1 - a - 2 a (y - a) - (y - a)) x \]

\[ + (1 - a - 2 a (y - a) - (y - a)) x + \ldots \]

\( \text{taylorlogexpand} \)

Default value: \texttt{true}

\( \text{taylorlogexpand} \) controls expansions of logarithms in \texttt{taylor} series.

When \( \text{taylorlogexpand} \) is \texttt{true}, all logarithms are expanded fully so that zero-recognition problems involving logarithmic identities do not disturb the expansion process. However, this scheme is not always mathematically correct since it ignores branch information.

When \( \text{taylorlogexpand} \) is set to \texttt{false}, then the only expansion of logarithms that occur is that necessary to obtain a formal power series.
**taylor_order_coefficients**

[Option variable]

Default value: true

taylor_order_coefficients controls the ordering of coefficients in a Taylor series. When taylor_order_coefficients is true, coefficients of taylor series are ordered canonically.

**taylor_simplifier (expr)**

[Function]

Simplifies coefficients of the power series expr. taylor calls this function.

**taylor_truncate_polynomials**

[Option variable]

Default value: true

When taylor_truncate_polynomials is true, polynomials are truncated based upon the input truncation levels.

Otherwise, polynomials input to taylor are considered to have infinite precision.

**taytorat (expr)**

[Function]

Converts expr from taylor form to canonical rational expression (CRE) form. The effect is the same as rat (ratdisrep (expr)), but faster.

**trunc (expr)**

[Function]

Annotates the internal representation of the general expression expr so that it is displayed as if its sums were truncated Taylor series. expr is not otherwise modified.

Example:

```
(%i1) expr: x^2 + x + 1;
       2
(%o1) x + x + 1
(%i2) trunc (expr);
       2
(%o2) 1 + x + x + . . .
(%i3) is (expr = trunc (expr));
(%o3) true
```

**unsum (f, n)**

[Function]

Returns the first backward difference \( f(n) - f(n - 1) \). Thus unsum in a sense is the inverse of sum.

See also nusum.

Examples:

```
(%i1) g(p) := p*4^n/binomial(2*n,n);

(%o1) g(p) := ------------------
      binomial(2 n, n)

(%i2) g(n^4);
       4  n
       4
(%o2) ------------------
```

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\( \binom{2n}{n} \)

(%i3) \text{nusum (%, n, 0, n);}
\[ \frac{4^n}{693} \binom{2n}{n} \]

(%o3) \[
\frac{4^3}{693} \binom{2n}{n} - \frac{2^n}{3^n} \]

(%i4) \text{unsum (%, n);}
\[ \frac{n}{\binom{2n}{n}} \]

(%o4) \[ \frac{n^4}{\binom{2n}{n}} \]

verbose \[ \text{[Option variable]} \]
Default value: false

When \text{verbose} is true, \text{powerseries} prints progress messages.

28.4 Introduction to Fourier series

The \text{fourie} package comprises functions for the symbolic computation of Fourier series. There are functions in the \text{fourie} package to calculate Fourier integral coefficients and some functions for manipulation of expressions.

28.5 Functions and Variables for Fourier series

\text{equalp (x, y)} \[ \text{[Function]} \]
Returns true if \text{equal (x, y)} otherwise false (doesn't give an error message like \text{equal (x, y)} would do in this case).

\text{remfun (f, expr)} \[ \text{[Function]} \]
\text{remfun (f, expr, x)} \[ \text{[Function]} \]
\text{remfun (f, expr)} replaces all occurrences of \text{f (arg)} by \text{arg} in \text{expr}.
\text{remfun (f, expr, x)} replaces all occurrences of \text{f (arg)} by \text{arg} in \text{expr} only if \text{arg} contains the variable \text{x}.

\text{funp (f, expr)} \[ \text{[Function]} \]
\text{funp (f, expr, x)} \[ \text{[Function]} \]
\text{funp (f, expr)} returns true if \text{expr} contains the function \text{f}.
\text{funp (f, expr, x)} returns true if \text{expr} contains the function \text{f} and the variable \text{x} is somewhere in the argument of one of the instances of \text{f}.

\text{absint (f, x, halfplane)} \[ \text{[Function]} \]
\text{absint (f, x)} \[ \text{[Function]} \]
\text{absint (f, x, a, b)} \[ \text{[Function]} \]
\text{absint (f, x, halfplane)} returns the indefinite integral of \text{f} with respect to \text{x} in the given halfplane (pos, neg, or both). \text{f} may contain expressions of the form \text{abs (x)}, \text{abs (sin (x))}, \text{abs (a) * exp (-abs (b) * abs (x))}.
absint \( (f, x) \) is equivalent to \( \text{absint} (f, x, \text{pos}) \).

\( \text{absint} (f, x, a, b) \) returns the definite integral of \( f \) with respect to \( x \) from \( a \) to \( b \).
\( f \) may include absolute values.

\texttt{fourier} \( (f, x, p) \) \quad \text{[Function]}

Returns a list of the Fourier coefficients of \( f(x) \) defined on the interval \([-p, p]\).

\texttt{foursimp} \( (l) \) \quad \text{[Function]}

Simplifies \( \sin (n \%\pi) \) to \( 0 \) if \( \text{sinnpiflag} \) is \text{true} and \( \cos (n \%\pi) \) to \((-1)^n\) if \( \text{cosnpiflag} \) is \text{true}.

\texttt{sinnpiflag} \quad \text{[Option variable]}

Default value: \text{true}
See \texttt{foursimp}.

\texttt{cosnpiflag} \quad \text{[Option variable]}

Default value: \text{true}
See \texttt{foursimp}.

\texttt{fourexpand} \( (l, x, p, \text{limit}) \) \quad \text{[Function]}

Constructs and returns the Fourier series from the list of Fourier coefficients \( l \) up through \( \text{limit} \) terms (\( \text{limit} \) may be \text{inf}). \( x \) and \( p \) have same meaning as in \texttt{fourier}.

\texttt{fourcos} \( (f, x, p) \) \quad \text{[Function]}

Returns the Fourier cosine coefficients for \( f(x) \) defined on \([0, p]\).

\texttt{foursin} \( (f, x, p) \) \quad \text{[Function]}

Returns the Fourier sine coefficients for \( f(x) \) defined on \([0, p]\).

\texttt{totalfourier} \( (f, x, p) \) \quad \text{[Function]}

Returns \texttt{fourexpand} \((\texttt{foursimp} \( (\texttt{fourier} \( (f, x, p) \)) \)), x, p, 'inf)\).

\texttt{fourint} \( (f, x) \) \quad \text{[Function]}

Constructs and returns a list of the Fourier integral coefficients of \( f(x) \) defined on \([\text{minf}, \text{inf}]\).

\texttt{fourintcos} \( (f, x) \) \quad \text{[Function]}

Returns the Fourier cosine integral coefficients for \( f(x) \) on \([0, \text{inf}]\).

\texttt{fourintsin} \( (f, x) \) \quad \text{[Function]}

Returns the Fourier sine integral coefficients for \( f(x) \) on \([0, \text{inf}]\).

28.6 Functions and Variables for Poisson series

\texttt{intopois} \( (a) \) \quad \text{[Function]}

Converts \( a \) into a Poisson encoding.

\texttt{outofpois} \( (a) \) \quad \text{[Function]}

Converts \( a \) from Poisson encoding to general representation. If \( a \) is not in Poisson form, \texttt{outofpois} carries out the conversion, i.e., the return value is \texttt{outofpois} \((\texttt{intopois} \( (a) \)))\). This function is thus a canonical simplifier for sums of powers of sine and cosine terms of a particular type.
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poisdiff (a, b) [Function]
Differentiates a with respect to b. b must occur only in the trig arguments or only in the coefficients.

poisexpt (a, b) [Function]
Functionally identical to intopois (a \^ b). b must be a positive integer.

poisint (a, b) [Function]
Integrates in a similarly restricted sense (to poisdiff). Non-periodic terms in b are dropped if b is in the trig arguments.

poislim [Option variable]
Default value: 5
poislim determines the domain of the coefficients in the arguments of the trig functions. The initial value of 5 corresponds to the interval [-2^((5-1)+1), 2^((5-1))], or [-15,16], but it can be set to [-2^((n-1)+1), 2^((n-1))].

poismap (series, sinfn, cosfn) [Function]
will map the functions sinfn on the sine terms and cosfn on the cosine terms of the Poisson series given. sinfn and cosfn are functions of two arguments which are a coefficient and a trigonometric part of a term in series respectively.

poisplus (a, b) [Function]
Is functionally identical to intopois (a + b).

poissimp (a) [Function]
Converts a into a Poisson series for a in general representation.

poisson [Special symbol]
The symbol /P/ follows the line label of Poisson series expressions.

poissubst (a, b, c) [Function]
Substitutes a for b in c. c is a Poisson series.
(1) Where B is a variable u, v, w, x, y, or z, then a must be an expression linear in those variables (e.g., 6*u + 4*v).
(2) Where b is other than those variables, then a must also be free of those variables, and furthermore, free of sines or cosines.
poissubst (a, b, c, d, n) is a special type of substitution which operates on a and b as in type (1) above, but where d is a Poisson series, expands \( \cos(d) \) and \( \sin(d) \) to order n so as to provide the result of substituting a + d for b in c. The idea is that d is an expansion in terms of a small parameter. For example, poissubst (u, v, cos(v), %e, 3) yields \( \cos(u)*(1 - %e^2/2) - \sin(u)*(%e - %e^3/6) \).

poistimes (a, b) [Function]
Is functionally identical to intopois (a*b).

poistrim () [Function]
is a reserved function name which (if the user has defined it) gets applied during Poisson multiplication. It is a predicate function of 6 arguments which are the coefficients of the u, v, ..., z in a term. Terms for which poistrim is true (for the coefficients of that term) are eliminated during multiplication.
printpois (a)  

Prints a Poisson series in a readable format. In common with outofpois, it will convert a into a Poisson encoding first, if necessary.
29 Number Theory

29.1 Functions and Variables for Number Theory

bern (n)  [Function]
Returns the \( n \)'th Bernoulli number for integer \( n \). Bernoulli numbers equal to zero are suppressed if \texttt{zerobern} is \texttt{false}.

See also \texttt{burn}.

\begin{verbatim}
(%i1) zerobern: true$
(%i2) map (bern, [0, 1, 2, 3, 4, 5, 6, 7, 8]);
    1 1 1 1 1 1
(%o2) [1, - -, - , 0, - --, 0, - --, - ]
    2 6 30 42 30
(%i3) zerobern: false$
(%i4) map (bern, [0, 1, 2, 3, 4, 5, 6, 7, 8]);
    1 1 1 1 1 5 691 7
(%o4) [1, - -, - , - --, --, - --, - ----, - ]
    2 6 30 42 30 66 2730 6
\end{verbatim}

bernpoly (x, n)  [Function]
Returns the \( n \)'th Bernoulli polynomial in the variable \( x \).

bfzeta (s, n)  [Function]
Returns the Riemann zeta function for the argument \( s \). The return value is a big float (bfloat); \( n \) is the number of digits in the return value.

bfhzeta (s, h, n)  [Function]
Returns the Hurwitz zeta function for the arguments \( s \) and \( h \). The return value is a big float (bfloat); \( n \) is the number of digits in the return value.

The Hurwitz zeta function is defined as

\[
\zeta(s, h) = \sum_{k=0}^{\infty} \frac{1}{(k + h)^s}
\]

\texttt{load ("bffac")} loads this function.

burn (n)  [Function]
Returns a rational number, which is an approximation of the \( n \)'th Bernoulli number for integer \( n \). \texttt{burn} exploits the observation that (rational) Bernoulli numbers can be approximated by (transcendental) zetas with tolerable efficiency:

\[
B(2 \, n) = \frac{n - 1 \, 1 - 2 \, n}{(-1) \, 2 \, \zeta(2 \, n) \, (2 \, n)!} \frac{2 \, n}{\pi}
\]
burn may be more efficient than bern for large, isolated \( n \) as bern computes all the Bernoulli numbers up to index \( n \) before returning. burn invokes the approximation for even integers \( n > 255 \). For odd integers and \( n \leq 255 \) the function bern is called.

load ("bffac") loads this function. See also bern.

chinese ([\( r_1, \ldots, r_n \)], [\( m_1, \ldots, m_n \)]) [Function]
Solves the system of congruences \( x \equiv r_1 \mod m_1 \), \ldots, \( x \equiv r_n \mod m_n \). The remainders \( r_n \) may be arbitrary integers while the moduli \( m_n \) have to be positive and pairwise coprime integers.

\[
\%i1 \quad \text{mods} : [1000, 1001, 1003, 1007];
\%o1 \quad [1000, 1001, 1003, 1007]
\%i2 \quad \text{lreduce ('gcd, mods)};
\%o2 \quad 1
\%i3 \quad x : \text{random(apply("*", mods))};
\%o3 \quad 685124877004
\%i4 \quad \text{rems} : \text{map(lambda([z], \text{mod}(x, z)), mods)};
\%o4 \quad [4, 568, 54, 624]
\%i5 \quad \text{chinese(rems, mods)};
\%o5 \quad 685124877004
\%i6 \quad \text{chinese([1, 2], [3, n])};
\%o6 \quad \text{chinese([1, 2], [3, n])}
\%i7 \quad \%, \ n = 4;
\%o7 \quad 10
\]

 cf (expr) [Function]
Computes a continued fraction approximation. expr is an expression comprising continued fractions, square roots of integers, and literal real numbers (integers, rational numbers, ordinary floats, and bigfloats). cf computes exact expansions for rational numbers, but expansions are truncated at ratepsilon for ordinary floats and \( 10^{-\text{fpprec}} \) for bigfloats.

Operands in the expression may be combined with arithmetic operators. Maxima does not know about operations on continued fractions outside of cf.

cf evaluates its arguments after binding listarith to false. cf returns a continued fraction, represented as a list.

A continued fraction \( a + 1/(b + 1/(c + \ldots)) \) is represented by the list \([a, b, c, \ldots]\). The list elements \( a, b, c, \ldots \) must evaluate to integers. expr may also contain sqrt (\( n \)) where \( n \) is an integer. In this case cf will give as many terms of the continued fraction as the value of the variable cflength times the period.

A continued fraction can be evaluated to a number by evaluating the arithmetic representation returned by cfdisrep. See also cfexpand for another way to evaluate a continued fraction.

See also cfdisrep, cfexpand, and cflength.

Examples:
- expr is an expression comprising continued fractions and square roots of integers.

\[
\%i11 \quad \text{cf ([5, 3, 1]*[11, 9, 7] + [3, 7]/[4, 3, 2])};
\]
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(%o1) \[59, 17, 2, 1, 1, 27\]
(%i2) cf ((3/17)*[1, -2, 5]/sqrt(11) + (8/13));
(%o2) \[0, 1, 1, 3, 2, 1, 4, 1, 9, 1, 9, 2\]

- **cflength** controls how many periods of the continued fraction are computed for algebraic, irrational numbers.

  (%i1) cflength: 1$
  (%i2) cf ((1 + sqrt(5))/2);
  (%o2) \[1, 1, 1, 2\]

- A continued fraction can be evaluated by evaluating the arithmetic representation returned by **cfdisrep**.

  (%i1) cflength: 3$
  (%i2) cfdisrep (cf (sqrt(3)))$
  (%i3) ev (%, numer);
  (%o3) 1.731707317073171

- Maxima does not know about operations on continued fractions outside of **cf**.

  (%i1) cf ([1,1,1,1,1,2] * 3);
  (%o1) \[4, 1, 5, 2\]

**cfdisrep** (list)

Constructs and returns an ordinary arithmetic expression of the form a + 1/(b + 1/(c + ...)) from the list representation of a continued fraction [a, b, c, ...].

  (%i1) cf ([1, 2, -3] + [1, -2, 1]);
  (%o1) \[1, 1, 2\]

**cfexpand** (x)

Returns a matrix of the numerators and denominators of the last (column 1) and next-to-last (column 2) convergents of the continued fraction \(x\).

  (%i1) cf (rat (ev (%pi, numer)));

  'rat' replaced 3.141592653589793 by 103993/33102 =3.141592653011902
(%o1) [3, 7, 15, 1, 292]
(%i2) cfexpand (%);
    [ 103993  355 ]
(%o2) [ ]
    [ 33102  113 ]
(%i3) %-[1,1]/%-[2,1], numer;
(%o3) 3.141592653011902

cf
length
[Option variable]
Default value: 1
cf
length
controls the number of terms of the continued fraction the function cf will give, as the value cf
length
times the period. Thus the default is to give one period.

(%i1) cf
length
: 1$
(%i2) cf ((1 + sqrt(5))/2);
    [1, 1, 1, 1, 2]
(%o2) [ ]

(%i3) cf
length
: 2$
(%i4) cf ((1 + sqrt(5))/2);
    [1, 1, 1, 1, 1, 1, 1, 2]
(%o4) [ ]

(%i5) cf
length
: 3$
(%i6) cf ((1 + sqrt(5))/2);
    [1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 2]

divsum
[Function]
divsum (n, k)
divsum (n)
divsum (n, k) returns the sum of the divisors of n raised to the k'th power.
divsum (n) returns the sum of the divisors of n.

(%i1) divsum (12);
(%o1) 28
(%i2) 1 + 2 + 3 + 4 + 6 + 12;
(%o2) 28
(%i3) divsum (12, 2);
(%o3) 210
(%i4) 1^-2 + 2^-2 + 3^-2 + 4^-2 + 6^-2 + 12^-2;
(%o4) 210
euler (n)
[Function]
Returns the n'th Euler number for nonnegative integer n. Euler numbers equal to zero are suppressed if zerobern is false.

For the Euler-Mascheroni constant, see %gamma.

(%i1) zerobern: true$
(%i2) map (euler, [0, 1, 2, 3, 4, 5, 6]);
(%o2) [1, 0, -1, 0, 5, 0, -61]
(%i3) zerobern: false$
(%i4) map (euler, [0, 1, 2, 3, 4, 5, 6]);
(%o4) [1, -1, 5, -61, 1385, -50521, 2702765]
factors_only
[Option variable]

Default value: false

Controls the value returned by ifactors. The default false causes ifactors to provide information about multiplicities of the computed prime factors. If factors_only is set to true, ifactors returns nothing more than a list of prime factors.

Example: See ifactors.

fib (n)
[Function]

Returns the n’th Fibonacci number. fib(0) is equal to 0 and fib(1) equal to 1, and fib (-n) equal to (-1)^n * fib(n).

After calling fib, prevfib is equal to fib(n - 1), the Fibonacci number preceding the last one computed.

(%i1) map (fib, [-4, -3, -2, -1, 0, 1, 2, 3, 4, 5, 6, 7, 8]);
(%o1) [- 3, 2, - 1, 1, 0, 1, 1, 2, 3, 5, 8, 13, 21]

fibtophi (expr)
[Function]

Expresses Fibonacci numbers in expr in terms of the constant %phi, which is (1 + sqrt(5))/2, approximately 1.61803399.

Examples:

(%i1) fibtophi (fib (n));
   n n
(%o1) %phi - (1 - %phi)

(%i2) fib (n-1) + fib (n) - fib (n+1);
(%o2) - fib(n + 1) + fib(n) + fib(n - 1)

(%i3) fibtophi (%);
   n + 1   n + 1   n n
(%o3) %phi - (1 - %phi) %phi - (1 - %phi)
       2 %phi - 1

(%i4) ratsimp (%);
(%o4) 0

ifactors (n)
[Function]

For a positive integer n returns the factorization of n. If n=p1^e1..pk^nk is the decomposition of n into prime factors, ifactors returns [[p1, e1], ..., [pk, ek]].

Factorization methods used are trial divisions by primes up to 9973, Pollard’s rho and p-1 method and elliptic curves.

If the variable ifactor_verbose is set to true ifactor produces detailed output about what it is doing including immediate feedback as soon as a factor has been found.
The value returned by \texttt{ifactors} is controlled by the option variable \texttt{factors_only}. The default \texttt{false} causes \texttt{ifactors} to provide information about the multiplicities of the computed prime factors. If \texttt{factors_only} is set to \texttt{true}, \texttt{ifactors} simply returns the list of prime factors.

\begin{verbatim}
(%i1) ifactors(51575319651600);
(%o1) [[2, 4], [3, 2], [5, 2], [1583, 1], [9050207, 1]]
(%i2) apply("*", map(lambda([u], u[1]^u[2]), %));
(%o2) 51575319651600
(%i3) ifactors(51575319651600), factors_only : true;
(%o3) [2, 3, 5, 1583, 9050207]
\end{verbatim}

\textbf{igcdex} \((n, k)\)  \quad \textbf{[Function]}

Returns a list \([a, b, u]\) where \(u\) is the greatest common divisor of \(n\) and \(k\), and \(u\) is equal to \(a n + b k\). The arguments \(n\) and \(k\) must be integers.

\texttt{igcdex} implements the Euclidean algorithm. See also \texttt{gcdex}.

The command \texttt{load(gcdex)} loads the function.

Examples:

\begin{verbatim}
(%i1) load(gcdex)$
(%i2) igcdex(30,18);
(%o2) [- 1, 2, 6]
(%i3) igcdex(1526757668, 783562735736);
(%o3) [845922341123, - 164826435, 4]
(%i4) igcdex(fib(20), fib(21));
(%o4) [4181, - 2584, 1]
\end{verbatim}

\textbf{inrt} \((x, n)\)  \quad \textbf{[Function]}

Returns the integer \(n'\text{th root of the absolute value of} \ x\).  

\begin{verbatim}
(%i1) l: [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12]$
(%i2) map (lambda ([a], inrt (10^a, 3)), l);
(%o2) [2, 4, 10, 21, 46, 100, 215, 464, 1000, 2154, 4641, 10000]
\end{verbatim}

\textbf{inv_mod} \((n, m)\)  \quad \textbf{[Function]}

Computes the inverse of \(n\) modulo \(m\). \texttt{inv_mod (n,m)} returns \texttt{false}, if \(n\) is a zero divisor modulo \(m\).

\begin{verbatim}
(%i1) inv_mod(3, 41);
(%o1) 14
(%i2) ratsimp(3^-1), modulus = 41;
(%o2) 14
(%i3) inv_mod(3, 42);
(%o3) false
\end{verbatim}

\textbf{isqrt} \((x)\)  \quad \textbf{[Function]}

Returns the "integer square root" of the absolute value of \(x\), which is an integer.

\textbf{jacobi} \((p, q)\)  \quad \textbf{[Function]}

Returns the Jacobi symbol of \(p\) and \(q\).

\begin{verbatim}
(%i1) l: [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12]$
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(%i2) map (lambda ([a], jacobi (a, 9)), 1);
(%o2) [1, 1, 0, 1, 0, 1, 1, 0, 1, 1, 0, 1, 1, 0]

lcm (expr_1, ..., expr_n) [Function]
Returns the least common multiple of its arguments. The arguments may be general expressions as well as integers.
load ("functs") loads this function.

lucas (n) [Function]
Returns the n'th Lucas number. lucas(0) is equal to 2 and lucas(1) equal to 1, and lucas(-n) equal to (-1)^(-n) * lucas(n).
(%i1) map (lucas, [-4, -3, -2, -1, 0, 1, 2, 3, 4, 5, 6, 7, 8]);
(%o1) [7, - 4, 3, - 1, 2, 1, 3, 4, 7, 11, 18, 29, 47]

After calling lucas, the global variable next_lucas is equal to lucas (n + 1), the Lucas number following the last returned. The example shows how Fibonacci numbers can be computed via lucas and next_lucas.

(%i1) fib_via_lucas(n) := block([lucas : lucas(n)],
                    signum(n) * (2*next_lucas - lucas)/5 )$
(%i2) map (fib_via_lucas, [-4, -3, -2, -1, 0, 1, 2, 3, 4, 5, 6, 7, 8]);
(%o2) [- 3, 2, - 1, 1, 0, 1, 1, 2, 3, 5, 8, 13, 21]

mod (x, y) [Function]
If x and y are real numbers and y is nonzero, return x - y * floor(x / y). Further for all real x, we have mod (x, 0) = x. For a discussion of the definition mod (x, 0) = x, see Section 3.4, of "Concrete Mathematics," by Graham, Knuth, and Patashnik.
The function mod (x, 1) is a sawtooth function with period 1 with mod (1, 1) = 0 and mod (0, 1) = 0.

To find the principal argument (a number in the interval (-%pi, %pi]) of a complex number, use the function x |-> %pi - mod (%pi - x, 2*%pi), where x is an argument.

When x and y are constant expressions (10 * %pi, for example), mod uses the same big float evaluation scheme that floor and ceiling uses. Again, it's possible, although unlikely, that mod could return an erroneous value in such cases.

For nonnumerical arguments x or y, mod knows several simplification rules:

(%i1) mod (x, 0);
(%o1) x
(%i2) mod (a*x, a*y);
(%o2) a mod(x, y)
(%i3) mod (0, x);
(%o3) 0

next_prime (n) [Function]
Returns the smallest prime bigger than n.
(%i1) next_prime(27);
(%o1) 29
partfrac (expr, var)

Expands the expression expr in partial fractions with respect to the main variable var. partfrac does a complete partial fraction decomposition. The algorithm employed is based on the fact that the denominators of the partial fraction expansion (the factors of the original denominator) are relatively prime. The numerators can be written as linear combinations of denominators, and the expansion falls out.

(%i1) 1/(1+x)^2 - 2/(1+x) + 2/(2+x);

\[ \frac{2}{x+2} - \frac{2}{x+1} + \frac{2}{x+1} \]

(%o1) \frac{2}{x+2} - \frac{2}{x+1} + \frac{2}{x+1}

(%i2) ratsimp (%);

\[ \frac{x}{3} \]

(%o2) \frac{x}{3}

(%i3) partfrac (% , x);

\[ \frac{2}{x+2} - \frac{2}{x+1} + \frac{2}{x+1} \]

(%o3) \frac{2}{x+2} - \frac{2}{x+1} + \frac{2}{x+1}

power_mod (a, n, m)

Uses a modular algorithm to compute \( a^n \mod m \) where \( a \) and \( n \) are integers and \( m \) is a positive integer. If \( n \) is negative, inv_mod is used to find the modular inverse.

(%i1) power_mod(3, 15, 5);

(%o1) 2

(%i2) mod(3^15,5);

(%o2) 2

(%i3) power_mod(2, -1, 5);

(%o3) 3

(%i4) inv_mod(2,5);

(%o4) 3

primep (n)

Primality test. If primep (n) returns false, \( n \) is a composite number and if it returns true, \( n \) is a prime number with very high probability.

For \( n \) less than 341550071728321 a deterministic version of Miller-Rabin’s test is used. If primep (n) returns true, then \( n \) is a prime number.

For \( n \) bigger than 341550071728321 primep uses primep_number_of_tests Miller-Rabin’s pseudo-primality tests and one Lucas pseudo-primality test. The probability that a non-prime \( n \) will pass one Miller-Rabin test is less than 1/4. Using the default value 25 for primep_number_of_tests, the probability of \( n \) being composite is much smaller that \( 10^{-15} \).

primep_number_of_tests

Default value: 25
Number of Miller-Rabin's tests used in \texttt{primep}.

\textbf{primes (start, end)}

Returns the list of all primes from \texttt{start} to \texttt{end}.

\begin{verbatim}
(%i1) primes(3, 7);
(%o1) [3, 5, 7]
\end{verbatim}

\textbf{prev_prime (n)}

Returns the greatest prime smaller than \texttt{n}.

\begin{verbatim}
(%i1) prev_prime(27);
(%o1) 23
\end{verbatim}

\textbf{qunit (n)}

Returns the principal unit of the real quadratic number field \texttt{sqrt (n)} where \texttt{n} is an integer, i.e., the element whose norm is unity. This amounts to solving Pell's equation \(a^2 - n b^2 = 1\).

\begin{verbatim}
(%i1) qunit (17);
(%o1) sqrt(17) + 4
(%i2) expand (% * (sqrt(17) - 4));
(%o2) 1
\end{verbatim}

\textbf{totient (n)}

Returns the number of integers less than or equal to \texttt{n} which are relatively prime to \texttt{n}.

\textbf{zerobern}

Default value: \texttt{true}

When \texttt{zerobern} is \texttt{false}, \texttt{bern} excludes the Bernoulli numbers and \texttt{euler} excludes the Euler numbers which are equal to zero. See \texttt{bern} and \texttt{euler}.

\textbf{zeta (n)}

Returns the Riemann zeta function. If \texttt{n} is a negative integer, 0, or a positive even integer, the Riemann zeta function simplifies to an exact value. For a positive even integer the option variable \texttt{zeta%pi} has to be \texttt{true} in addition (See \texttt{zeta%pi}). For a floating point or bigfloat number the Riemann zeta function is evaluated numerically. Maxima returns a noun form \texttt{zeta (n)} for all other arguments, including rational noninteger, and complex arguments, or for even integers, if \texttt{zeta%pi} has the value \texttt{false}.

\texttt{zeta(1)} is undefined, but Maxima knows the limit \texttt{limit(zeta(x), x, 1)} from above and below.

The Riemann zeta function distributes over lists, matrices, and equations.

See also \texttt{bfzeta} and \texttt{zeta%pi}.

Examples:

\begin{verbatim}
(%i1) zeta([-2, -1, 0, 0.5, 2, 3, 1+%i]);  2
    1   1
1 + pi
(%o1) [0, - --, - --, - 1.460354508809586, ----, zeta(3),
        2
          2
\end{verbatim}
\[
\begin{align*}
(\%i2) & \text{ limit(zeta(x),x,1,plus);} \\
(\%o2) & \text{ inf} \\
(\%i3) & \text{ limit(zeta(x),x,1,minus);} \\
(\%o3) & \text{ minf}
\end{align*}
\]

\textbf{zeta\%pi}

Default value: \texttt{true}

When \texttt{zeta\%pi} is \texttt{true}, \texttt{zeta} returns an expression proportional to \(\%pi^n\) for even integer \(n\). Otherwise, \texttt{zeta} returns a noun form \texttt{zeta(n)} for even integer \(n\).

Examples:

\begin{align*}
(\%i1) & \text{ zeta\%pi: true$} \\
(\%i2) & \text{ zeta (4);} \\
(\%o2) & \frac{4}{\%pi} \\
(\%i3) & \text{ zeta\%pi: false$} \\
(\%i4) & \text{ zeta (4);} \\
(\%o4) & \text{ zeta(4)}
\end{align*}

\textbf{zn_add_table (n)}

Shows an addition table of all elements in \((\mathbb{Z}/n\mathbb{Z})\).

See also \texttt{zn_mult_table}, \texttt{zn_power_table}.

\textbf{zn_characteristic_factors (n)}

Returns a list containing the characteristic factors of the totient of \(n\).

Using the characteristic factors a multiplication group modulo \(n\) can be expressed as a group direct product of cyclic subgroups.

In case the group itself is cyclic the list only contains the totient and using \texttt{zn_primroot} a generator can be computed. If the totient splits into more than one characteristic factors \texttt{zn_factor_generators} finds generators of the corresponding subgroups.

Each of the \(r\) factors in the list divides the right following factors. For the last factor \(f_r\) therefore holds \(a^{f_r} \equiv 1 \pmod{n}\) for all \(a\) coprime to \(n\). This factor is also known as Carmichael function or Carmichael lambda.

If \(n > 2\), then \texttt{totient(n)/2^r} is the number of quadratic residues, and each of these has \(2^r\) square roots.

See also \texttt{totient}, \texttt{zn_primroot}, \texttt{zn_factor_generators}.

Examples:

The multiplication group modulo 14 is cyclic and its 6 elements can be generated by a primitive root.

\begin{align*}
(\%i1) & \text{ [zn_characteristic_factors(14), phi: totient(14)];} \\
(\%o1) & \text{ [[6], 6]}
\end{align*}
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(%i2) [zn_factor_generators(14), g: zn_primroot(14)];
(%o2) 
[ [3, 3], 3]

(%i3) M14: makelist(power_mod(g,i,14), i,0,phi-1);
(%o3) 
[1, 3, 9, 13, 11, 5]

The multiplication group modulo 15 is not cyclic and its 8 elements can be generated by two factor generators.

(%i1) [[f1,f2]: zn_characteristic_factors(15), totient(15)];
(%o1) 
[ [2, 4], 8]

(%i2) [[g1,g2]: zn_factor_generators(15), zn_primroot(15)];
(%o2) 
[ [11, 7], false]

(%i3) UG1: makelist(power_mod(g1,i,15), i,0,f1-1);
(%o3) 
[1, 11]

(%i4) UG2: makelist(power_mod(g2,i,15), i,0,f2-1);
(%o4) 
[1, 7, 4, 13]

(%i5) M15: create_list(mod(i*j,15), i,UG1, j,UG2);
(%o5) 
[1, 7, 4, 13, 11, 2, 14, 8]

For the last characteristic factor 4 it holds that \(a^4 = 1 \mod 15\) for all \(a\) in M15.

M15 has two characteristic factors and therefore \(8/2^2\) quadratic residues, and each of these has \(2^2\) square roots.

(%i6) zn_power_table(15);

\[
\begin{array}{cccc}
1 & 1 & 1 & 1 \\
2 & 4 & 8 & 1 \\
4 & 1 & 4 & 1 \\
7 & 4 & 13 & 1 \\
8 & 4 & 2 & 1 \\
11 & 1 & 11 & 1 \\
13 & 4 & 7 & 1 \\
14 & 1 & 14 & 1 \\
\end{array}
\]

(%o6) 

(%i7) map(lambda([i], zn_nth_root(i,2,15)), [1,4]);
(%o7) 
[[1, 4, 11, 14], [2, 7, 8, 13]]

zn_carmichael_lambda (n) [Function]
Returns 1 if \(n\) is 1 and otherwise the greatest characteristic factor of the totient of \(n\).

For remarks and examples see zn_characteristic_factors.

zn_determinant (matrix, p) [Function]
Uses the technique of LU-decomposition to compute the determinant of matrix over \((\mathbb{Z}/p\mathbb{Z})\). \(p\) must be a prime.
However if the determinant is equal to zero the LU-decomposition might fail. In that case \texttt{zn_determinant} computes the determinant non-modular and reduces thereafter.

See also \texttt{zn_invert_by_lu}.

Examples:

\begin{verbatim}
(\%i1) m : matrix([1,3],[2,4]);
[ 1 3 ]
(\%o1) [ ]
[ 2 4 ]
(\%i2) zn_determinant(m, 5);
(\%o2) 3
(\%i3) m : matrix([2,4,1],[3,1,4],[4,3,2]);
[ 2 4 1 ]
[ ]
[ 3 1 4 ]
[ ]
[ 4 3 2 ]
(\%o3) [ ]
(\%i4) zn_determinant(m, 5);
(\%o4) 0
\end{verbatim}

\textbf{zn_factor_generators} \texttt{(n)} \[\text{Function}\]

Returns a list containing factor generators corresponding to the characteristic factors of the totient of \texttt{n}.

For remarks and examples see \texttt{zn_characteristic_factors}.

\textbf{zn_invert_by_lu} \texttt{(matrix, p)} \[\text{Function}\]

Uses the technique of LU-decomposition to compute the modular inverse of \texttt{matrix} over \((\mathbb{Z}/p\mathbb{Z})\). \texttt{p} must be a prime and \texttt{matrix} invertible. \texttt{zn_invert_by_lu} returns \texttt{false} if \texttt{matrix} is not invertible.

See also \texttt{zn_determinant}.

Example:

\begin{verbatim}
(\%i1) m : matrix([1,3],[2,4]);
[ 1 3 ]
(\%o1) [ ]
[ 2 4 ]
(\%i2) zn_determinant(m, 5);
(\%o2) 3
(\%i3) mi : zn_invert_by_lu(m, 5);
[ 3 4 ]
(\%o3) [ ]
[ 1 2 ]
(\%i4) matrixmap(lambda([a], mod(a, 5)), m . mi);
[ 1 0 ]
(\%o4) [ ]
[ 0 1 ]
\end{verbatim}
zn_log

\[\text{zn_log}(a, g, n)\]
\[\text{zn_log}(a, g, n, [[p_1, e_1], \ldots, [p_k, e_k]])\]

Computes the discrete logarithm. Let \((\mathbb{Z}/n\mathbb{Z})^*\) be a cyclic group, \(g\) a primitive root modulo \(n\) and let \(a\) be a member of this group. \(\text{zn_log}(a, g, n)\) then solves the congruence \(g^x = a \mod n\).

The applied algorithm needs a prime factorization of \(\text{totient}(n)\). This factorization might be time consuming as well and in some cases it can be useful to factor first and then to pass the list of factors to \(\text{zn_log}\) as the fourth argument. The list must be of the same form as the list returned by \(\text{ifactors(totient(n))}\) using the default option \(\text{factors_only : false}\).

The algorithm uses a Pohlig-Hellman-reduction and Pollard’s Rho-method for discrete logarithms. The run time of \(\text{zn_log}\) primarily depends on the bitlength of the totient’s greatest prime factor.

See also \(\text{zn_primroot}, \text{zn_order}, \text{ifactors}, \text{totient}\).

Examples:

\(\text{zn_log}(a, g, n)\) solves the congruence \(g^x = a \mod n\).

\(\%i1\) \(n\) : 22$
\(\%i2\) \(g\) : \(\text{zn_primroot}(n)\);
\(\%o2\) 7
\(\%i3\) \(\text{ord}_7 : \text{zn_order}(7, n)\);
\(\%o3\) 10
\(\%i4\) \(\text{powers}_7 : \text{makelist}(\text{power_mod}(g, x, n), x, 0, \text{ord}_7 - 1)\);
\(\%o4\) \([1, 7, 5, 13, 3, 21, 15, 17, 9, 19]\)
\(\%i5\) \(\text{zn_log}(21, g, n)\);
\(\%o5\) 5
\(\%i6\) \(\text{map}(\lambda([x], \text{zn_log}(x, g, n)), \text{powers}_7)\);
\(\%o6\) \([0, 1, 2, 3, 4, 5, 6, 7, 8, 9]\)

The optional fourth argument must be of the same form as the list returned by \(\text{ifactors(totient(n))}\). The run time primarily depends on the bitlength of the totient’s greatest prime factor.

\(\%i1\) \(p\) : \(2^{127} - 1\), \(\text{primep}(p)\);
\(\%o1\) true
\(\%i2\) \(\text{ifs} : \text{ifactors}(p - 1)\)$
\(\%i3\) \(g\) : \(\text{zn_primroot}(p, \text{ifs})\);
\(\%o3\) 43
\(\%i4\) \(a\) : \(\text{power_mod}(g, 1234567890, p)\)$
\(\%i5\) \(\text{zn_log}(a, g, p, \text{ifs})\);
\(\%o5\) 1234567890
\(\%i6\) \(\text{time}(\%o5)\);
\(\%o6\) \([1.204]\)
\(\%i7\) \(f_{\text{max}} : \text{last}(\text{ifs})\);
\(\%o7\) \([77158673929, 1]\)
\(\%i8\) \(\text{slength( printf(false, "\-b", f_{\text{max}}[1]) )}\);
\(\%o8\) 37
zn_mult_table

zn_mult_table(n)
zn_mult_table(n, gcd)

Without the optional argument gcd zn_mult_table(n) shows a multiplication table of all elements in (Z/nZ)* which are all elements coprime to n.

The optional second argument gcd allows to select a specific subset of (Z/nZ). If gcd is an integer, a multiplication table of all residues x with gcd(x,n) = gcd are returned. Additionally row and column headings are added for better readability. If necessary, these can be easily removed by submatrix(1, table, 1).

If gcd is set to all, the table is printed for all non-zero elements in (Z/nZ).

The second example shows an alternative way to create a multiplication table for subgroups.

See also zn_add_table, zn_power_table.

Examples:
The default table shows all elements in (Z/nZ)* and allows to demonstrate and study basic properties of modular multiplication groups. E.g. the principal diagonal contains all quadratic residues, each row and column contains every element, the tables are symmetric, etc..

If gcd is set to all, the table is printed for all non-zero elements in (Z/nZ).

(%i1) zn_mult_table(8);

(%o1) 

(%i2) zn_mult_table(8, all);

(%o2)

If gcd is an integer, row and column headings are added for better readability.
If the subset chosen by gcd is a group there is another way to create a multiplication table. An isomorphic mapping from a group with 1 as identity builds a table which is easy to read. The mapping is accomplished via CRT.
In the second version of T36_4 the identity, here 28, is placed in the top left corner, just like in table T9.

(%i1) T36_4: zn_mult_table(36,4);  

(%o1) [1 2 4 5 7 8 ]
     [ ]
     [2 4 8 1 5 7 ]
     [ ]
     [4 8 7 2 1 5 ]

(%i2) T9: zn_mult_table(36/4);  

(%o2) [28 20 4 32 16 8 ]
     [ ]
     [20 4 8 28 32 16 ]
     [ ]
     [4 8 16 20 28 32 ]

(%i3) T36_4: matrixmap(lambda([x], chinese([0,x],[4,9])), T9);  

(%o3) [28 20 4 32 16 8 ]
     [ ]
     [20 4 8 28 32 16 ]
     [ ]
     [4 8 16 20 28 32 ]

zn_nth_root  
zn_nth_root (x, n, m)
zn_nth_root (x, n, m, [[p1, e1], ..., [pk, ek]])
Returns a list with all n-th roots of x from the multiplication subgroup of \((\mathbb{Z}/m\mathbb{Z})\) which contains x, or false, if x is no n-th power modulo m or not contained in any multiplication subgroup of \((\mathbb{Z}/m\mathbb{Z})\).
$x$ is an element of a multiplication subgroup modulo $m$, if the greatest common divisor $g = \text{gcd}(x,m)$ is coprime to $m/g$.

\textbf{zn\_nth\_root} is based on an algorithm by Adleman, Manders and Miller and on theorems about modulo multiplication groups by Daniel Shanks.

The algorithm needs a prime factorization of the modulus $m$. So in case the factorization of $m$ is known, the list of factors can be passed as the fourth argument. This optional argument must be of the same form as the list returned by \texttt{ifactors(m)} using the default option \texttt{factors\_only: false}.

Examples:

A power table of the multiplication group modulo 14 followed by a list of lists containing all $n$-th roots of 1 with $n$ from 1 to 6.

\begin{verbatim}
(%i1) zn_power_table(14);
[ 1 1 1 1 1 1 ]
[             ]
[ 3 9 13 11 5 1 ]
[             ]
[ 5 11 13 9 3 1 ]
(%o1)
[             ]
[ 9 11 1 9 11 1 ]
[             ]
[ 11 9 1 11 9 1 ]
[             ]
[ 13 1 13 1 13 1 ]

(%i2) makelist(zn_nth_root(1,n,14), n,1,6);
(%o2) [[1], [1, 13], [1, 9, 11], [1, 13], [1], [1, 3, 5, 9, 11, 13]]
\end{verbatim}

In the following example $x$ is not coprime to $m$, but is a member of a multiplication subgroup of $(\mathbb{Z}/m\mathbb{Z})$ and any $n$-th root is a member of the same subgroup. The residue class 3 is no member of any multiplication subgroup of $(\mathbb{Z}/63\mathbb{Z})$ and is therefore not returned as a third root of 27.

Here \texttt{zn\_power\_table} shows all residues $x$ in $(\mathbb{Z}/63\mathbb{Z})$ with $\text{gcd}(x,63) = 9$. This subgroup is isomorphic to $(\mathbb{Z}/7\mathbb{Z})^*$ and its identity 36 is computed via CRT.

\begin{verbatim}
(%i1) m: 7*9$
(%i2) zn_power_table(m,9);
[ 9 18 36 9 18 36 ]
[             ]
[ 18 9 36 18 9 36 ]
[             ]
[ 27 36 27 36 27 36 ]
(%o2)
[             ]
[ 36 36 36 36 36 36 ]
[             ]
[ 45 9 27 18 54 36 ]
[             ]
[ 54 18 27 9 45 36 ]
\end{verbatim}
In the following RSA-like example, where the modulus \( N \) is squarefree, i.e. it splits into exclusively first power factors, every \( x \) from 0 to \( N-1 \) is contained in a multiplication subgroup.

The process of decryption needs the \( e \)-th root. \( e \) is coprime to \( \text{totient}(N) \) and therefore the \( e \)-th root is unique. In this case \texttt{zn_nth_root} effectively performs CRT-RSA. (Please note that \texttt{flatten} removes braces but no solutions.)

\[
\begin{align*}
(\%i1) & \quad [p, q, e] : [5, 7, 17] \quad N : p \cdot q \\
(\%i3) & \quad \text{xs} : \text{makelist}(x, x, 0, N-1) \\
(\%i4) & \quad \text{ys} : \text{map}(\lambda[x], \text{power_mod}(x, e, N)), \text{xs}) \\
(\%i5) & \quad \text{zs} : \text{flatten}(\text{map}(\lambda[y], \text{zn_nth_root}(y, e, N)), \text{ys}) \\
(\%i6) & \quad \text{is}(\text{zs} = \text{xs}; \\
(\%o6) & \quad \text{true}
\end{align*}
\]

In the following example the factorization of the modulus is known and passed as the fourth argument.

\[
\begin{align*}
(\%i1) & \quad p : 2^{107} - 1 \quad q : 2^{127} - 1 \quad N : p \cdot q \\
(\%i4) & \quad \text{ibase} : \text{obase} : 16 \\
(\%i5) & \quad \text{msg} : 11223344556677889900aabbccddeeff \\
(\%i6) & \quad \text{enc} : \text{power_mod}(\text{msg}, 10001, N); \\
(\%o6) & \quad \text{1a8db7892a589bd2be25dd507a425001fe9c82161abc673241c8b388} \\
(\%i7) & \quad \text{zn_nth_root}(\text{enc}, 10001, N, [[p, 1], [q, 1]]) \\
(\%o7) & \quad [11223344556677889900aabbccddeeff]
\end{align*}
\]

\texttt{zn_order}

\[
\begin{align*}
\text{zn_order} & \quad (x, n) \\
\text{zn_order} & \quad (x, n, [[p1, e1], \ldots, [pk, ek]])
\end{align*}
\]

Returns the order of \( x \) if it is a unit of the finite group \((\mathbb{Z}/n\mathbb{Z})^*\) or returns \texttt{false}. \( x \) is a unit modulo \( n \) if it is coprime to \( n \).

The applied algorithm needs a prime factorization of \texttt{totient(n)}. This factorization might be time consuming in some cases and it can be useful to factor first and then to pass the list of factors to \texttt{zn_log} as the third argument. The list must be of the same form as the list returned by \texttt{ifactors(totient(n))} using the default option \texttt{factors_only : false}.

See also \texttt{zn_primroot}, \texttt{ifactors}, \texttt{totient}.

Examples:
zn_order computes the order of the unit x in (Z/nZ)*.

(%i1) n : 22$
(%i2) g : zn_primroot(n);
(%o2) 7$
(%i3) units_22 : sublist(makelist(i,i,1,21), lambda([x], gcd(x, n) = 1));
(%o3) [1, 3, 5, 7, 9, 13, 15, 17, 19, 21]
(%i4) (ord_7 : zn_order(7, n)) = totient(n);
(%o4) 10 = 10
(%i5) powers_7 : makelist(power_mod(g,i,n), i,0,ord_7 - 1);
(%o5) [1, 7, 5, 13, 3, 21, 15, 17, 9, 19]
(%i6) map(lambda([x], zn_order(x, n)), powers_7);
(%o6) [1, 10, 5, 10, 5, 2, 5, 10, 5, 10]
(%i7) map(lambda([x], ord_7/gcd(x, ord_7)), makelist(i,i,0,ord_7 - 1));
(%o7) [1, 10, 5, 10, 5, 2, 5, 10, 5, 10]
(%i8) totient(totient(n));
(%o8) 4

The optional third argument must be of the same form as the list returned by ifactors(totient(n)).

(%i1) (p : 2^142 + 217, primep(p));
(%o1) true
(%i2) ifs : ifactors( totient(p) )$
(%i3) g : zn_primroot(p, ifs);
(%o3) 3
(%i4) is( (ord_3 : zn_order(g, p, ifs)) = totient(p) );
(%o4) true
(%i5) map(lambda([x], ord_3/zn_order(x, p, ifs)), makelist(i,i,2,15));
(%o5) [22, 1, 44, 10, 5, 2, 22, 2, 8, 2, 1, 1, 20, 1]

zn_power_table

zn_power_table (n)
zn_power_table (n, gcd)
zn_power_table (n, gcd, max_exp)

Without any optional argument zn_power_table(n) shows a power table of all elements in (Z/nZ)* which are all residue classes coprime to n. The exponent loops from 1 to the greatest characteristic factor of totient(n) (also known as Carmichael function or Carmichael lambda) and the table ends with a column of ones on the right side.

The optional second argument gcd allows to select powers of a specific subset of (Z/nZ). If gcd is an integer, powers of all residue classes x with gcd(x,n) = gcd are returned, i.e. the default value for gcd is 1. If gcd is set to all, the table contains powers of all elements in (Z/nZ).

If the optional third argument max_exp is given, the exponent loops from 1 to max_exp.

See also zn_add_table, zn_mult_table.

Examples:
The default which is \( \gcd = 1 \) allows to demonstrate and study basic theorems of e.g. Fermat and Euler.

The argument \( \gcd \) allows to select subsets of \( \mathbb{Z}/n\mathbb{Z} \) and to study multiplication subgroups and isomorphisms. E.g. the groups \( \mathbb{G}_{10} \) and \( \mathbb{G}_{10,2} \) are under multiplication both isomorphic to \( \mathbb{G}_5 \). 1 is the identity in \( \mathbb{G}_5 \). So are 1 resp. 6 the identities in \( \mathbb{G}_{10} \) resp. \( \mathbb{G}_{10,2} \). There are corresponding mappings for primitive roots, n-th roots, etc..

\[
\begin{align*}
\text{(%i1) zn_power_table(10);} & \quad [1 
 & 1 
 & 1 
 & 1 ] \\
& \quad [ ] \\
& \quad [3 
 & 9 
 & 7 
 & 1 ] \\
& \quad [ ] \\
& \quad [7 
 & 9 
 & 3 
 & 1 ] \\
& \quad [ ] \\
& \quad [9 
 & 1 
 & 9 
 & 1 ] \\
\text{(%o1)} & \\
\text{(%i2) zn_power_table(10,2);} & \quad [2 
 & 4 
 & 8 
 & 6 ] \\
& \quad [ ] \\
& \quad [4 
 & 6 
 & 4 
 & 6 ] \\
& \quad [ ] \\
\text{(%o2)} & \\
\text{(%i3) zn_power_table(10,5);} & \quad [5 
 & 5 
 & 5 
 & 5 ] \\
\text{(%o3)} & \\
\text{(%i4) zn_power_table(10,10);} & \quad [0 
 & 0 
 & 0 
 & 0 ] \\
\text{(%o4)} & \\
\text{(%i5) G5: [1,2,3,4];} & \quad [1, 
 & 2, 
 & 3, 
 & 4] \\
\text{(%o5)} & \\
\text{(%i6) G10_2: map(lambda([x], chinese([0,x],[2,5])), G5);} & \quad [6, 
 & 2, 
 & 8, 
 & 4] \\
\text{(%o6)} & \\
\text{(%i7) G10: map(lambda([x], power_mod(3, zn_log(x,2,5), 10)), G5);} & \quad [1, 
 & 3, 
 & 7, 
 & 9] \\
\text{(%o7)} & \\
\end{align*}
\]

If \( \gcd \) is set to all, the table contains powers of all elements in \( \mathbb{Z}/n\mathbb{Z} \).

The third argument \( \max_exp \) allows to set the highest exponent. The following table shows a very small example of RSA.

\[
\begin{align*}
\text{(%i1) N:2*5$ \phi:totient(N)$ e:7$ d:inv_mod(e,\phi)$} \\
\text{(%i5) zn_power_table(N, all, e*d);} & \quad [0 
 & 0 
 & 0 
 & 0 
 & 0 
 & 0 
 & 0 
 & 0 
 & 0 
 & 0 
 & 0 
 & 0 
 & 0 
 & 0 
 & 0 
 & 0 
 & 0 
 & 0 
 & 0 
 & 0 
 & 0 
 & 0 ] \\
& \quad [ ] \\
& \quad [1 
 & 1 
 & 1 
 & 1 
 & 1 
 & 1 
 & 1 
 & 1 
 & 1 
 & 1 
 & 1 
 & 1 ] \\
& \quad [ ] \\
& \quad [2 
 & 4 
 & 8 
 & 6 
 & 2 
 & 4 
 & 8 
 & 6 
 & 2 
 & 4 
 & 8 
 & 6 
 & 2 ] \\
& \quad [ ] \\
& \quad [3 
 & 9 
 & 7 
 & 1 
 & 3 
 & 9 
 & 7 
 & 1 
 & 3 
 & 9 
 & 7 
 & 1 
 & 3 ] \\
\text{(%o5)} & \\
\end{align*}
\]
zn_primroot

zn_primroot (n)

zn_primroot (n, [[p1, e1], . . ., [pk, ek]])

If the multiplicative group \((\mathbb{Z}/n\mathbb{Z})^*\) is cyclic, \(zn\_primroot\) computes the smallest primitive root modulo \(n\). \((\mathbb{Z}/n\mathbb{Z})^*\) is cyclic if \(n\) is equal to \(2, 4, p^k\) or \(2p^k\), where \(p\) is prime and greater than \(2\) and \(k\) is a natural number. \(zn\_primroot\) performs an according pretest if the option variable \(zn\_primroot\_pretest\) (default: false) is set to true. In any case the computation is limited by the upper bound \(zn\_primroot\_limit\).

If \((\mathbb{Z}/n\mathbb{Z})^*\) is not cyclic or if there is no primitive root up to \(zn\_primroot\_limit\), \(zn\_primroot\) returns false.

The applied algorithm needs a prime factorization of \(totient(n)\). This factorization might be time consuming in some cases and it can be useful to factor first and then to pass the list of factors to \(zn\_log\) as an additional argument. The list must be of the same form as the list returned by \(ifactors(totient(n))\) using the default option \(factors\_only:\) false.

See also \(zn\_primroot\_p, zn\_order, ifactors, totient\).

Examples:

\(zn\_primroot\) computes the smallest primitive root modulo \(n\) or returns false.

\(\%i1\) \(n\) : 14$
\(\%i2\) \(g\) : \(zn\_primroot(n)\);
\(\%o2\) 3
\(\%i3\) \(zn\_order(g, n) = totient(n)\);
\(\%o3\) 6 = 6
\(\%i4\) \(n\) : 15$
\(\%i5\) \(zn\_primroot(n)\);
\(\%o5\) false

The optional second argument must be of the same form as the list returned by \(ifactors(totient(n))\).

\(\%i1\) \(p\) : \(2^142 + 217\), \(primep(p)\);
\(\%o1\) true
\(\%i2\) \(ifs\) : \(ifactors(totient(p))\) $
\(\%i3\) \(g\) : \(zn\_primroot(p, ifs)\);
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(%o3) 3
(%i4) [time(%o2), time(%o3)];
(%o4) [15.556972, 0.004]
(%i5) is(zn_order(g, p, ifs) = p - 1);
(%o5) true
(%i6) n : 2^142 + 216$
(%i7) ifs : ifactors(totient(n))$
(%i8) zn_primroot(n, ifs),
zn_primroot_limit : 200, zn_primroot_verbose : true;
'zn_primroot' stopped at zn_primroot_limit = 200
(%o8) false

zn_primroot_limit
[Option variable] Default value: 1000
If zn_primroot cannot find a primitive root, it stops at this upper bound. If the option variable zn_primroot_verbose (default: false) is set to true, a message will be printed when zn_primroot_limit is reached.

zn_primroot_p
[Function]
zn_primroot_p(x, n)
z_primroot_p(x, n, [p1, e1], ..., [pk, ek])
Checks whether x is a primitive root in the multiplicative group (Z/nZ)*.
The applied algorithm needs a prime factorization of totient(n). This factorization might be time consuming and in case zn_primroot_p will be consecutively applied to a list of candidates it can be useful to factor first and then to pass the list of factors to zn_log as a third argument. The list must be of the same form as the list returned by ifactors(totient(n)) using the default option factors_only: false.
See also zn_primroot, zn_order, ifactors, totient.
Examples:
zn_primroot_p as a predicate function.
(%i1) n : 14$
(%i2) units_14 : sublist(makelist(i, i, 1, 13), lambda([i], gcd(i, n) = 1));
(%o2) [1, 3, 5, 9, 11, 13]
(%i3) zn_primroot_p(13, n);
(%o3) false
(%i4) sublist(units_14, lambda([x], zn_primroot_p(x, n)));
(%o4) [3, 5]
(%i5) map(lambda([x], zn_order(x, n)), units_14);
(%o5) [1, 6, 6, 3, 3, 2]
The optional third argument must be of the same form as the list returned by ifactors(totient(n)).
(%i1) (p : 2^142 + 217, primep(p));
(%o1) true
(%i2) ifs : ifactors(totient(p));
(%i3) sublist(makelist(i, i, 1, 50), lambda([x], zn_primroot_p(x, p, ifs)));
(%o3) [3, 12, 13, 15, 21, 24, 26, 27, 29, 33, 38, 42, 48]
(%i4) [time(%o2), time(%o3)];
(%o4) [[7.748484], [0.036002]]

zn_primroot_pretest [Option variable]
  Default value: false
  The multiplicative group \((\mathbb{Z}/n\mathbb{Z})^*\) is cyclic if \(n\) is equal to 2, 4, \(p^k\) or \(2p^k\), where \(p\) is prime and greater than 2 and \(k\) is a natural number.
  zn_primroot_pretest controls whether \texttt{zn_primroot} will check if one of these cases occur before it computes the smallest primitive root. Only if \texttt{zn_primroot_pretest} is set to \texttt{true} this pretest will be performed.

zn_primroot_verbose [Option variable]
  Default value: false
  Controls whether \texttt{zn_primroot} prints a message when reaching \texttt{zn_primroot_limit}.
30 Symmetries

30.1 Introduction to Symmetries

sym is a package for working with symmetric groups of polynomials.

It was written for Macsyma-Symbolics by Annick Valibouze\(^1\). The algorithms are described in the following papers:


30.2 Functions and Variables for Symmetries

30.2.1 Changing bases

**comp2pui** \((n, L)\) [Function]

implements passing from the complete symmetric functions given in the list \(L\) to the elementary symmetric functions from 0 to \(n\). If the list \(L\) contains fewer than \(n+1\) elements, it will be completed with formal values of the type \(h1, h2, \ldots\). If the first element of the list \(L\) exists, it specifies the size of the alphabet, otherwise the size is set to \(n\).

\[
\text{(%i1)} \quad \text{comp2pui} (3, [4, g]); \\
\text{(%o1)} \quad [4, g, 2 h2 - g, 3 h3 - g h2 + g (g - 2 h2)]
\]

**ele2pui** \((m, L)\) [Function]

goes from the elementary symmetric functions to the complete functions. Similar to \(\text{comp2ele}\) and \(\text{comp2pui}\).

Other functions for changing bases: \(\text{comp2ele}\).

**ele2comp** \((m, L)\) [Function]

Goes from the elementary symmetric functions to the compete functions. Similar to \(\text{comp2ele}\) and \(\text{comp2pui}\).

Other functions for changing bases: \(\text{comp2ele}\).

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\(^1\) [www-calfor.lip6.fr/~avb](http://www-calfor.lip6.fr/~avb)


\(^3\) [www-calfor.lip6.fr/~avb/DonneesTelechargeables/MesArticles/issac89ACMValibouze.pdf](http://www-calfor.lip6.fr/~avb/DonneesTelechargeables/MesArticles/issac89ACMValibouze.pdf)

elem (ele, sym, lvar)

decomposes the symmetric polynomial sym, in the variables contained in the list lvar, in terms of the elementary symmetric functions given in the list ele. If the first element of ele is given, it will be the size of the alphabet, otherwise the size will be the degree of the polynomial sym. If values are missing in the list ele, formal values of the type e1, e2, etc. will be added. The polynomial sym may be given in three different forms: contracted (elem should then be 1, its default value), partitioned (elem should be 3), or extended (i.e. the entire polynomial, and elem should then be 2). The function pui is used in the same way.

On an alphabet of size 3 with e1, the first elementary symmetric function, with value 7, the symmetric polynomial in 3 variables whose contracted form (which here depends on only two of its variables) is \(x^4 - 2xy\) decomposes as follows in elementary symmetric functions:

\[
(\%i1) \text{elem ([3, 7], x^4 - 2xy, [x, y]);} \\
(\%o1) 7 (e3 - 7 e2 + 7 (49 - e2)) + 21 e3 \\
+ (- 2 (49 - e2) - 2) e2
\]

\[
(\%i2) \text{ratsimp (%);} \\
(\%o2) 28 e3 + 2 e2 - 198 e2 + 2401
\]

Other functions for changing bases: comp2ele.

mon2schur (L)

The list L represents the Schur function \(S_L\): we have \(L = [i_1, i_2, \ldots, i_q]\), with \(i_1 \leq i_2 \leq \ldots \leq i_q\). The Schur function \(S_{i_1, i_2, \ldots, i_q}\) is the minor of the infinite matrix \(h_{i-j}\), \(i \geq 1, j \geq 1\), consisting of the \(q\) first rows and the columns \(i_1 + 1, i_2 + 2, \ldots, i_q + q\).

This Schur function can be written in terms of monomials by using treinat and kostka. The form returned is a symmetric polynomial in a contracted representation in the variables \(x_1, x_2, \ldots\)

\[
(\%i1) \text{mon2schur ([1, 1, 1]);} \\
(\%o1) x1 x2 x3 \\
(\%i2) \text{mon2schur ([3]);} \\
(\%o2) x1 x2 x3 + x1 x2 + x1 \\
(\%i3) \text{mon2schur ([1, 2]);} \\
(\%o3) 2 x1 x2 x3 + x1 x2
\]

which means that for 3 variables this gives:

\[
2 x1 x2 x3 + x1^2 x2 + x2^2 x1 + x1^2 x3 + x3^2 x1 + x2^2 x3 + x3^2 x2
\]

Other functions for changing bases: comp2ele.

multi_elem (l_elem, multi_pc, l_var)

decomposes a multi-symmetric polynomial in the multi-contracted form multi_pc in the groups of variables contained in the list of lists l_var in terms of the elementary symmetric functions contained in l_elem.
(\%i1) multi_elem ([[2, e1, e2], [2, f1, f2]], a*x + a^2 + x^3, [[x, y], [a, b]]);
  3
(\%o1) - 2 f2 + f1 (f1 + e1) - 3 e1 e2 + e1
(\%i2) ratsimp (%);
  2 3
(\%o2) - 2 f2 + f1 + e1 f1 - 3 e1 e2 + e1

Other functions for changing bases: \texttt{comp2ele}, \texttt{comp2pui}.

\texttt{multi\_pui} \hspace{1cm} \textbf{[Function]}

is to the function \texttt{pui} what the function \texttt{multi\_elem} is to the function \texttt{elem}.

(\%i1) multi\_pui ([[2, p1, p2], [2, t1, t2]], a*x + a^2 + x^3, [[x, y], [a, b]]);
  3
(\%o1) t2 + p1 t1 + -------- - ---
  2 2

\texttt{pui} \hspace{0.5cm} \textbf{[Function]}

decomposes the symmetric polynomial \texttt{sym}, in the variables in the list \texttt{lvar}, in terms of the power functions in the list \texttt{L}. If the first element of \texttt{L} is given, it will be the size of the alphabet, otherwise the size will be the degree of the polynomial \texttt{sym}. If values are missing in the list \texttt{L}, formal values of the type \texttt{p1}, \texttt{p2}, etc. will be added. The polynomial \texttt{sym} may be given in three different forms: contracted (\texttt{elem} should then be 1, its default value), partitioned (\texttt{elem} should be 3), or extended (i.e. the entire polynomial, and \texttt{elem} should then be 2). The function \texttt{pui} is used in the same way.

(\%i1) pui;
(\%o1) 1
(\%i2) pui ([3, a, b], u*x*y*z, [x, y, z]);
  2
(\%o2) a (a - b) u (a b - p3) u
  6 3
(\%i3) ratsimp (%);
(\%o3)

Other functions for changing bases: \texttt{comp2ele}.

\texttt{pui2comp} \hspace{0.5cm} \textbf{[Function]}

renders the list of the first \texttt{n} complete functions (with the length first) in terms of the power functions given in the list \texttt{lpu}. If the list \texttt{lpu} is empty, the cardinal is \texttt{n}, otherwise it is its first element (as in \texttt{comp2ele} and \texttt{comp2pui}).
(%i1) pui2comp (2, []);
       2
2
p2 + p1
(%o1) [2, p1, --------]
       2

(%i2) pui2comp (3, [2, a1]);
2
  a1 (p2 + a1 )
2
  p3 + ------------------- + a1 p2
  p2 + a1
(%o2) [2, a1, -------, ---------------------------------]
       2
       3

(%i3) ratsimp (%);
2
  p2 + a1
2
  3
  p3 + 3 a1 p2 + a1
(%o3) [2, a1, -------, --------------------]
       2
       6

Other functions for changing bases: comp2ele.

pui2ele (n, lpui) [Function]
effects the passage from power functions to the elementary symmetric functions. If
the flag pui2ele is girard, it will return the list of elementary symmetric functions
from 1 to n, and if the flag is close, it will return the n-th elementary symmetric
function.

Other functions for changing bases: comp2ele.

puireduc (n, lpui) [Function]
lpui is a list whose first element is an integer m. pui2ele reduces gives the first n power
functions in terms of the first m.

(%i1) pui2ele (3, [2]);
2
p1 (p1 - p2)
(%o1) [2, p1, p2, p1 p2 - -----------------]

(%i2) ratsimp (%);
3
3
p1 p2 - p1
(%o2) [2, p1, p2, -----------------]

2

schur2comp (P, L_var) [Function]
P is a polynomial in the variables of the list L_var. Each of these variables represents
a complete symmetric function. In L_var the i-th complete symmetric function is
represented by the concatenation of the letter h and the integer i: h_i. This function
expresses P in terms of Schur functions.

(%i1) schur2comp (h1*h2 - h3, [h1, h2, h3]);
(%o1)
  s
1, 2
30.2.2 Changing representations

**cont2part** \((pc, lvar)\)

returns the partitioned polynomial associated to the contracted form \(pc\) whose variables are in \(lvar\).

```lisp
(%i11) pc: 2*a^3*b*x^4*y + x^5;
   \end{verbatim}
```

\(3\quad 4\quad 5\)

```lisp
   \end{verbatim}
```

\(2\quad a\quad b\quad x\quad y\quad +\quad x\)

```lisp
(%i12) cont2part (pc, [x, y]);
```

\(3\)

```lisp
   \end{verbatim}
```

\([[1, 5, 0], [2\quad a\quad b, 4, 1]]\)

**contract** \((psym, lvar)\)

returns a contracted form (i.e. a monomial orbit under the action of the symmetric group) of the polynomial \(psym\) in the variables contained in the list \(lvar\). The function **explose** performs the inverse operation. The function **tcontract** tests the symmetry of the polynomial.

```lisp
(%i1) psym: explose (2*a^3*b*x^4*y, [x, y, z]);
   \end{verbatim}
```

\(3\quad 4\quad 3\quad 4\quad 3\quad 4\quad +\quad 2\quad a\quad b\quad x\quad y\quad +\quad 2\quad a\quad b\quad x\quad y\)

\(3\quad 4\quad 3\quad 4\)

```lisp
(%i2) contract (psym, [x, y, z]);
```

\(2\quad a\quad b\quad x\quad y\)

**explose** \((pc, lvar)\)

returns the symmetric polynomial associated with the contracted form \(pc\). The list \(lvar\) contains the variables.

```lisp
(%i1) explode (a*x + 1, [x, y, z]);
   \end{verbatim}
```

\(a\quad z\quad +\quad a\quad y\quad +\quad a\quad x\quad +\quad 1\)

```lisp
(%i2) contract (psym, [x, y, z]);
```

\(2\quad a\quad b\quad x\quad y\)

**part2cont** \((ppart, lvar)\)

goess from the partitioned form to the contracted form of a symmetric polynomial. The contracted form is rendered with the variables in \(lvar\).

```lisp
(%i1) part2cont ([[2*a^3*b, 4, 1]], [x, y]);
```

\(2\quad a\quad b\quad x\quad y\)

**partpol** \((psym, lvar)\)

\(psym\) is a symmetric polynomial in the variables of the list \(lvar\). This function returns its partitioned representation.

```lisp
(%i1) partpol (-a*(x + y) + 3*x*y, [x, y]);
```

\([[3, 1, 1], [-\quad a, 1, 0]]\)
tcontract (pol, lvar)  
[Function] 
tests if the polynomial pol is symmetric in the variables of the list lvar. If so, it returns a contracted representation like the function contract.

tpartpol (pol, lvar)  
[Function] 
tests if the polynomial pol is symmetric in the variables of the list lvar. If so, it returns its partitioned representation like the function partpol.

30.2.3 Groups and orbits

direct ([p_1, ..., p_n], y, f, [lvar_1, ..., lvar_n])  
[Function] 
calculates the direct image (see M. Giusti, D. Lazard et A. Valibouze, ISSAC 1988, Rome) associated to the function f, in the lists of variables lvar_1, ..., lvar_n, and in the polynomials p_1, ..., p_n in a variable y. The arity of the function f is important for the calculation. Thus, if the expression for f does not depend on some variable, it is useless to include this variable, and not including it will also considerably reduce the amount of computation.

(%i1) direct ([z^2 - e1* z + e2, z^2 - f1* z + f2], z, b*v + a*u, [[u, v], [a, b]]);

2
2 2 2 2
2 2
2
(%o1) y - e1 f1 y + (e1 - 4 e2) f2 + e2 f1

(%i2) ratsimp (%);

2 2 2
2
2 2
(%o2) y - e1 f1 y + (e1 - 4 e2) f2 + e2 f1
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(%i3) ratsimp (direct ([z^3-e1*z^2+e2*z-e3,z^2 - f1* z + f2],
   z, b*v + a*u, [[u, v], [a, b]]));

\[
\begin{align*}
6 & 5 & 2 & 2 & 2 & 4 \\
\text{(%o3)} & y & - 2 & e1 & f1 & y & + ((2 & e1 & - 6 & e2) & f2 & + (2 & e2 & + e1) & f1) & y \\
   & 3 & 3 & 3
\end{align*}
\]

\[
\begin{align*}
2 & 2 & 4 & 2 \\
+ ((9 & e3 & + 5 & e1 & e2 & - 2 & e1) & f1 & f2 & + (-2 & e3 & - 2 & e1 & e2) & f1) & y
\end{align*}
\]

\[
\begin{align*}
2 & 2 & 4 & 2 \\
+ ((9 & e2 & - 6 & e1 & e2 & + e1) & f2)
\end{align*}
\]

\[
\begin{align*}
2 & 2 & 2 & 2 & 4 \\
+ (-9 & e1 & e3 & - 6 & e2 & + 3 & e1 & e2) & f1 & f2 & + (2 & e1 & e3 & + e2) & f1)
\end{align*}
\]

\[
\begin{align*}
y & + (((9 & e1 & - 27 & e2) & e3 & + 3 & e1 & e2 & - e1 & e2) & f1 & f2 & + (15 & e2 & - 2 & e1) & e3 & - e1 & e2) & f1 & f2
\end{align*}
\]

\[
\begin{align*}
2 & 2 & 3 & 5 \\
+ (-27 & e3 & + (18 & e1 & e2 & - 4 & e1) & e3 & - 4 & e2 & + e1 & e2) & f2
\end{align*}
\]

\[
\begin{align*}
2 & 3 & 3 & 2 & 2 & 3 \\
+ (27 & e3 & + (e1 & - 9 & e1 & e2) & e3 & + e2) & f1 & f2
\end{align*}
\]

\[
\begin{align*}
2 & 4 & 2 & 6 \\
+ (e1 & e2 & e3 & - 9 & e3) & f1 & f2 & + e3 & f1
\end{align*}
\]

Finding the polynomial whose roots are the sums \(a+u\) where \(a\) is a root of \(z^2-e_1 z+e_2\) and \(u\) is a root of \(z^2-f_1 z+f_2\).

(%i1) ratsimp (direct ([z^2-e1* z + e2, z^2 - f1* z + f2],
   z, a + u, [[u], [a]]));

\[
\begin{align*}
4 & 3 & 2 \\
\text{(%o1)} & y & + (-2 & f1 & - 2 & e1) & y & + (2 & f2 & + f1 & + 3 & e1 & f1 & + 2 & e2 \\
   & 2 & 2 & 2 & 2 & 2 \\
+ e1) & y & + ((-2 & f1 & - 2 & e1) & f2 & - e1 & f1 & + (-2 & e2 & - e1) & f1
\end{align*}
\]

\[
\begin{align*}
2 & 2 & 2 \\
- 2 & e1 & e2) & y & + f2 & + (e1 & f1 & - 2 & e2 & + e1) & f2 & + e2 & f1 & + e1 & e2 & f1
\end{align*}
\]

\[
\begin{align*}
2 & + e2
\end{align*}
\]

direct accepts two flags: elementaires and puissances (default) which allow decomposing the symmetric polynomials appearing in the calculation into elementary symmetric functions, or power functions, respectively.
Functions of sym used in this function:
   multi_orbit (so orbit), pui_direct, multi_elem (so elem), multi_pui (so pui),
   pui2ele, ele2pui (if the flag direct is in puissances).

multi_orbit (P, [lvar_1, lvar_2, ..., lvar_p])  [Function]
   P is a polynomial in the set of variables contained in the lists lvar_1, lvar_2, ..., lvar_p.
   This function returns the orbit of the polynomial P under the action of the product
   of the symmetric groups of the sets of variables represented in these p lists.
   (%i1) multi_orbit (a*x + b*y, [[x, y], [a, b]]);
   (%o1) [b y + a x, a y + b x]
   (%i2) multi_orbit (x + y + 2*a, [[x, y], [a, b, c]]);
   (%o2) [y + x + 2 c, y + x + 2 b, y + x + 2 a]
   Also see: orbit for the action of a single symmetric group.

multsym (ppart_1, ppart_2, n)  [Function]
returns the product of the two symmetric polynomials in n variables by working only modulo
the action of the symmetric group of order n. The polynomials are in their partioned form.
Given the 2 symmetric polynomials in x, y: 3*(x + y) + 2*x*y and 5*(x^2 + y^2)
whose partitioned forms are [[3, 1], [2, 1, 1]] and [[5, 2]], their product will be
   (%i1) multsym ([[3, 1], [2, 1, 1]], [[5, 2]], 2);
   (%o1) [[10, 3, 1], [15, 3, 0], [15, 2, 1]]
that is 10*(x^3*y + y^3*x) + 15*(x^2*y + y^2*x) + 15*(x^3 + y^3).
Functions for changing the representations of a symmetric polynomial:
contract, cont2part, explose, part2cont, partpol, tcontract, tpartpol.

orbit (P, lvar)  [Function]
computes the orbit of the polynomial P in the variables in the list lvar under the
action of the symmetric group of the set of variables in the list lvar.
   (%i1) orbit (a*x + b*y, [x, y]);
   (%o1) [a y + b x, b y + a x]
   (%i2) orbit (2*x + x^2, [x, y]);
   (%o2) [y + 2 y, x + 2 x]
See also multi_orbit for the action of a product of symmetric groups on a polynomial.

pui_direct (orbite, [lvar_1, ..., lvar_n], [d_1, d_2, ..., d_n])  [Function]
Let f be a polynomial in n blocks of variables lvar_1, ..., lvar_n. Let c_i be the number
of variables in lvar_i, and SC be the product of n symmetric groups of degree c_1, ...
, c_n. This group acts naturally on f. The list orbite is the orbit, denoted SC(f),
of the function f under the action of SC. (This list may be obtained by the function
multi_orbit.) The di are integers s.t. 0 <= d_i <= c_i, 1 <= d_1, ..., c_n.
Let SD be the product of the symmetric groups S_{d_1} * S_{d_2} * ... * S_{d_n}. The function
pui_direct returns the first n power functions of SD(f) deduced from the power functions of SC(f), where n is the size of SD(f).
The result is in multi-contracted form w.r.t. \( SD \), i.e. only one element is kept per orbit, under the action of \( SD \).

\[
\begin{align*}
\text{(%i1)} & \quad l : [[x, y], [a, b]]; \\
\text{(%o1)} & \quad [[x, y], [a, b]] \\
\text{(%i2)} & \quad \text{pui_direct (multi_orbit (a*x + b*y, 1), l, [2, 2]);} \\
& \quad \quad 2 \quad 2 \\
\text{(%o2)} & \quad [a \ x, 4 \ a \ b \ x \ y + a \ x] \\
\text{(%i3)} & \quad \text{pui_direct (multi_orbit (a*x + b*y, 1), l, [3, 2]);} \\
& \quad \quad 2 \quad 2 \quad 2 \quad 2 \quad 3 \quad 3 \\
\text{(%o3)} & \quad [2 \ a \ x, 4 \ a \ b \ x \ y + 2 \ a \ x, 3 \ a \ b \ x \ y + 2 \ a \ x,} \\
& \quad \quad 2 \quad 2 \quad 2 \quad 3 \quad 3 \quad 3 \quad 4 \quad 4 \\
& \quad 12 \ a \ b \ x \ y + 4 \ a \ b \ x \ y + 2 \ a \ x, \\
& \quad \quad 3 \quad 2 \quad 3 \quad 2 \quad 4 \quad 4 \quad 5 \quad 5 \\
& \quad 10 \ a \ b \ x \ y + 5 \ a \ b \ x \ y + 2 \ a \ x, \\
& \quad \quad 3 \quad 3 \quad 3 \quad 3 \quad 4 \quad 2 \quad 4 \quad 2 \quad 5 \quad 5 \quad 6 \quad 6 \\
& \quad 40 \ a \ b \ x \ y + 15 \ a \ b \ x \ y + 6 \ a \ b \ x \ y + 2 \ a \ x] \\
\text{(%i4)} & \quad \text{pui_direct ([y + x + 2*c, y + x + 2*b, y + x + 2*a],} \\
& \quad \quad [[x, y], [a, b, c]], [2, 3]); \\
& \quad \quad 2 \quad 2 \\
\text{(%o4)} & \quad [3 \ x + 2 \ a, 6 \ x \ y + 3 \ x + 4 \ a \ x + 4 \ a,} \\
& \quad \quad 2 \quad 2 \quad 2 \quad 3 \\
& \quad 9 \ x \ y + 12 \ a \ x \ y + 3 \ x + 6 \ a \ x + 12 \ a \ x + 8 \ a]
\end{align*}
\]

### 30.2.4 Partitions

**kostka** \((\text{part}_1, \text{part}_2)\)  
written by P. Esperet, calculates the Kostka number of the partition \(\text{part}_1\) and \(\text{part}_2\).

\[
\begin{align*}
\text{(%i1)} & \quad \text{kostka ([3, 3, 3], [2, 2, 1, 1, 1]);} \\
\text{(%o1)} & \quad 6
\end{align*}
\]

**lgtreillis** \((n, m)\)  
returns the list of partitions of weight \(n\) and length \(m\).

\[
\begin{align*}
\text{(%i1)} & \quad \text{lgtreillis (4, 2);} \\
\text{(%o1)} & \quad [[[3, 1], [2, 2]]]
\end{align*}
\]

Also see: ltreillis, treillis and treinat.

**ltreillis** \((n, m)\)  
returns the list of partitions of weight \(n\) and length less than or equal to \(m\).

\[
\begin{align*}
\text{(%i1)} & \quad \text{ltreillis (4, 2);} \\
\text{(%o1)} & \quad [[[4, 0], [3, 1], [2, 2]]]
\end{align*}
\]

Also see: lgtreillis, treillis and treinat.
treillis \((n)\)  \hspace{0.5cm} \text{[Function]}  
returns all partitions of weight \(n\). 

\begin{verbatim}
(%i1) treillis (4)
(%o1) [[4], [3, 1], [2, 2], [2, 1, 1], [1, 1, 1, 1]]
\end{verbatim}

See also: lgtreillis, ltreillis and treinat.

treinat \((\text{part})\)  \hspace{0.5cm} \text{[Function]}  
retruns the list of partitions inferior to the partition \(\text{part}\) w.r.t. the natural order.

\begin{verbatim}
(%i1) treinat ([5])
(%o1) [[5]]
(%i2) treinat ([1, 1, 1, 1, 1])
(%o2) [[5], [4, 1], [3, 2], [3, 1, 1], [2, 2, 1], [2, 1, 1, 1], [1, 1, 1, 1, 1]]
(%i3) treinat ([3, 2])
(%o3) [[5], [4, 1], [3, 2]]
\end{verbatim}

See also: lgtreillis, ltreillis and treillis.

### 30.2.5 Polynomials and their roots

ele2polynome \((L, z)\)  \hspace{0.5cm} \text{[Function]}  
returns the polynomial in \(z\) s.t. the elementary symmetric functions of its roots are in the list \(L = [n, \ e_1, \ldots, \ e_n]\), where \(n\) is the degree of the polynomial and \(e_i\) the \(i\)-th elementary symmetric function.

\begin{verbatim}
(%i1) ele2polynome ([2, e1, e2], z)
(%o1) 2
(%i2) polynome2ele (x^7 - 14*x^5 + 56*x^3 - 56*x + 22, x)
(%o2) [7, 0, -14, 0, 56, 0, -56, -22]
(%i3) ele2polynome ([7, 0, -14, 0, 56, 0, -56, -22], x)
(%o3) 7 5 3
\end{verbatim}

The inverse: polynome2ele \((P, z)\).

Also see: polynome2ele, pui2polynome.

polynome2ele \((P, x)\)  \hspace{0.5cm} \text{[Function]}  
gives the list \(l = [n, \ e_1, \ldots, \ e_n]\) where \(n\) is the degree of the polynomial \(P\) in the variable \(x\) and \(e_i\) is the \(i\)-th elementary symmetric function of the roots of \(P\).

\begin{verbatim}
(%i1) polynome2ele (x^7 - 14*x^5 + 56*x^3 - 56*x + 22, x)
(%o1) [7, 0, -14, 0, 56, 0, -56, -22]
(%i2) ele2polynome ([7, 0, -14, 0, 56, 0, -56, -22], x)
(%o2) 7 5 3
\end{verbatim}

The inverse: ele2polynome \((1, x)\)
prodrac \((L, k)\)  
\(L\) is a list containing the elementary symmetric functions on a set \(A\). prodrac returns the polynomial whose roots are the \(k\) by \(k\) products of the elements of \(A\).  
Also see somrac.

pui2polynome \((x, \text{lpuui})\)  
calculates the polynomial in \(x\) whose power functions of the roots are given in the list \(\text{lpuui}\).

\[
\begin{align*}
%i1 &\text{ pui;} \\
%o1 &1 \\
%i2 &\text{kill(labels);} \\
%o0 &\text{done} \\
%i1 &\text{polynome2ele } (x^3 - 4x^2 + 5x - 1, x); \\
%o1 &[3, 4, 5, 1] \\
%i2 &\text{ele2pui } (3, %); \\
%o2 &[3, 4, 6, 7] \\
%i3 &\text{pui2polynome } (x, %); \\
%o3 &3 2 \\
&x - 4 x + 5 x - 1
\end{align*}
\]

See also: polynome2ele, ele2polynome.

somrac \((L, k)\)  
The list \(L\) contains elementary symmetric functions of a polynomial \(P\). The function computes the polynomial whose roots are the \(k\) by \(k\) distinct sums of the roots of \(P\).  
Also see prodrac.

30.2.6 Resolvents

resolvante \((P, x, f, [x_{_1}, \ldots, x_{_d}]\)  
calculates the resolvent of the polynomial \(P\) in \(x\) of degree \(n \geq d\) by the function \(f\) expressed in the variables \(x_{_1}, \ldots, x_{_d}\). For efficiency of computation it is important to not include in the list \([x_{_1}, \ldots, x_{_d}]\) variables which do not appear in the transformation function \(f\).  
To increase the efficiency of the computation one may set flags in resolvante so as to use appropriate algorithms:

If the function \(f\) is unitary:
- A polynomial in a single variable,
- linear,
- alternating,
- a sum,
- symmetric,
- a product,
- the function of the Cayley resolvent (usable up to degree 5)

\[
(x_{1}x_{2} + x_{2}x_{3} + x_{3}x_{4} + x_{4}x_{5} + x_{5}x_{1} - \\
(x_{1}x_{3} + x_{3}x_{5} + x_{5}x_{2} + x_{2}x_{4} + x_{4}x_{1}))^2
\]

general,
the flag of \texttt{resolvante} may be, respectively:

- unitaire,
- lineaire,
- alternee,
- somme,
- produit,
- cayley,
- generale.

\begin{verbatim}
(%i1) resolvante: unitaire$
(%i2) resolvante (x^7 - 14*x^5 + 56*x^3 - 56*x + 22, x, x^3 - 1, [x]);
" resolvante unitaire " [7, 0, 28, 0, 168, 0, 1120, - 154, 7840, - 2772, 56448, - 33880, 413952, - 352352, 3076668, - 3363360, 23114112, - 30494464, 175230832, - 267412992, 1338886528, - 2292126760] 3 6 3 9 6 3 [x - 1, x - 2 x + 1, x - 3 x + 3 x - 1, 12 9 6 3 15 12 9 6 3 x - 4 x + 6 x - 4 x + 1, x - 5 x + 10 x - 10 x + 5 x 18 15 12 9 6 3 - 1, x - 6 x + 15 x - 20 x + 15 x - 6 x + 1, 21 18 15 12 9 6 3 x - 7 x + 21 x - 35 x + 35 x - 21 x + 7 x - 1] [- 7, 1127, - 6139, 431767, - 5472047, 201692519, - 3603982011] 7 6 5 4 3 2 (\%o2) y + 7 y - 539 y - 1841 y + 51443 y + 376999 y + 125253 4
(%i3) resolvante: lineaire$
(%i4) resolvante (x^4 - 1, x, x1 + 2*x2 + 3*x3, [x1, x2, x3]);
" resolvante lineaire "
24 20 16 12 8 (\%o4) y + 80 y + 7520 y + 1107200 y + 49475840 y + 344489984 y + 655360000
(%i5) resolvante: general$
\end{verbatim}
(%i6) resolvante (x^4 - 1, x, x1 + 2*x2 + 3*x3, [x1, x2, x3]);
" resolvante generale "
\begin{align*}
&\begin{array}{cccccc}
24 & 20 & 16 & 12 & 8
\end{array} \\
&y + 80y + 7520y + 1107200y + 49475840y \\
&\quad + 344489984y + 655360000
\end{align*}

(%i7) resolvante (x^4 - 1, x, x1 + 2*x2 + 3*x3, [x1, x2, x3, x4]);
" resolvante generale "
\begin{align*}
&\begin{array}{cccccc}
24 & 20 & 16 & 12 & 8
\end{array} \\
&y + 80y + 7520y + 1107200y + 49475840y \\
&\quad + 344489984y + 655360000
\end{align*}

(%i8) direct ([x^4 - 1], x, x1 + 2*x2 + 3*x3, [[x1, x2, x3]]);
" resolvante lineaire "
\begin{align*}
&y - 1
\end{align*}

(%i9) resolvante: lineaire$

(%i10) resolvante (x^4 - 1, x, x1 + x2 + x3, [x1, x2, x3]);
" resolvante symetrique "
\begin{align*}
&y - 1
\end{align*}

(%i11) resolvante: symetrique$

(%i12) resolvante (x^4 - 1, x, x1 + x2 + x3, [x1, x2, x3]);
" resolvante symetrique "
\begin{align*}
&y - 1
\end{align*}

(%i13) resolvante (x^4 + x + 1, x, x1 - x2, [x1, x2]);
" resolvante alternee "
\begin{align*}
&y - 4y - 1
\end{align*}

(%i14) resolvante: alternee$

(%i15) resolvante (x^4 + x + 1, x, x1 - x2, [x1, x2]);
" resolvante alternee "
\begin{align*}
&y + 8y + 26y - 112y + 216y + 229
\end{align*}

(%i16) resolvante: produit$
(%i17) resolvante (x^7 - 7*x + 3, x, x1*x2*x3, [x1, x2, x3]);

" resolvante produit "

(%o17) y - 7 y - 1029 y + 135 y + 7203 y - 756 y

+ 1323 y + 352947 y - 46305 y - 2463339 y + 324135 y

- 30618 y - 453789 y - 40246444 y + 282225202 y

- 44274492 y + 155098503 y + 12252303 y + 2893401 y

- 171532242 y + 6751269 y + 2657205 y - 94517766 y

- 3720087 y + 26040609 y + 14348907

(%i18) resolvante: symetrique$

(%i19) resolvante (x^7 - 7*x + 3, x, x1*x2*x3, [x1, x2, x3]);

" resolvante symetrique "

(%o19) y - 7 y - 1029 y + 135 y + 7203 y - 756 y

+ 1323 y + 352947 y - 46305 y - 2463339 y + 324135 y

- 30618 y - 453789 y - 40246444 y + 282225202 y

- 44274492 y + 155098503 y + 12252303 y + 2893401 y

- 171532242 y + 6751269 y + 2657205 y - 94517766 y

- 3720087 y + 26040609 y + 14348907

(%i20) resolvante: cayley$
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(%i21) resolvante (x^5 - 4*x^2 + x + 1, x, a, []);

" resolvante de Cayley "

6 5 4 3 2
(%o21) x - 40 x + 4080 x - 92928 x + 3772160 x + 37880832 x + 93392896

For the Cayley resolvent, the 2 last arguments are neutral and the input polynomial must necessarily be of degree 5.

See also:
resolvante_bipartite, resolvante_produit_sym,
resolvante_unitaire, resolvante_alternee1, resolvante_klein,
resolvante_klein3, resolvante_vierer, resolvante_diedrale.

resolvante_alternee1 (P, x) [Function]
calculates the transformation $P(x)$ of degree $n$ by the function $\prod_{1 \leq i < j \leq n-1} (x_i - x_j)$.

See also:
resolvante_produit_sym, resolvante_unitaire,
resolvante, resolvante_klein, resolvante_klein3,
resolvante_vierer, resolvante_diedrale, resolvante_bipartite.

resolvante_bipartite (P, x) [Function]
calculates the transformation of $P(x)$ of even degree $n$ by the function $x_1 x_2 \cdots x_{n/2} + x_{n/2+1} \cdots x_n$.

(%i1) resolvante_bipartite (x^6 + 108, x);

10 8 6
(%o1) y - 972 y + 314928 y - 34012224 y

See also:
resolvante_produit_sym, resolvante_unitaire,
resolvante, resolvante_klein, resolvante_klein3,
resolvante_vierer, resolvante_diedrale, resolvante_alternee1.

resolvante_diedrale (P, x) [Function]
calculates the transformation of $P(x)$ by the function $x_1 x_2 + x_3 x_4$.

(%i1) resolvante_diedrale (x^5 - 3*x^4 + 1, x);

15 12 11 10 9 8 7 6 5 4 3 2
(%o1) x - 21 x - 81 x - 21 x + 207 x + 1134 x + 2331 x

- 945 x - 4970 x - 18333 x - 29079 x - 20745 x - 25326 x

- 697

See also:
resolvante_produit_sym, resolvante_unitaire,
resolvante_alternee1, resolvante_klein, resolvante_klein3,
resolvante_vierer, resolvante.
\texttt{resolvante\_klein} \((P, x)\) \hspace{1cm} \textbf{[Function]} \\
calculates the transformation of \(P(x)\) by the function \(x_1 x_2 x_4 + x_4\).

See also:
\texttt{resolvante\_produit\_sym}, \texttt{resolvante\_unitaire},
\texttt{resolvante\_alternee1}, \texttt{resolvante}, \texttt{resolvante\_klein3},
\texttt{resolvante\_vierer}, \texttt{resolvante\_diedrale}.

\texttt{resolvante\_klein3} \((P, x)\) \hspace{1cm} \textbf{[Function]} \\
calculates the transformation of \(P(x)\) by the function \(x_1 x_2 x_4 + x_4\).

See also:
\texttt{resolvante\_produit\_sym}, \texttt{resolvante\_unitaire},
\texttt{resolvante\_alternee1}, \texttt{resolvante\_klein}, \texttt{resolvante},
\texttt{resolvante\_vierer}, \texttt{resolvante\_diedrale}.

\texttt{resolvante\_produit\_sym} \((P, x)\) \hspace{1cm} \textbf{[Function]} \\
calculates the list of all product resolvents of the polynomial \(P(x)\).

\begin{verbatim}
(%i1) resolvante\_produit\_sym (x^5 + 3*x^4 + 2*x - 1, x);
5 4 10 8 7 6 5
(%o1) \[y + 3 y + 2 y - 1, y - 2 y - 21 y - 31 y - 14 y
4 3 2 10 8 7 6 5 4
-y + 14 y + 3 y + 1, y + 3 y + 14 y - y - 14 y - 31 y
3 2 5 4
- 21 y - 2 y + 1, y - 2 y - 3 y - 1, y - 1]\n(%i2) resolvante: produit$
(%i3) resolvante (x^5 + 3*x^4 + 2*x - 1, x, a*b*c, [a, b, c]);
" resolvante produit "
10 8 7 6 5 4 3 2
(%o3) y + 3 y + 14 y - y - 14 y - 31 y - 21 y - 2 y + 1
\end{verbatim}

See also:
\texttt{resolvante}, \texttt{resolvante\_unitaire},
\texttt{resolvante\_alternee1}, \texttt{resolvante\_klein}, \texttt{resolvante\_klein3},
\texttt{resolvante\_vierer}, \texttt{resolvante\_diedrale}.

\texttt{resolvante\_unitaire} \((P, Q, x)\) \hspace{1cm} \textbf{[Function]} \\
computes the resolvent of the polynomial \(P(x)\) by the polynomial \(Q(x)\).

See also:
\texttt{resolvante\_produit\_sym}, \texttt{resolvante},
\texttt{resolvante\_alternee1}, \texttt{resolvante\_klein}, \texttt{resolvante\_klein3},
\texttt{resolvante\_vierer}, \texttt{resolvante\_diedrale}.

\texttt{resolvante\_vierer} \((P, x)\) \hspace{1cm} \textbf{[Function]} \\
computes the transformation of \(P(x)\) by the function \(x_1 x_2 - x_3 x_4\).
See also:

resolvante_produit_sym, resolvante_unitaire,
resolvante_alternee1, resolvante_klein, resolvante_klein3,
resolvante, resolvante_diedrale.

30.2.7 Miscellaneous

**multinomial** \((r, part)\) [Function]

where \(r\) is the weight of the partition \(part\). This function returns the associate multinomial coefficient: if the parts of \(part\) are \(i_1, i_2, \ldots, i_k\), the result is \(r!/(i_1! \ i_2! \ldots \ i_k!)\).

**permut** \((L)\) [Function]

returns the list of permutations of the list \(L\).
31 Groups

31.1 Functions and Variables for Groups

todd_coxeter

\[ \text{todd_coxeter} \quad \text{[Function]} \]
\[ \text{todd_coxeter} \left( \text{relations, subgroup} \right) \]
\[ \text{todd_coxeter} \left( \text{relations} \right) \]

Find the order of \( G/H \) where \( G \) is the Free Group modulo \( \text{relations} \), and \( H \) is the subgroup of \( G \) generated by \( \text{subgroup} \). \( \text{subgroup} \) is an optional argument, defaulting to [] . In doing this it produces a multiplication table for the right action of \( G \) on \( G/H \), where the cosets are enumerated \([H,Hg_2,Hg_3,...]\). This can be seen internally in the variable \( \text{todd_coxeter_state} \).

Example:

\[
\%i1 \) \text{symet}(n):=\text{create_list}(\text{if} \ (j - i) = 1 \text{ then } \left(p(i,j)\right)^3 \text{ else} \)
\[
\text{if} \ (\text{not} \ i = j) \text{ then } \left(p(i,j)\right)^2 \text{ else} \)
\[
p(i,i) \), \ j, 1, n-1, i, 1, j); \]
\[
\%o1 \) \text{symet}(n) := \text{create_list}(\text{if} \ j - i = 1 \text{ then } p(i, j) \)
\[
\text{else} \ (\text{if} \ \text{not} \ i = j \text{ then } p(i, j) \text{ else} \ p(i, i)), \ j, 1, n - 1, \)
\[
i, 1, j) \]
\[
\%i2 \) \text{p}(i,j) := \text{concat}(x,i) . \text{concat}(x,j); \]
\[
\%o2 \) \text{p}(i, j) := \text{concat}(x, i) . \text{concat}(x, j) \]
\[
\%i3 \) \text{symet}(5); \]
\[
\%o3 \) \text{[x1 , (x1 . x2) , x2 , (x1 . x3) , (x2 . x3) ,}
\[
\text{x3 , (x1 . x4) , (x2 . x4) , (x3 . x4) , x4 ]} \]
\[
\%i4 \) \text{todd_coxeter}(\%o3); \]

\text{Rows tried 426} \quad \%o4 \)
\[
\%i5 \) \text{todd_coxeter}(\%o3,[x1]); \]

\text{Rows tried 213} \quad \%o5 \)
\[
\%i6 \) \text{todd_coxeter}(\%o3,[x1,x2]); \]

\text{Rows tried 71} \quad \%o6 \)
32 Runtime Environment

32.1 Introduction for Runtime Environment

maxima-init.mac is a file which is loaded automatically when Maxima starts. You can use maxima-init.mac to customize your Maxima environment. maxima-init.mac, if it exists, is typically placed in the directory named by maxima_userdir, although it can be in any directory searched by the function file_search.

Here is an example maxima-init.mac file:

    setup_autoload("specfun.mac", ultraspherical, assoc_legendre_p);
    showtime:all;

In this example, setup_autoload tells Maxima to load the specified file (specfun.mac) if any of the functions (ultraspherical, assoc_legendre_p) are called but not yet defined. Thus you needn’t remember to load the file before calling the functions.

The statement showtime: all tells Maxima to set the showtime variable. The maxima-init.mac file can contain any other assignments or other Maxima statements.

32.2 Interrupts

The user can stop a time-consuming computation with the ^C (control-C) character. The default action is to stop the computation and print another user prompt. In this case, it is not possible to restart a stopped computation.

If the Lisp variable *debugger-hook* is set to nil, by executing

    :lisp (setq *debugger-hook* nil)

then upon receiving ^C, Maxima will enter the Lisp debugger, and the user may use the debugger to inspect the Lisp environment. The stopped computation can be restarted by entering continue in the Lisp debugger. The means of returning to Maxima from the Lisp debugger (other than running the computation to completion) is different for each version of Lisp.

On Unix systems, the character ^Z (control-Z) causes Maxima to stop altogether, and control is returned to the shell prompt. The fg command causes Maxima to resume from the point at which it was stopped.

32.3 Functions and Variables for Runtime Environment

maxima_tempdir [System variable]

maxima_tempdir names the directory in which Maxima creates some temporary files. In particular, temporary files for plotting are created in maxima_tempdir.

The initial value of maxima_tempdir is the user’s home directory, if Maxima can locate it; otherwise Maxima makes a guess about a suitable directory.

maxima_tempdir may be assigned a string which names a directory.

maxima_userdir [System variable]

maxima_userdir names a directory which Maxima searches to find Maxima and Lisp files. (Maxima searches some other directories as well; file_search_maxima and file_search_lisp are the complete lists.)
The initial value of `maxima_userdir` is a subdirectory of the user’s home directory, if Maxima can locate it; otherwise Maxima makes a guess about a suitable directory. `maxima_userdir` may be assigned a string which names a directory. However, assigning to `maxima_userdir` does not automatically change `file_search_maxima` and `file_search_lisp`; those variables must be changed separately.

```
room
   room ()
   room (true)
   room (false)
```

Prints out a description of the state of storage and stack management in Maxima. `room` calls the Lisp function of the same name.
- `room ()` prints out a moderate description.
- `room (true)` prints out a verbose description.
- `room (false)` prints out a terse description.

```
sstatus (keyword, item)
```

When `keyword` is the symbol `feature`, `item` is put on the list of system features. After `sstatus (keyword, item)` is executed, `status (feature, item)` returns `true`. If `keyword` is the symbol `nofeature`, `item` is deleted from the list of system features. This can be useful for package writers, to keep track of what features they have loaded in.

See also `status`.

```
status
   status (feature)
   status (feature, item)
```

Returns information about the presence or absence of certain system-dependent features.
- `status (feature)` returns a list of system features. These include Lisp version, operating system type, etc. The list may vary from one Lisp type to another.
- `status (feature, item)` returns `true` if `item` is on the list of items returned by `status (feature)` and `false` otherwise. `status` quotes the argument `item`. The quote-quote operator `''` defeats quotation. A feature whose name contains a special character, such as a hyphen, must be given as a string argument. For example, `status (feature, "ansi-cl")`.

See also `sstatus`.

The variable `features` contains a list of features which apply to mathematical expressions. See `features` and `featurep` for more information.

```
system (command)
```

Executes `command` as a separate process. The command is passed to the default shell for execution. `system` is not supported by all operating systems, but generally exists in Unix and Unix-like environments.

Supposing `_hist.out` is a list of frequencies which you wish to plot as a bar graph using `xgraph`.

```
(%i1) (with_stdout("_hist.out",
```
for i:1 thru length(hist) do (  
  print(i,hist[i])),  
  system("xgraph -bar -brw .7 -nl < _hist.out");

In order to make the plot be done in the background (returning control to Maxima) and remove the temporary file after it is done do:

system("(xgraph -bar -brw .7 -nl < _hist.out; rm -f _hist.out)&")

\[ \text{Function} \ \text{time} (\%o1, \%o2, \%o3, \ldots) \]

Returns a list of the times, in seconds, taken to compute the output lines \%o1, \%o2, \%o3, \ldots The time returned is Maxima’s estimate of the internal computation time, not the elapsed time. time can only be applied to output line variables; for any other variables, time returns unknown.

Set showtime: true to make Maxima print out the computation time and elapsed time with each output line.

\[ \text{Function} \ \text{timedate} \]\n
\[ \text{timedate} (T, tz\_offset) \]

\[ \text{timedate} (T) \]

\[ \text{timedate} () \]

\[ \text{timedate}(T, tz\_offset) \] returns a string representing the time \( T \) in the time zone \( tz\_offset \). The string format is YYYY-MM-DD HH:MM:SS.NNN[+|−]ZZ:ZZ (using as many digits as necessary to represent the fractional part) if \( T \) has a nonzero fractional part, or YYYY-MM-DD HH:MM:SS[+|−]ZZ:ZZ if its fractional part is zero.

\( T \) measures time, in seconds, since midnight, January 1, 1900, in the GMT time zone. \( tz\_offset \) measures the offset of the time zone, in hours, east (positive) or west (negative) of GMT. \( tz\_offset \) must be an integer, rational, or float between -24 and 24, inclusive. If \( tz\_offset \) is not a multiple of 1/60, it is rounded to the nearest multiple of 1/60.

\[ \text{timedate}(T) \] is equivalent to \[ \text{timedate}(T, tz\_offset) \] with \( tz\_offset \) equal to the offset of the local time zone.

\[ \text{timedate}() \] is equivalent to \[ \text{timedate}(\text{absolute\_real\_time}()) \]. That is, it returns the current time in the local time zone.

Example:

\[ \text{timedate} \] with no argument returns a string representing the current time and date.

\[
\begin{align*}
\%i1 \ & : \ \text{timedate} () ; \\
\%o1 \ & : \ 2010-06-08 04:08:09+01:00 \\
\%i2 \ & : \ \text{print ("timedate reports current time", d) } $ \\
\%o2 \ & : \ \text{timedate reports current time 2010-06-08 04:08:09+01:00}
\end{align*}
\]

\[ \text{timedate} \] with an argument returns a string representing the argument.

\[
\begin{align*}
\%i1 \ & : \ \text{timedate} (0) ; \\
\%o1 \ & : \ 1900-01-01 01:00:00+01:00 \\
\%i2 \ & : \ \text{timedate} (\text{absolute\_real\_time} ()) - 7*24*3600) ; \\
\%o2 \ & : \ 2010-06-01 04:19:51+01:00
\end{align*}
\]
timedate with optional timezone offset.

(%i1) timedate (1000000000, -9.5);
(%o1) 1931-09-09 16:16:40-09:30

parse_timedate

parse_timedate (S)

Parses a string $S$ representing a date or date and time of day and returns the number of seconds since midnight, January 1, 1900 GMT. If there is a nonzero fractional part, the value returned is a rational number, otherwise, it is an integer. parse_timedate returns false if it cannot parse $S$ according to any of the allowed formats.

The string $S$ must have one of the following formats, optionally followed by a timezone designation:

- YYYY-MM-DD [T]hh:mm:ss[,.]nnn
- YYYY-MM-DD [T]hh:mm:ss
- YYYY-MM-DD

where the fields are year, month, day, hours, minutes, seconds, and fraction of a second, and square brackets indicate acceptable alternatives. The fraction may contain one or more digits.

Except for the fraction of a second, each field must have exactly the number of digits indicated: four digits for the year, and two for the month, day of the month, hours, minutes, and seconds.

A timezone designation must have one of the following forms:

- [+-]hh:mm
- [+-]hmm
- [+-]hh
- Z

where hh and mm indicate hours and minutes east (+) or west (-) of GMT. The timezone may be from +24 hours (inclusive) to -24 hours (inclusive).

A literal character Z is equivalent to +00:00 and its variants, indicating GMT.

If no timezone is indicated, the time is assumed to be in the local time zone.

Any leading or trailing whitespace (space, tab, newline, and carriage return) is ignored, but any other leading or trailing characters cause parse_timedate to fail and return false.

See also timedate and absolute_real_time.

Examples:

Midnight, January 1, 1900, in the local time zone, in each acceptable format. The result is the number of seconds the local time zone is ahead (negative result) or behind (positive result) GMT. In this example, the local time zone is 8 hours behind GMT.

(%i1) parse_timedate ("1900-01-01 00:00:00,000");
(%o1) 28800
(%i2) parse_timedate ("1900-01-01 00:00:00.000");
(%o2) 28800
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(%i3) parse_timedate ("1900-01-01T00:00:00,000");
(%o3) 28800
(%i4) parse_timedate ("1900-01-01T00:00:00.000");
(%o4) 28800
(%i5) parse_timedate ("1900-01-01 00:00:00");
(%o5) 28800
(%i6) parse_timedate ("1900-01-01T00:00:00");
(%o6) 28800
(%i7) parse_timedate ("1900-01-01");
(%o7) 28800

Midnight, January 1, 1900, GMT, in different indicated time zones.

(%i1) parse_timedate ("1900-01-01 19:00:00+19:00");
(%o1) 0
(%i2) parse_timedate ("1900-01-01 07:00:00+07:00");
(%o2) 0
(%i3) parse_timedate ("1900-01-01 01:00:00+01:00");
(%o3) 0
(%i4) parse_timedate ("1900-01-01Z");
(%o4) 0
(%i5) parse_timedate ("1899-12-31 21:00:00-03:00");
(%o5) 0
(%i6) parse_timedate ("1899-12-31 13:00:00-11:00");
(%o6) 0
(%i7) parse_timedate ("1899-12-31 08:00:00-16:00");
(%o7) 0

encode_time

encode_time (year, month, day, hours, minutes, seconds, tz_offset)

Given a time and date specified by year, month, day, hours, minutes, and seconds, encode_time returns the number of seconds (possibly including a fractional part) since midnight, January 1, 1900 GMT.

year must be an integer greater than or equal to 1899. However, 1899 is allowed only if the resulting encoded time is greater than or equal to 0.

month must be an integer from 1 to 12, inclusive.

day must be an integer from 1 to n, inclusive, where n is the number of days in the month specified by month.

hours must be an integer from 0 to 23, inclusive.

minutes must be an integer from 0 to 59, inclusive.

seconds must be an integer, rational, or float greater than or equal to 0 and less than 60. When seconds is not an integer, encode_time returns a rational, such that the fractional part of the return value is equal to the fractional part of seconds. Otherwise, seconds is an integer, and the return value is likewise an integer.

tz_offset measures the offset of the time zone, in hours, east (positive) or west (negative) of GMT. tz_offset must be an integer, rational, or float between -24 and 24,
inclusive. If \( tz\_offset \) is not a multiple of \( 1/3600 \), it is rounded to the nearest multiple of \( 1/3600 \).

If \( tz\_offset \) is not present, the offset of the local time zone is assumed.

See also \texttt{decode_time}.

Examples:

\begin{verbatim}
(%i1) encode_time (1900, 1, 1, 0, 0, 0, 0);
(%o1) 0
(%i2) encode_time (1970, 1, 1, 0, 0, 0, 0);
(%o2) 2208988800
(%i3) encode_time (1970, 1, 1, 8, 30, 0, 8.5);
(%o3) 2208988800
(%i4) encode_time (1969, 12, 31, 16, 0, 0, -8);
(%o4) 2208988800
(%i5) encode_time (1969, 12, 31, 16, 0, 1/1000, -8); 2208988800001
(%o5) ------------
1000
(%i6) % - 2208988800;
1
(%o6) ----
1000
\end{verbatim}

\texttt{decode_time} \hspace{1cm} [Function]

\texttt{decode_time (T, tz\_offset)}

\texttt{decode_time (T)}

Given the number of seconds (possibly including a fractional part) since midnight, January 1, 1900 GMT, returns the date and time as represented by a list comprising the year, month, day of the month, hours, minutes, seconds, and time zone offset. \( tz\_offset \) measures the offset of the time zone, in hours, east (positive) or west (negative) of GMT. \( tz\_offset \) must be an integer, rational, or float between -24 and 24, inclusive. If \( tz\_offset \) is not a multiple of \( 1/3600 \), it is rounded to the nearest multiple of \( 1/3600 \).

If \( tz\_offset \) is not present, the offset of the local time zone is assumed.

See also \texttt{encode_time}.

Examples:

\begin{verbatim}
(%i1) decode_time (0, 0);  
(%o1) [1900, 1, 1, 0, 0, 0, 0]
(%i2) decode_time (0);  
(%o2) [1899, 12, 31, 16, 0, 0, - 8]
(%i3) decode_time (2208988800, 9.25);  
(%o3) [1970, 1, 1, 9, 15, 0, --]
37
(%i4) decode_time (2208988800);  
(%o4) [1969, 12, 31, 16, 0, 0, - 8]
\end{verbatim}
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```lisp
(%i5) decode_time (2208988800 + 1729/1000, -6);
   1729
(%o5) [1969, 12, 31, 18, 0, ----, - 6]
    1000
(%i6) decode_time (2208988800 + 1729/1000);
   1729
(%o6) [1969, 12, 31, 16, 0, ----, - 8]
    1000
```

**absolute_real_time ()**

Returns the number of seconds since midnight, January 1, 1900 GMT. The return value is an integer.

See also `elapsed_real_time` and `elapsed_run_time`.

Example:

```lisp
(%i1) absolute_real_time ();
   3385045277
(%o1) 3385045277
(%i2) 1900 + absolute_real_time () / (365.25 * 24 * 3600);
   2007.265612087104
(%o2) 2007.265612087104
```

**elapsed_real_time ()**

Returns the number of seconds (including fractions of a second) since Maxima was most recently started or restarted. The return value is a floating-point number.

See also `absolute_real_time` and `elapsed_run_time`.

Example:

```lisp
(%i1) elapsed_real_time ();
   2.559324
(%o1) 2.559324
(%i2) expand ((a + b)^500)$
(%i3) elapsed_real_time ();
   7.552087
(%o3) 7.552087
```

**elapsed_run_time ()**

Returns an estimate of the number of seconds (including fractions of a second) which Maxima has spent in computations since Maxima was most recently started or restarted. The return value is a floating-point number.

See also `absolute_real_time` and `elapsed_real_time`.

Example:

```lisp
(%i1) elapsed_run_time ();
   0.04
(%o1) 0.04
(%i2) expand ((a + b)^500)$
(%i3) elapsed_run_time ();
   1.26
(%o3) 1.26
```
33 Miscellaneous Options

33.1 Introduction to Miscellaneous Options

In this section various options are discussed which have a global effect on the operation of Maxima. Also various lists such as the list of all user defined functions, are discussed.

33.2 Share

The Maxima "share" directory contains programs and other files of interest to Maxima users, but not part of the core implementation of Maxima. These programs are typically loaded via load or setup_autoload.

: lisp *maxima-sharedir* displays the location of the share directory within the user’s file system.

printfile ("share.usg") prints an out-of-date list of share packages. Users may find it more informative to browse the share directory using a file system browser.

33.3 Functions and Variables for Miscellaneous Options

askexp

[System variable]

When asksign is called, askexp is the expression asksign is testing.

At one time, it was possible for a user to inspect askexp by entering a Maxima break with control-A.

genindex

[Option variable]

Default value: i

genindex is the alphabetic prefix used to generate the next variable of summation when necessary.

gensumnum

[Option variable]

Default value: 0

gensumnum is the numeric suffix used to generate the next variable of summation. If it is set to false then the index will consist only of genindex with no numeric suffix.

gensym

[Function]

gensym ()
gensym (x)

gensym() creates and returns a fresh symbol.

The name of the new symbol is the concatenation of a prefix, which defaults to "g", and a suffix, which is an integer that defaults to the value of an internal counter.

If x is supplied, and is a string, then that string is used as a prefix instead of "g" for this call to gensym only.

If x is supplied, and is a nonnegative integer, then that integer, instead of the value of the internal counter, is used as the suffix for this call to gensym only.

If and only if no explicit suffix is supplied, the internal counter is incremented after it is used.
Examples:

(\%i1) gensym();
\%o1 g887
(\%i2) gensym("new");
\%o2 new888
(\%i3) gensym(123);
\%o3 g123

packagefile [Option variable]
Default value: false
Package designers who use save or translate to create packages (files) for others to use may want to set packagefile: true to prevent information from being added to Maxima’s information-lists (e.g. values, functions) except where necessary when the file is loaded in. In this way, the contents of the package will not get in the user’s way when he adds his own data. Note that this will not solve the problem of possible name conflicts. Also note that the flag simply affects what is output to the package file. Setting the flag to true is also useful for creating Maxima init files.

remvalue [Function]

remvalue (name_1, \ldots, name_n)
remvalue remvalue (all)
Removes the values of user variables name_1, \ldots, name_n (which can be subscripted) from the system.
remvalue (all) removes the values of all variables in values, the list of all variables given names by the user (as opposed to those which are automatically assigned by Maxima).
See also values.

rncombine (expr) [Function]
Transforms expr by combining all terms of expr that have identical denominators or denominators that differ from each other by numerical factors only. This is slightly different from the behavior of combine, which collects terms that have identical denominators.
Setting pfeformat: true and using combine yields results similar to those that can be obtained with rncombine, but rncombine takes the additional step of cross-multiplying numerical denominator factors. This results in neater forms, and the possibility of recognizing some cancellations.
load(rncomb) loads this function.

setup_autoload (filename, function_1, \ldots, function_n) [Function]
Specifies that if any of function_1, \ldots, function_n are referenced and not yet defined, filename is loaded via load. filename usually contains definitions for the functions specified, although that is not enforced.
setup_autoload does not work for array functions.
setup_autoload quotes its arguments.
Example:

(\%i1) legendre_p (1, \%pi);
(%o1) \text{legendre\_p}(1, \pi)
(%i2) \text{setup\_autoload} ("specfun\_mac", \text{legendre\_p}, \text{ultraspherical});
(%o2) done
(%i3) \text{ultraspherical} (2, 1/2, \pi);
Warning - you are redefining the Macsyma function \text{ultraspherical}
Warning - you are redefining the Macsyma function \text{legendre\_p}
\begin{align*}
2 & \quad \frac{3}{2} \left(\frac{1}{\pi} - 1\right) + 3 \left(\frac{1}{\pi} - 1\right) + 1 \\
\text{o3) } & \quad \frac{2}{3} \\
\end{align*}
(%i4) \text{legendre\_p} (1, \pi);
(%o4) \pi
(%i5) \text{legendre\_q} (1, \pi);
\begin{align*}
\text{i} & \quad \frac{\%\pi + 1}{\%\pi \log\left(\frac{1}{1 - \%\pi}\right)} - 1 \\
\text{o5) } & \quad \frac{2}{3} \\
\end{align*}
tcl\_output
[Function]
tcl\_output (list, i0, skip)
tcl\_output (list, i0)
tcl\_output ([list\_1, \ldots, list\_n], i)
Prints elements of a list enclosed by curly braces \{\}, suitable as part of a program in the Tcl/Tk language.
tcl\_output (list, i0, skip) prints list, beginning with element i0 and printing elements i0 + skip, i0 + 2 skip, etc.
tcl\_output (list, i0) is equivalent to tcl\_output (list, i0, 2).
tcl\_output ([list\_1, \ldots, list\_n], i) prints the i’th elements of list\_1, \ldots, list\_n.
Examples:
(%i11) tcl\_output ([1, 2, 3, 4, 5, 6], 1, 3)$
\begin{align*}
\{1 & \quad \{1.000000000 & \quad 4.000000000 \\
& \} \\
\text{i2)} & \quad tcl\_output ([1, 2, 3, 4, 5, 6], 2, 3)$
\begin{align*}
\{2 & \quad \{2.000000000 & \quad 5.000000000 \\
& \} \\
\text{i3)} & \quad tcl\_output ([3/7, 5/9, 11/13, 13/17], 1)$
\begin{align*}
\{( & \quad ((\text{RAT SIMP}) 3 7) ((\text{RAT SIMP}) 11 13) \\
& \} \\
\text{i4)} & \quad tcl\_output ([x1, y1, x2, y2, x3, y3], 2)$
\begin{align*}
\{ & \quad \{x1 & \quad y1 \quad x2 \quad y2 \quad x3 \quad y3\} \\
\text{Y1} & \quad \text{Y2} & \quad \text{Y3} \}
(5i5) tcl_output ([[1, 2, 3], [11, 22, 33]], 1)

{SIMP 1.000000000 11.000000000
}

34 Rules and Patterns

34.1 Introduction to Rules and Patterns

This section describes user-defined pattern matching and simplification rules. There are two groups of functions which implement somewhat different pattern matching schemes. In one group are \texttt{tellsimp}, \texttt{tellsimpafter}, \texttt{defmatch}, \texttt{defrule}, \texttt{apply1}, \texttt{applyb1}, and \texttt{apply2}. In the other group are \texttt{let} and \texttt{letsimp}. Both schemes define patterns in terms of pattern variables declared by \texttt{matchdeclare}.

Pattern-matching rules defined by \texttt{tellsimp} and \texttt{tellsimpafter} are applied automatically by the Maxima simplifier. Rules defined by \texttt{defmatch}, \texttt{defrule}, and \texttt{let} are applied by an explicit function call.

There are additional mechanisms for rules applied to polynomials by \texttt{tellrat}, and for commutative and noncommutative algebra in \texttt{affine} package.

34.2 Functions and Variables for Rules and Patterns

\texttt{apply1 (expr, rule_1, \ldots, rule_n)} \hspace{1cm} \text{[Function]}

Repeatedly applies \texttt{rule_1} to \texttt{expr} until it fails, then repeatedly applies the same rule to all subexpressions of \texttt{expr}, left to right, until \texttt{rule_1} has failed on all subexpressions. Call the result of transforming \texttt{expr} in this manner \texttt{expr_2}. Then \texttt{rule_2} is applied in the same fashion starting at the top of \texttt{expr_2}. When \texttt{rule_n} fails on the final subexpression, the result is returned.

\texttt{maxapplydepth} is the depth of the deepest subexpressions processed by \texttt{apply1} and \texttt{apply2}.

See also \texttt{applyb1}, \texttt{apply2} and \texttt{let}.

\texttt{apply2 (expr, rule_1, \ldots, rule_n)} \hspace{1cm} \text{[Function]}

If \texttt{rule_1} fails on a given subexpression, then \texttt{rule_2} is repeatedly applied, etc. Only if all rules fail on a given subexpression is the whole set of rules repeatedly applied to the next subexpression. If one of the rules succeeds, then the same subexpression is reprocessed, starting with the first rule.

\texttt{maxapplydepth} is the depth of the deepest subexpressions processed by \texttt{apply1} and \texttt{apply2}.

See also \texttt{apply1} and \texttt{let}.

\texttt{applyb1 (expr, rule_1, \ldots, rule_n)} \hspace{1cm} \text{[Function]}

Repeatedly applies \texttt{rule_1} to the deepest subexpression of \texttt{expr} until it fails, then repeatedly applies the same rule one level higher (i.e., larger subexpressions), until \texttt{rule_1} has failed on the top-level expression. Then \texttt{rule_2} is applied in the same fashion to the result of \texttt{rule_1}. After \texttt{rule_n} has been applied to the top-level expression, the result is returned.

\texttt{applyb1} is similar to \texttt{apply1} but works from the bottom up instead of from the top down.

\texttt{maxapplyheight} is the maximum height which \texttt{applyb1} reaches before giving up.

See also \texttt{apply1}, \texttt{apply2} and \texttt{let}.
current\_let\_rule\_package

Default value: default\_let\_rule\_package

current\_let\_rule\_package is the name of the rule package that is used by functions in the let package (letsimp, etc.) if no other rule package is specified. This variable may be assigned the name of any rule package defined via the let command.

If a call such as letsimp (expr, rule\_pkg\_name) is made, the rule package rule\_pkg\_name is used for that function call only, and the value of current\_let\_rule\_package is not changed.

default\_let\_rule\_package

Default value: default\_let\_rule\_package

default\_let\_rule\_package is the name of the rule package used when one is not explicitly set by the user with let or by changing the value of current\_let\_rule\_package.

defmatch

```lisp
(defun defmatch (prognm, pattern, x_1, ..., x_n)
defmatch (prognm, pattern)
  Defines a function \texttt{prognm}([\texttt{expr}, x_1, ..., x_n]) which tests \texttt{expr} to see if it matches \texttt{pattern}.

  \texttt{pattern} is an expression containing the pattern arguments \texttt{x}_1, \ldots, \texttt{x}_n (if any) and some pattern variables (if any). The pattern arguments are given explicitly as arguments to defmatch while the pattern variables are declared by the matchdeclare function. Any variable not declared as a pattern variable in matchdeclare or as a pattern argument in defmatch matches only itself.

  The first argument to the created function \texttt{prognm} is an expression to be matched against the pattern and the other arguments are the actual arguments which correspond to the dummy variables \texttt{x}_1, \ldots, \texttt{x}_n in the pattern.

  If the match is successful, \texttt{prognm} returns a list of equations whose left sides are the pattern arguments and pattern variables, and whose right sides are the subexpressions which the pattern arguments and variables matched. The pattern variables, but not the pattern arguments, are assigned the subexpressions they match. If the match fails, \texttt{prognm} returns false.

  A literal pattern (that is, a pattern which contains neither pattern arguments nor pattern variables) returns true if the match succeeds.

  See also matchdeclare, defrule, tellsimp and tellsimpafter.

Examples:

Define a function \texttt{linearp}([\texttt{expr}, \texttt{x}]) which tests \texttt{expr} to see if it is of the form \texttt{a*}\texttt{x} + \texttt{b} such that \texttt{a} and \texttt{b} do not contain \texttt{x} and \texttt{a} is nonzero. This match function matches expressions which are linear in any variable, because the pattern argument \texttt{x} is given to defmatch.

  (%i1) matchdeclare (a, lambda ([e], e\#0 and freeof(x, e)), b, freeof(x));
  (%o1) done
  (%i2) defmatch (linearp, a*\texttt{x} + \texttt{b}, \texttt{x});
```
Define a function `linearp(expr)` which tests `expr` to see if it is of the form `a*x + b` such that `a` and `b` do not contain `x` and `a` is nonzero. This match function only matches expressions linear in `x`, not any other variable, because no pattern argument is given to `defmatch`.

```
(%i11) matchdeclare (a, lambda ([e], e#0 and freeof(x, e)), b, freeof(x));
(%o11) done
(%i12) defmatch (linearp, a*x + b);
(%o12) linearp
(%i13) linearp (3*z + (y + 1)*z + y^2, z);
(%o13) [b = y , a = y + 4, x = z]
(%i14) a;
(%o14) y + 4
(%i15) b;
(%o15) 2
(%i16) x;
(%o16) x
```

Define a function `checklimits(expr)` which tests `expr` to see if it is a definite integral.

```
(%i17) simp : false;
(%o17) false
(%i18) 'integrate (sin(t), t, %pi + x, 2*%pi + x);
```
\[
\frac{x + 2 \, \pi}{x + \pi}
\]

\[
\int \sin(t) \, dt
\]

\[
\frac{x + \pi}{x + \pi}
\]

(%i9) checklimits (%);
(%o9) \[
[b = x + 2 \, \pi, a = x + \pi, x = t, f = \sin(t)]
\]

\textbf{defrule (ruleame, pattern, replacement)}

Defines and names a replacement rule for the given pattern. If the rule named \textit{ruleame} is applied to an expression (by \texttt{apply1}, \texttt{applyb1}, or \texttt{apply2}), every subexpression matching the pattern will be replaced by the replacement. All variables in the replacement which have been assigned values by the pattern match are assigned those values in the replacement which is then simplified.

The rules themselves can be treated as functions which transform an expression by one operation of the pattern match and replacement. If the match fails, the rule function returns \texttt{false}.

\textbf{disprule}

\[
\text{disprule (ruleame}_1, \ldots, \text{ruleame}_n)
\]

\[
\text{disprule (all)}
\]

Display rules with the names \textit{ruleame}_1, \ldots, \textit{ruleame}_n, as returned by \texttt{defrule}, \texttt{tellsimp}, or \texttt{tellsimpafter}, or a pattern defined by \texttt{defmatch}. Each rule is displayed with an intermediate expression label (\%t).

\texttt{disprule (all)} displays all rules.

\texttt{disprule} quotes its arguments. \texttt{disprule} returns the list of intermediate expression labels corresponding to the displayed rules.

See also \texttt{letrules}, which displays rules defined by \texttt{let}.

Examples:

(%i1) tellsimpafter (foo(x, y), bar(x) + baz(y));
(%o1) \[
[\text{foorule}_1, \text{false}]
\]

(%i2) tellsimpafter (x + y, special_add (x, y));
(%o2) \[
[\text{+rule}_1, \text{special_add}]
\]

(%i3) defmatch (quux, mumble (x));
(%o3) quux

(%i4) disprule (foorule1, ?\text{\textasciitilde}+rule1, quux);
(%t4) foorule1 : foo(x, y) -> baz(y) + bar(x)

(%t5) +rule1 : y + x -> \text{special_add}(x, y)

(%t6) quux : mumble(x) -> []

(%o6) \[
[\%5, \%5, \%6]
\]

(%i6) ev(%);
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(%i6) [foorule1 : foo(x, y) -> baz(y) + bar(x),
   +rule1 : y + x -> special_add(x, y), quux : mumble(x) -> []]

let (prod, repl, predname, arg_1, ..., arg_n)
let ([prod, repl, predname, arg_1, ..., arg_n], package_name)

Defines a substitution rule for letsimp such that prod is replaced by repl. prod is a
product of positive or negative powers of the following terms:

- Atoms which letsimp will search for literally unless previous to calling letsimp
  the matchdeclare function is used to associate a predicate with the atom. In
  this case letsimp will match the atom to any term of a product satisfying the
  predicate.
- Kernels such as sin(x), n!, f(x,y), etc. As with atoms above letsimp will look
  for a literal match unless matchdeclare is used to associate a predicate with the
  argument of the kernel.

A term to a positive power will only match a term having at least that power. A
term to a negative power on the other hand will only match a term with a power at
least as negative. In the case of negative powers in prod the switch letrat must be
set to true. See also letrat.

If a predicate is included in the let function followed by a list of arguments, a
tentative match (i.e. one that would be accepted if the predicate were omitted) is
accepted only if predname (arg_1', ..., arg_n') evaluates to true where arg_i' is
the value matched to arg_i. The arg_i may be the name of any atom or the argument
of any kernel appearing in prod. repl may be any rational expression. If any of the
atoms or arguments from prod appear in repl the appropriate substitutions are made.

The global flag letrat controls the simplification of quotients by letsimp. When
letrat is false, letsimp simplifies the numerator and denominator of expr separately,
and does not simplify the quotient. Substitutions such as n!/n goes to (n-1)! then fail. When letrat is true, then the numerator, denominator, and the quotient
are simplified in that order.

These substitution functions allow you to work with several rule packages at once.
Each rule package can contain any number of let rules and is referenced by a
user-defined name. The command let ([prod, repl, predname, arg_1, ..., arg_n], package_name) adds the rule predname to the rule package package_name.
The command letsimp (expr, package_name) applies the rules in package_name.
letsimp (expr, package_name1, package_name2, ...) is equivalent to letsimp (expr, package_name1) followed by letsimp (%, package_name2), ...

current Let_rule_package is the name of the rule package that is presently being
used. This variable may be assigned the name of any rule package defined via
the let command. Whenever any of the functions comprising the let package are
called with no package name, the package named by current Let_rule_package is
used. If a call such as letsimp (expr, rule_pkg_name) is made, the rule package
rule_pkg_name is used for that letsimp command only, and current Let_rule_
package is not changed. If not otherwise specified, current Let_rule_package de-
defaults to default_Let_rule_package.

(%i11) matchdeclare ([a, a1, a2], true)$
(%i2) oneless (x, y) := is (x = y-1)$
(%i3) let (a1*a2!, a1!, oneless, a2, a1);
(%o3) a1 a2! --> a1! where oneless(a2, a1)
(%i4) letrat: true$
(%i5) let (a1!/a1, (a1-1)!);

--- --> (a1 - 1)!

(%i6) letsimp (n*m!*(n-1)!/m);
(%o6) (m - 1)! n!
(%i7) let (sin(a)^2, 1 - cos(a)^2);
2 2

sin (a) --> 1 - cos (a)
(%i8) letsimp (sin(x)^4);
4 2

cos (x) - 2 cos (x) + 1

letrat

[Option variable]

Default value: false

When letrat is false, letsimp simplifies the numerator and denominator of a ratio separately, and does not simplify the quotient.

When letrat is true, the numerator, denominator, and their quotient are simplified in that order.

(%i1) matchdeclare (n, true)$
(%i2) let (n!/n, (n-1)!);

--- --> (n - 1)!

(%i3) letrat: false$
(%i4) letsimp (a!/a);

---

(%i5) letrat: true$
(%i6) letsimp (a!/a);

(a - 1)!

letrules

[Function]

letrules ()
letrules (package_name)

Displays the rules in a rule package. letrules () displays the rules in the current rule package. letrules (package_name) displays the rules in package_name.

The current rule package is named by current_let_rule_package. If not otherwise specified, current_let_rule_package defaults to default_let_rule_package.

See also disprule, which displays rules defined by tellsimp and tellsimpafter.
letsimp  

letsimp (expr)  
letsimp (expr, package_name)  
letsimp (expr, package_name_1, ..., package_name_n)

Repeatedly applies the substitution rules defined by let until no further change is made to expr.

letsimp (expr) uses the rules from current_let_rule_package.

letsimp (expr, package_name) uses the rules from package_name without changing current_let_rule_package.

letsimp (expr, package_name_1, ..., package_name_n) is equivalent to letsimp (expr, package_name_1), followed by letsimp (%, package_name_2), and so on.

let_rule_packages  [Option variable]

Default value: [default_let_rule_package]

let_rule_packages is a list of all user-defined let rule packages plus the default package default_let_rule_package.

matchdeclare (a_1, pred_1, ..., a_n, pred_n)  [Function]

Associates a predicate pred_k with a variable or list of variables a_k so that a_k matches expressions for which the predicate returns anything other than false.

A predicate is the name of a function, or a lambda expression, or a function call or lambda call missing the last argument, or true or all. Any expression matches true or all. If the predicate is specified as a function call or lambda call, the expression to be tested is appended to the list of arguments; the arguments are evaluated at the time the match is evaluated. Otherwise, the predicate is specified as a function name or lambda expression, and the expression to be tested is the sole argument. A predicate function need not be defined when matchdeclare is called; the predicate is not evaluated until a match is attempted.

A predicate may return a Boolean expression as well as true or false. Boolean expressions are evaluated by is within the constructed rule function, so it is not necessary to call is within the predicate.

If an expression satisfies a match predicate, the match variable is assigned the expression, except for match variables which are operands of addition + or multiplication *. Only addition and multiplication are handled specially; other n-ary operators (both built-in and user-defined) are treated like ordinary functions.

In the case of addition and multiplication, the match variable may be assigned a single expression which satisfies the match predicate, or a sum or product (respectively) of such expressions. Such multiple-term matching is greedy: predicates are evaluated in the order in which their associated variables appear in the match pattern, and a term which satisfies more than one predicate is taken by the first predicate which it satisfies. Each predicate is tested against all operands of the sum or product before the next predicate is evaluated. In addition, if 0 or 1 (respectively) satisfies a match predicate, and there are no other terms which satisfy the predicate, 0 or 1 is assigned to the match variable associated with the predicate.
The algorithm for processing addition and multiplication patterns makes some match results (for example, a pattern in which a "match anything" variable appears) dependent on the ordering of terms in the match pattern and in the expression to be matched. However, if all match predicates are mutually exclusive, the match result is insensitive to ordering, as one match predicate cannot accept terms matched by another.

Calling \texttt{matchdeclare} with a variable \texttt{a} as an argument changes the \texttt{matchdeclare} property for \texttt{a}, if one was already declared; only the most recent \texttt{matchdeclare} is in effect when a rule is defined. Later changes to the \texttt{matchdeclare} property (via \texttt{matchdeclare} or \texttt{remove}) do not affect existing rules.

\texttt{propvars (matchdeclare)} returns the list of all variables for which there is a \texttt{matchdeclare} property. \texttt{printprops (a, matchdeclare)} returns the predicate for variable \texttt{a}. \texttt{printprops (all, matchdeclare)} returns the list of predicates for all \texttt{matchdeclare} variables. \texttt{remove (a, matchdeclare)} removes the \texttt{matchdeclare} property from \texttt{a}.

The functions \texttt{defmatch}, \texttt{defrule}, \texttt{tellsimp}, \texttt{tellsimpafter}, and \texttt{let} construct rules which test expressions against patterns.

\texttt{matchdeclare} quotes its arguments. \texttt{matchdeclare} always returns \texttt{done}.

Examples:

A predicate is the name of a function, or a lambda expression, or a function call or lambda call missing the last argument, or \texttt{true} or \texttt{all}.

\begin{verbatim}
(%i1) matchdeclare (aa, integerp);
(%o1) done
(%i2) matchdeclare (bb, lambda ([x], x > 0));
(%o2) done
(%i3) matchdeclare (cc, freeof (%e, %pi, %i));
(%o3) done
(%i4) matchdeclare (dd, lambda ([x, y], gcd (x, y) = 1) (1728));
(%o4) done
(%i5) matchdeclare (ee, true);
(%o5) done
(%i6) matchdeclare (ff, all);
(%o6) done
\end{verbatim}

If an expression satisfies a match predicate, the match variable is assigned the expression.

\begin{verbatim}
(%i1) matchdeclare (aa, integerp, bb, atom);
(%o1) done
(%i2) defrule (r1, bb^aa, ["integer" = aa, "atom" = bb]);
    aa
(%o2) r1 : bb -> [integer = aa, atom = bb]
(%i3) r1 (%pi^8);
(%o3) [integer = 8, atom = %pi]
\end{verbatim}

In the case of addition and multiplication, the match variable may be assigned a single expression which satisfies the match predicate, or a sum or product (respectively) of such expressions.
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(%i1) matchdeclare (aa, atom, bb, lambda ([x], not atom(x)));
(%o1) done
(%i2) defrule (r1, aa + bb, ["all atoms" = aa, "all nonatoms" = bb]);
bb + aa partitions 'sum'
(%o2) r1 : bb + aa -> [all atoms = aa, all nonatoms = bb]
(%i3) r1 (8 + a*b + sin(x));
(%o3) [all atoms = 8, all nonatoms = sin(x) + a b]
(%i4) defrule (r2, aa * bb, ["all atoms" = aa, "all nonatoms" = bb]);
bb aa partitions 'product'
(%o4) r2 : aa bb -> [all atoms = aa, all nonatoms = bb]
(%i5) r2 (8 * (a + b) * sin(x));
(%o5) [all atoms = 8, all nonatoms = (b + a) sin(x)]

When matching arguments of + and *, if all match predicates are mutually exclusive, the match result is insensitive to ordering, as one match predicate cannot accept terms matched by another.

(%i1) matchdeclare (aa, atom, bb, lambda ([x], not atom(x)));  
(%o1) done
(%i2) defrule (r1, aa + bb, ["all atoms" = aa, "all nonatoms" = bb]);
(%o2) r1 : bb + aa -> [all atoms = aa, all nonatoms = bb]
(%i3) r1 (8 + a*b + %pi + sin(x) - c + 2^n);
(%o3) [all atoms = %pi + 8, all nonatoms = sin(x) + 2 - c + a b]
(%i4) defrule (r2, aa * bb, ["all atoms" = aa, "all nonatoms" = bb]);
(%o4) r2 : aa bb -> [all atoms = aa, all nonatoms = bb]
(%i5) r2 (8 * (a + b) * %pi * sin(x) / c * 2^n);
(%o5) [all atoms = 8 %pi, all nonatoms = ------------------]

The functions propvars and printprops return information about match variables.

(%i1) matchdeclare ([aa, bb, cc], atom, [dd, ee], integerp);
(%o1) done
(%i2) matchdeclare (ff, floatnump, gg, lambda ([x], x > 100));
(%o2) done
(%i3) propvars (matchdeclare);
(%o3) [aa, bb, cc, dd, ee, ff, gg]
(%i4) printprops (ee, matchdeclare);
(%o4) [integerp(ee)]
(%i5) printprops (gg, matchdeclare);
(%o5) [lambda([x], x > 100, gg)]
(%i6) printprops (all, matchdeclare);
(%o6) [lambda([x], x > 100, gg), floatnump(ff), integerp(ee), integerp(dd), atom(cc), atom(bb), atom(aa)]

maxapplydepth
Default value: 10000
maxapplydepth is the maximum depth to which apply1 and apply2 will delve.

maxapplyheight
Default value: 10000
maxapplyheight is the maximum height to which applyb1 will reach before giving up.

remlet
remlet (prod, name)
remlet ()
remlet (all)
remlet (all, name)
Deletes the substitution rule, prod \rightarrow repl, most recently defined by the let function. If name is supplied the rule is deleted from the rule package name.

remlet() and remlet(all) delete all substitution rules from the current rule package. If the name of a rule package is supplied, e.g. remlet (all, name), the rule package name is also deleted.

If a substitution is to be changed using the same product, remlet need not be called, just redefine the substitution using the same product (literally) with the let function and the new replacement and/or predicate name. Should remlet (prod) now be called the original substitution rule is revived.

See also remrule, which removes a rule defined by tellsimp or tellsimpafter.

remrule
remrule (op, rulename)
remrule (op, all)
Removes rules defined by tellsimp or tellsimpafter.

remrule (op, rulename) removes the rule with the name rulename from the operator op. When op is a built-in or user-defined operator (as defined by infix, prefix, etc.), op and rulename must be enclosed in double quote marks.

remrule (op, all) removes all rules for the operator op.

See also remlet, which removes a rule defined by let.

Examples:

(%i11) tellsimp (foo (aa, bb), bb - aa);
(%o11) [foorule1, false]
(%i12) tellsimpafter (aa + bb, special_add (aa, bb));
(%o12) [+rule1, simplus]
(%i3) infix ("@@");
(%o3) @@
(%i4) tellsimp (aa @@ bb, bb/aa);
tellsimp (pattern, replacement) is similar to tellsimpafter but places new information before old so that it is applied before the built-in simplification rules.

tellsimp is used when it is important to modify the expression before the simplifier works on it, for instance if the simplifier "knows" something about the expression, but what it returns is not to your liking. If the simplifier "knows" something about the main operator of the expression, but is simply not doing enough for you, you probably want to use tellsimpafter.

The pattern may not be a sum, product, single variable, or number.

The system variable rules is the list of rules defined by defrule, defmatch, tellsimp, and tellsimpafter.

Examples:
(%i11) matchdeclare (x, freeof (%i));
    done
(%i12) %iargs: false$
(%i13) tellsimp (sin(%i*x), %i*sinh(x));
    [sinrule1, simp-%sin]
(%i14) trigexpand (sin (%i*y + x));
    sin(x) cos(%i y) + %i cos(x) sinh(y)
(%i15) %iargs:true$
(%i16) errcatch(0^0);
    0
    0 has been generated
(%o6) []
(%i7) ev (tellsimp (0^0, 1), simp: false);
(%o7) ["rule1, simpexpt]
(%i8) 0^0;
(%o8) 1
(%i9) remrule ("^", %th(2)[1]);
(%o9) ^
(%i10) tellsimp (sin(x)^2, 1 - cos(x)^2);
(%o10) ["rule2, simpexpt]
(%i11) (1 + sin(x))^2;
(%o11) (sin(x) + 1)
(%i12) expand (%);
(%o12) 2 sin(x) - cos (x) + 2
(%i13) sin(x)^2;
(%o13) 1 - cos (x)
(%i14) kill (rules);
(%o14) done
(%i15) matchdeclare (a, true);
(%o15) done
(%i16) tellsimp (sin(a)^2, 1 - cos(a)^2);
(%o16) ["rule3, simpexpt]
(%i17) sin(y)^2;
(%o17) 1 - cos (y)

**tellsimpafter (pattern, replacement)**

[Function]

Defines a simplification rule which the Maxima simplifier applies after built-in simplification rules. *pattern* is an expression, comprising pattern variables (declared by *matchdeclare*) and other atoms and operators, considered literals for the purpose of pattern matching. *replacement* is substituted for an actual expression which matches *pattern*; pattern variables in *replacement* are assigned the values matched in the actual expression.

*pattern* may be any nonatomic expression in which the main operator is not a pattern variable; the simplification rule is associated with the main operator. The names of functions (with one exception, described below), lists, and arrays may appear in *pattern* as the main operator only as literals (not pattern variables); this rules out expressions such as *aa*(x) and *bb*[y] as patterns, if *aa* and *bb* are pattern variables. Names of functions, lists, and arrays which are pattern variables may appear as operators other than the main operator in *pattern*.

There is one exception to the above rule concerning names of functions. The name of a subscripted function in an expression such as *aa*[x](y) may be a pattern variable, because the main operator is not *aa* but rather the Lisp atom *mqapply*. This is a consequence of the representation of expressions involving subscripted functions.
Simplification rules are applied after evaluation (if not suppressed through quotation or the flag \texttt{noeval}). Rules established by \texttt{tellsimpafter} are applied in the order they were defined, and after any built-in rules. Rules are applied bottom-up, that is, applied first to subexpressions before application to the whole expression. It may be necessary to repeatedly simplify a result (for example, via the quote-quote operator \texttt{''} or the flag \texttt{infeval}) to ensure that all rules are applied.

Pattern variables are treated as local variables in simplification rules. Once a rule is defined, the value of a pattern variable does not affect the rule, and is not affected by the rule. An assignment to a pattern variable which results from a successful rule match does not affect the current assignment (or lack of it) of the pattern variable. However, as with all atoms in Maxima, the properties of pattern variables (as declared by \texttt{put} and related functions) are global.

The rule constructed by \texttt{tellsimpafter} is named after the main operator of \texttt{pattern}. Rules for built-in operators, and user-defined operators defined by \texttt{infix}, \texttt{prefix}, \texttt{postfix}, \texttt{matchfix}, and \texttt{nofix}, have names which are Lisp identifiers. Rules for other functions have names which are Maxima identifiers.

The treatment of noun and verb forms is slightly confused. If a rule is defined for a noun (or verb) form and a rule for the corresponding verb (or noun) form already exists, the newly-defined rule applies to both forms (noun and verb). If a rule for the corresponding verb (or noun) form does not exist, the newly-defined rule applies only to the noun (or verb) form.

The rule constructed by \texttt{tellsimpafter} is an ordinary Lisp function. If the name of the rule is \texttt{$foorule1$}, the construct \texttt{:lisp (trace $foorule1$)} traces the function, and \texttt{:lisp (symbol-function '$foorule1$')} displays its definition. \texttt{tellsimpafter} quotes its arguments. \texttt{tellsimpafter} returns the list of rules for the main operator of \texttt{pattern}, including the newly established rule.

See also \texttt{matchdeclare}, \texttt{defmatch}, \texttt{defrule}, \texttt{tellsimp}, \texttt{let}, \texttt{kill}, \texttt{remrule} and \texttt{clear_rules}.

Examples:

\texttt{pattern} may be any nonatomic expression in which the main operator is not a pattern variable.

(\%i1) \texttt{matchdeclare (aa, atom, [ll, mm], listp, xx, true)}$

(\%i2) \texttt{tellsimpafter (sin (ll), map (sin, ll))};

(\%i3) \texttt{sin ([1/6, 1/4, 1/3, 1/2, 1]*\pi)};

(\%i4) \texttt{tellsimpafter (ll^mm, map ("^", ll, mm))};

(\%i5) \texttt{[a, b, c]^[1, 2, 3]};

(\%i6) \texttt{tellsimpafter (foo (aa (xx)), aa (foo (xx))看出);

(\%o6) \texttt{[foorule1, false]}
(%i7) foo (bar (u - v));
(%o7) \text{bar}(\text{foo}(u - v))

Rules are applied in the order they were defined. If two rules can match an expression, the rule which was defined first is applied.

(%i11) matchdeclare (aa, integerp); 
(%o11) done 
(%i12) tellsimpafter (foo (aa), bar_1 (aa)); 
(%o12) [foorule1, false] 
(%i13) tellsimpafter (foo (aa), bar_2 (aa)); 
(%o13) [foorule2, foorule1, false] 
(%i14) foo (42); 
(%o14) bar_1(42) 

Pattern variables are treated as local variables in simplification rules. (Compare to defmatch, which treats pattern variables as global variables.)

(%i11) matchdeclare (aa, integerp, bb, atom); 
(%o11) done 
(%i12) tellsimpafter (foo(aa, bb), bar('aa=aa, 'bb=bb)); 
(%o12) [foorule1, false] 
(%i13) bb: 12345; 
(%o13) 12345 
(%i14) foo (42, %e); 
(%o14) bar(aa = 42, bb = %e) 
(%i15) bb; 
(%o15) 12345 

As with all atoms, properties of pattern variables are global even though values are local. In this example, an assignment property is declared via define_variable. This is a property of the atom bb throughout Maxima.

(%i11) matchdeclare (aa, integerp, bb, atom); 
(%o11) done 
(%i12) tellsimpafter (foo(aa, bb), bar('aa=aa, 'bb=bb)); 
(%o12) [foorule1, false] 
(%i13) foo (42, %e); 
(%o13) bar(aa = 42, bb = %e) 
(%i14) define_variable (bb, true, boolean); 
(%o14) true 
(%i15) foo (42, %e); 
Error: bb was declared mode boolean, has value: %e 
-- an error. Quitting. To debug this try debugmode(true); 

Rules are named after main operators. Names of rules for built-in and user-defined operators are Lisp identifiers, while names for other functions are Maxima identifiers.

(%i11) tellsimpafter (foo (%pi + %e), 3*%pi); 
(%o11) [foorule1, false] 
(%i12) tellsimpafter (foo (%pi * %e), 17*%e); 
(%o12) [foorule2, foorule1, false] 
(%i13) tellsimpafter (foo (%i ^ %e), -42*%i);
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(%i3) [foorule3, foorule2, foorule1, false]
(%i4) tellsimpafter (foo (9) + foo (13), quux (22));
(%o4) [+rule1, simplus]
(%i5) tellsimpafter (foo (9) * foo (13), blurf (22));
(%o5) [*rule1, simptimes]
(%i6) tellsimpafter (foo (9) ^ foo (13), mumble (22));
(%o6) [^rule1, simpexpt]
(%i7) rules;
(%o7) [foorule1, foorule2, foorule3, +rule1, *rule1, ^rule1]
(%i8) foorule_name: first (%o1);
(%o8) foorule1
(%i9) plusrule_name: first (%o4);
(%o9) +rule1
(%i10) remrule (foo, foorule1);
(%o10) foo
(%i11) remrule ("^", ?\^rule1);
(%o11) ~
(%i12) rules;
(%o12) [foorule2, foorule3, +rule1, *rule1]

A worked example: anticommutative multiplication.

(%i1) gt (i, j) := integerp(j) and i < j;
(%o1) gt(i, j) := integerp(j) and i < j
(%i2) matchdeclare (i, integerp, j, gt(i));
(%o2) done
(%i3) tellsimpafter (s[i]^^2, 1);
(%o3) [^^rule1, simpncexpt]
(%i4) tellsimpafter (s[i] . s[j], -s[j] . s[i]);
(%o4) [.rule1, simpnct]
(%i5) s[1] . (s[1] + s[2]);
(%o5) s . (s + s )
   1  2  1
(%i6) expand (%);
(%o6) 1 - s . s
   2  1
(%i7) factor (expand (sum (s[i], i, 0, 9)^^5));
(%o7) 100 (s + s + s + s + s + s + s + s + s + s )
   9  8  7  6  5  4  3  2  1  0

clear_rules ()
[Function]
Executes kill (rules) and then resets the next rule number to 1 for addition +, multiplication *, and exponentiation ^.
35 Sets

35.1 Introduction to Sets

Maxima provides set functions, such as intersection and union, for finite sets that are defined by explicit enumeration. Maxima treats lists and sets as distinct objects. This feature makes it possible to work with sets that have members that are either lists or sets.

In addition to functions for finite sets, Maxima provides some functions related to combinatorics; these include the Stirling numbers of the first and second kind, the Bell numbers, multinomial coefficients, partitions of nonnegative integers, and a few others. Maxima also defines a Kronecker delta function.

35.1.1 Usage

To construct a set with members \( a_1, \ldots, a_n \), write \( \text{set}(a_1, \ldots, a_n) \) or \( \{a_1, \ldots, a_n\} \); to construct the empty set, write \( \text{set}() \) or \( \emptyset \). In input, \( \text{set}(\ldots) \) and \( \{ \ldots \} \) are equivalent. Sets are always displayed with curly braces.

If a member is listed more than once, simplification eliminates the redundant member.

\[
\begin{align*}
(\%i1) & \quad \text{set}(); \\
(\%o1) & \quad \{\} \\
(\%i2) & \quad \text{set}(a, b, a); \\
(\%o2) & \quad \{a, b\} \\
(\%i3) & \quad \text{set}(a, \text{set}(b)); \\
(\%o3) & \quad \{a, \{b\}\} \\
(\%i4) & \quad \text{set}(a, [b]); \\
(\%o4) & \quad \{a, [b]\} \\
(\%i5) & \quad \text{set}(); \\
(\%o5) & \quad \{\} \\
(\%i6) & \quad \{a, b, a\}; \\
(\%o6) & \quad \{a, b\} \\
(\%i7) & \quad \{a, \{b\}\}; \\
(\%o7) & \quad \{a, \{b\}\} \\
(\%i8) & \quad \{a, [b]\}; \\
(\%o8) & \quad \{a, [b]\}
\end{align*}
\]

Two would-be elements \( x \) and \( y \) are redundant (i.e., considered the same for the purpose of set construction) if and only if \( \text{is}(x = y) \) yields true. Note that \( \text{is}(\text{equal}(x, y)) \) can yield true while \( \text{is}(x = y) \) yields false; in that case the elements \( x \) and \( y \) are considered distinct.

\[
\begin{align*}
(\%i1) & \quad x: a/c + b/c; \\
(\%o1) & \quad \begin{array}{c}
  b \\
  a \\
  - + \\
  c \\
  c
\end{array} \\
(\%i2) & \quad y: a/c + b/c; \\
(\%o2) & \quad \begin{array}{c}
  b \\
  a \\
  - + \\
  c \\
  c
\end{array}
\end{align*}
\]
\(z: \frac{a + b}{c};\)

\[
\begin{align*}
\frac{b + a}{c}
\end{align*}
\]

\(\text{is } (x = y);\)

true

\(\text{is } (y = z);\)

false

\(\text{is } \text{equal} (y, z);\)

true

\(y - z;\)

\[
\begin{align*}
\frac{b + a}{c} - \frac{b}{c} + \frac{a}{c}
\end{align*}
\]

\(\text{ratsimp } \%;\)

0

\(\{x, y, z\};\)

\[
\begin{align*}
\frac{b + a}{c} - \frac{b}{c} + \frac{a}{c}
\end{align*}
\]

To construct a set from the elements of a list, use \texttt{setify}.

\(\text{setify } ([b, a]);\)

\{a, b\}

Set members \(x\) and \(y\) are equal provided \(\text{is}(x = y)\) evaluates to true. Thus \(\text{rat}(x)\) and \(x\) are equal as set members; consequently,

\(\{x, \text{rat}(x)\};\)

\{x\}

Further, since \(\text{is}((x - 1)*(x + 1) = x^2 - 1)\) evaluates to false, \((x - 1)*(x + 1)\) and \(x^2 - 1\) are distinct set members; thus

\(\{(x - 1)*(x + 1), x^2 - 1\};\)

\(\{(x - 1)*(x + 1), x^2 - 1\}\)

To reduce this set to a singleton set, apply \texttt{rat} to each set member:

\(\{(x - 1)*(x + 1), x^2 - 1\};\)

\(\{(x - 1)*(x + 1), x^2 - 1\}\)

\(\text{map } (\text{rat}, \%);\)

\(\{(x - 1), x - 1\}\)

To remove redundancies from other sets, you may need to use other simplification functions. Here is an example that uses \texttt{trigsimp}:

\(\{1, \cos(x)^2 + \sin(x)^2\};\)

\(\{1, \cos(x)^2 + \sin(x)^2\}\)

\(\text{map } (\text{trigsimp}, \%);\)
A set is simplified when its members are non-redundant and sorted. The current version of the set functions uses the Maxima function `orderlessp` to order sets; however, future versions of the set functions might use a different ordering function.

Some operations on sets, such as substitution, automatically force a re-simplification; for example,

```maxima
(%i1) s: {a, b, c}$
(%i2) subst (c=a, s);
(%o2) {a, b}
(%i3) subst ([a=x, b=x, c=x], s);
(%o3) {x}
(%i4) map (lambda ([x], x^2), set (-1, 0, 1));
(%o4) {0, 1}
```

Maxima treats lists and sets as distinct objects; functions such as `union` and `intersection` complain if any argument is not a set. If you need to apply a set function to a list, use the `setify` function to convert it to a set. Thus

```maxima
(%i1) union ([1, 2], {a, b});
Function union expects a set, instead found [1,2]
-- an error. Quitting. To debug this try debugmode(true);
(%i2) union (setify ([1, 2]), {a, b});
(%o2) {1, 2, a, b}
```

To extract all set elements of a set \(s\) that satisfy a predicate \(f\), use `subset(s, f)`. (A **predicate** is a boolean-valued function.) For example, to find the equations in a given set that do not depend on a variable \(z\), use

```maxima
(%i1) subset ({x + y + z, x - y + 4, x + y - 5}, lambda ([e], freeof (z, e)));
(%o1) {- y + x + 4, y + x - 5}
```

The section **Section 35.2 [Functions and Variables for Sets]**, page 568, has a complete list of the set functions in Maxima.

### 35.1.2 Set Member Iteration

There two ways to to iterate over set members. One way is the use `map`; for example:

```maxima
(%i1) map (f, {a, b, c});
(%o1) {f(a), f(b), f(c)}
```

The other way is to use `for x in s do`

```maxima
(%i1) s: {a, b, c};
(%o1) {a, b, c}
(%i2) for si in s do print (concat (si, 1));
a1
b1
c1
(%o2) done
```

The Maxima functions `first` and `rest` work correctly on sets. Applied to a set, `first` returns the first displayed element of a set; which element that is may be implementation-dependent. If \(s\) is a set, then `rest(s)` is equivalent to `disjoin(first(s), s)`. Currently,
there are other Maxima functions that work correctly on sets. In future versions of the set
functions, first and rest may function differently or not at all.

Maxima’s orderless and ordergreat mechanisms are incompatible with the set func-
tions. If you need to use either orderless or ordergreat, call those functions before
constructing any sets, and do not call unorder.

35.1.3 Authors
Stavros Macrakis of Cambridge, Massachusetts and Barton Willis of the University of Ne-
braska at Kearney (UNK) wrote the Maxima set functions and their documentation.

35.2 Functions and Variables for Sets

\textbf{adjoin} (x, a) \hspace{1cm} \text{[Function]}
\begin{itemize}
\item Returns the union of the set a with \{x\}.
\item adjoin complains if a is not a literal set.
\item adjoin(x, a) and \text{union}(\text{set}(x), a) are equivalent; however, \text{adjoin} may be some-
what faster than \text{union}.
\end{itemize}
See also \text{disjoin}.

Examples:
\begin{verbatim}
(%i1) adjoin (c, {a, b});
(%o1) {a, b, c}
(%i2) adjoin (a, {a, b});
(%o2) {a, b}
\end{verbatim}

\textbf{belln} (n) \hspace{1cm} \text{[Function]}
\begin{itemize}
\item Represents the \(n\)-th Bell number. belln(n) is the number of partitions of a set with \(n\)
members.
\item For nonnegative integers \(n\), belln(n) simplifies to the \(n\)-th Bell number. belln does
not simplify for any other arguments.
\item belln distributes over equations, lists, matrices, and sets.
\end{itemize}
Examples:
\begin{verbatim}
belln applied to nonnegative integers.
(%i1) makelist (belln (i), i, 0, 6);
(%o1) [1, 1, 2, 5, 15, 52, 203]
(%i2) is (cardinality (set_partitions ({})) = belln (0));
(%o2) true
(%i3) is (cardinality (set_partitions ({1, 2, 3, 4, 5, 6})) = belln (6));
(%o3) true
\end{verbatim}
belln applied to arguments which are not nonnegative integers.
\begin{verbatim}
(%i1) [belln (x), belln (sqrt(3)), belln (-9)];
(%o1) [belln(x), belln(sqrt(3)), belln(- 9)]
\end{verbatim}
cardinality (a)

Returns the number of distinct elements of the set a.

cardinality ignores redundant elements even when simplification is disabled.

Examples:

\[
\begin{align*}
%i1 & : \text{cardinality (\{\})}; \\
%o1 & : 0 \\
%i2 & : \text{cardinality (\{a, a, b, c\})}; \\
%o2 & : 3 \\
%i3 & : \text{simp : false}; \\
%o3 & : \text{false} \\
%i4 & : \text{cardinality (\{a, a, b, c\})}; \\
%o4 & : 3
\end{align*}
\]

cartesian_product (b_1, ... , b_n)

Returns a set of lists of the form \([x_1, \ldots, x_n]\), where \(x_1, \ldots, x_n\) are elements of the sets \(b_1, \ldots, b_n\), respectively.

cartesian_product complains if any argument is not a literal set.

Examples:

\[
\begin{align*}
%i1 & : \text{cartesian_product (\{0, 1\})}; \\
%o1 & : \{[0], [1]\} \\
%i2 & : \text{cartesian_product (\{0, 1\}, \{0, 1\})}; \\
%o2 & : \{[0, 0], [0, 1], [1, 0], [1, 1]\} \\
%i3 & : \text{cartesian_product (\{x\}, \{y\}, \{z\})}; \\
%o3 & : \{[x, y, z]\} \\
%i4 & : \text{cartesian_product (\{x\}, \{-1, 0, 1\})}; \\
%o4 & : \{[x, -1], [x, 0], [x, 1]\}
\end{align*}
\]

disjoin (x, a)

Returns the set a without the member x. If x is not a member of a, return a unchanged.

disjoin complains if a is not a literal set.

disjoin(x, a), delete(x, a), and setdifference(a, set(x)) are all equivalent. Of these, disjoin is generally faster than the others.

Examples:

\[
\begin{align*}
%i1 & : \text{disjoin (a, \{a, b, c, d\})}; \\
%o1 & : \{b, c, d\} \\
%i2 & : \text{disjoin (a + b, \{5, z, a + b, \%pi\})}; \\
%o2 & : \{5, \%pi, z\} \\
%i3 & : \text{disjoin (a - b, \{5, z, a + b, \%pi\})}; \\
%o3 & : \{5, \%pi, b + a, z\}
\end{align*}
\]

disjointp (a, b)

Returns true if and only if the sets a and b are disjoint.

disjointp complains if either a or b is not a literal set.
Examples:

(%i1) disjointp ({a, b, c}, {1, 2, 3});
(%o1) true
(%i2) disjointp ({a, b, 3}, {1, 2, 3});
(%o2) false

\[\text{divisors (n)}\]

Represents the set of divisors of \(n\).

\(\text{divisors(n)}\) simplifies to a set of integers when \(n\) is a nonzero integer. The set of divisors includes the members 1 and \(n\). The divisors of a negative integer are the divisors of its absolute value.

\(\text{divisors}\) distributes over equations, lists, matrices, and sets.

Examples:

We can verify that 28 is a perfect number: the sum of its divisors (except for itself) is 28.

(%i1) s: divisors(28);
(%o1) \{1, 2, 4, 7, 14, 28\}
(%i2) lreduce ("+", args(s)) - 28;
(%o2) 28

\(\text{divisors}\) is a simplifying function. Substituting 8 for \(a\) in \(\text{divisors(a)}\) yields the divisors without reevaluating \(\text{divisors(8)}\).

(%i1) divisors (a);
(%o1) \text{divisors(a)}
(%i2) subst (8, a, %);
(%o2) \{1, 2, 4, 8\}

\(\text{divisors}\) distributes over equations, lists, matrices, and sets.

(%i1) divisors (a = b);
(%o1) \text{divisors(a) = divisors(b)}
(%i2) divisors ([a, b, c]);
(%o2) \{\text{divisors(a), divisors(b), divisors(c)}\}
(%i3) divisors (matrix ([a, b], [c, d]));
(%o3) [[\text{divisors(a) divisors(b)}],
\text{[divisors(c) divisors(d)]}]
(%i4) divisors ({a, b, c});
(%o4) \{\text{divisors(a), divisors(b), divisors(c)}\}

\[\text{elementp (x, a)}\]

Returns \text{true} if and only if \(x\) is a member of the set \(a\).

\(\text{elementp}\) complains if \(a\) is not a literal set.

Examples:

(%i1) elementp (sin(1), {sin(1), sin(2), sin(3)});
(%o1) \text{true}
(%i2) elementp (sin(1), {cos(1), cos(2), cos(3)});
(%o2) \text{false}
emptyp (a)

Return true if and only if a is the empty set or the empty list.

Examples:

(%i1) map (emptyp, [{}, []]);
(%o1) [true, true]
(%i2) map (emptyp, [a + b, {}, %pi]);
(%o2) [false, false, false]

equiv_classes (s, F)

Returns a set of the equivalence classes of the set s with respect to the equivalence relation F.

F is a function of two variables defined on the Cartesian product of s with s. The return value of F is either true or false, or an expression expr such that is(expr) is either true or false.

When F is not an equivalence relation, equiv_classes accepts it without complaint, but the result is generally incorrect in that case.

Examples:

The equivalence relation is a lambda expression which returns true or false.

(%i1) equiv_classes ({1, 1.0, 2, 2.0, 3, 3.0},
            lambda ([x, y], is (equal (x, y))));
(%o1) {{1, 1.0}, {2, 2.0}, {3, 3.0}}

The equivalence relation is the name of a relational function which evaluates to true or false.

(%i1) equiv_classes ({1, 1.0, 2, 2.0, 3, 3.0}, equal);
(%o1) {{1, 1.0}, {2, 2.0}, {3, 3.0}}

The equivalence classes are numbers which differ by a multiple of 3.

(%i1) equiv_classes ({1, 2, 3, 4, 5, 6, 7},
            lambda ([x, y], remainder (x - y, 3) = 0));
(%o1) {{1, 4, 7}, {2, 5}, {3, 6}}

every

every (f, s)
every (f, L_1, ..., L_n)

Returns true if the predicate f is true for all given arguments.

Given one set as the second argument, every(f, s) returns true if is(f(a_i)) returns true for all a_i in s. every may or may not evaluate f for all a_i in s. Since sets are unordered, every may evaluate f(a_i) in any order.

Given one or more lists as arguments, every(f, L_1, ..., L_n) returns true if is(f(x_i, ..., x_n)) returns true for all x_i, ..., x_n in L_1, ..., L_n, respectively. every may or may not evaluate f for every combination x_1, ..., x_n. every evaluates lists in the order of increasing index.

Given an empty set {} or empty lists [] as arguments, every returns true.

When the global flag maperror is true, all lists L_1, ..., L_n must have equal lengths. When maperror is false, list arguments are effectively truncated to the length of the shortest list.
Return values of the predicate $f$ which evaluate (via `is`) to something other than `true` or `false` are governed by the global flag `prederror`. When `prederror` is `true`, such values are treated as `false`, and the return value from `every` is `false`. When `prederror` is `false`, such values are treated as `unknown`, and the return value from `every` is `unknown`.

Examples:

`every` applied to a single set. The predicate is a function of one argument.

```
(%i1) every (integerp, {1, 2, 3, 4, 5, 6});
(%o1) true
(%i2) every (atom, {1, 2, sin(3), 4, 5 + y, 6});
(%o2) false
```

`every` applied to two lists. The predicate is a function of two arguments.

```
(%i1) every ("=", [a, b, c], [a, b, c]);
(%o1) true
(%i2) every ("#", [a, b, c], [a, b, c]);
(%o2) false
```

`every` applied to a single set. The predicate is a function of one argument.

```
(%i1) prederror : false;
(%o1) false
(%i2) map (lambda ([a, b], is (a < b)), [x, y, z], [x^2, y^2, z^2]);
(%o2) [unknown, unknown, unknown]
(%i3) every ("<", [x, y, z], [x^2, y^2, z^2]);
(%o3) unknown
(%i4) prederror : true;
(%o4) true
(%i5) every ("<", [x, y, z], [x^2, y^2, z^2]);
(%o5) false
```

`extremal_subset`

```
extremal_subset (s, f, max)
extremal_subset (s, f, min)
```

Returns the subset of $s$ for which the function $f$ takes on maximum or minimum values.

`extremal_subset(s, f, max)` returns the subset of the set or list $s$ for which the real-valued function $f$ takes on its maximum value.

`extremal_subset(s, f, min)` returns the subset of the set or list $s$ for which the real-valued function $f$ takes on its minimum value.

Examples:

```
(%i1) extremal_subset ({-2, -1, 0, 1, 2}, abs, max);
(%o1) {-2, 2}
(%i2) extremal_subset ({sqrt(2), 1.57, %pi/2}, sin, min);
(%o2) {sqrt(2)}
```
flattened (expr)

Collects arguments of subexpressions which have the same operator as expr and constructs an expression from these collected arguments.

Subexpressions in which the operator is different from the main operator of expr are copied without modification, even if they, in turn, contain some subexpressions in which the operator is the same as for expr.

It may be possible for flatten to construct expressions in which the number of arguments differs from the declared arguments for an operator; this may provoke an error message from the simplifier or evaluator. flatten does not try to detect such situations.

Expressions with special representations, for example, canonical rational expressions (CRE), cannot be flattened; in such cases, flatten returns its argument unchanged.

Examples:

Applied to a list, flatten gathers all list elements that are lists.

(%i1) flatten ([a, b, [c, [d, e], f], [[g, h]], i, j]);
(%o1) [a, b, c, d, e, f, g, h, i, j]

Applied to a set, flatten gathers all members of set elements that are sets.

(%i1) flatten ({a, {b}, {{c}}});
(%o1) {a, b, c}
(%i2) flatten ({a, {[a]}, {a}});
(%o2) {a, [a]}

flatten is similar to the effect of declaring the main operator n-ary. However, flatten has no effect on subexpressions which have an operator different from the main operator, while an n-ary declaration affects those.

(%i1) expr: flatten (f (g (f (f (x)))));
(%o1) f(f(f(x)))
(%i2) declare (f, nary);
(%o2) done
(%i3) ev (expr);
(%o3) f(f(f(x)))

flatten treats subscripted functions the same as any other operator.

(%i1) flatten (f[5] (f[5] (x, y), z));
(%o1) f (x, y, z)

It may be possible for flatten to construct expressions in which the number of arguments differs from the declared arguments for an operator;

(%i1) 'mod (5, 'mod (7, 4));
(%o1) mod(5, mod(7, 4))
(%i2) flatten (%);
(%o2) mod(5, 7, 4)
(%i3) ''%, nouns;
Wrong number of arguments to mod
-- an error. Quitting. To debug this try debugmode(true);
full_listify (a)  [Function]
Replaces every set operator in a by a list operator, and returns the result. full_listify replaces set operators in nested subexpressions, even if the main operator is not set.

listify replaces only the main operator.

Examples:

(%i1) full_listify ({a, b, {c, d, e, f}, g});
(%o1) [a, b, [c, [d, e, f], g]]
(%i2) full_listify (F (G ({a, b, H({c, d, e})})));
(%o2) F(G([a, b, H([c, d, e])])

fullsetify (a)  [Function]
When a is a list, replaces the list operator with a set operator, and applies fullsetify to each member which is a set. When a is not a list, it is returned unchanged.

setify replaces only the main operator.

Examples:

In line (%o2), the argument of f isn’t converted to a set because the main operator of f([b]) isn’t a list.

(%i1) fullsetify ([a, [a]]);
(%o1) {a, {a}}
(%i2) fullsetify ([a, f([b])]);
(%o2) {a, f([b])}

identity (x)  [Function]
Returns x for any argument x.

Examples:

identity may be used as a predicate when the arguments are already Boolean values.

(%i1) every (identity, [true, true]);
(%o1) true

integer_partitions  [Function]

integer_partitions (n)
integer_partitions (n, len)

Returns integer partitions of n, that is, lists of integers which sum to n.

integer_partitions(n) returns the set of all partitions of the integer n. Each partition is a list sorted from greatest to least.

integer_partitions(n, len) returns all partitions that have length len or less; in this case, zeros are appended to each partition with fewer than len terms to make each partition have exactly len terms. Each partition is a list sorted from greatest to least.

A list [a_1, ..., a_m] is a partition of a nonnegative integer n when (1) each a_i is a nonzero integer, and (2) a_1 + ... + a_m = n. Thus 0 has no partitions.

Examples:

(%i1) integer_partitions (3);
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(\%o1) \{[1, 1, 1], [2, 1], [3]\}
(\%i2) s: integer_partitions (25)$
(\%i3) cardinality (s);
(\%o3) 1958
(\%i4) map (lambda ([x], apply ("+", x)), s);
(\%o4) \{25\}
(\%i5) integer_partitions (5, 3);
(\%o5) \{[2, 2, 1], [3, 1, 1], [3, 2, 0], [4, 1, 0], [5, 0, 0]\}
(\%i6) integer_partitions (5, 2);
(\%o6) \{[3, 2], [4, 1], [5, 0]\}

To find all partitions that satisfy a condition, use the function \texttt{subset}; here is an example that finds all partitions of 10 that consist of prime numbers.

(\%i1) s: integer_partitions (10)$
(\%i2) cardinality (s);
(\%o2) 42
(\%i3) xprimep(x) := integerp(x) and (x > 1) and primep(x)$
(\%i4) subset (s, lambda ([x], every (xprimep, x)));
(\%o4) \{[2, 2, 2, 2, 2], [3, 3, 2, 2], [5, 3, 2], [5, 5], [7, 3]\}

\texttt{intersect (a_1, ..., a_n)}

\texttt{intersect} is the same as \texttt{intersection}, which see.

\texttt{intersection (a_1, ..., a_n)}

Returns a set containing the elements that are common to the sets \texttt{a_1} through \texttt{a_n}.

\texttt{intersection} complains if any argument is not a literal set.

Examples:

(\%i1) S_1 : \{a, b, c, d\};
(\%o1) \{a, b, c, d\}
(\%i2) S_2 : \{d, e, f, g\};
(\%o2) \{d, e, f, g\}
(\%i3) S_3 : \{c, d, e, f\};
(\%o3) \{c, d, e, f\}
(\%i4) S_4 : \{u, v, w\};
(\%o4) \{u, v, w\}
(\%i5) intersection (S_1, S_2);
(\%o5) \{d\}
(\%i6) intersection (S_2, S_3);
(\%o6) \{d, e, f\}
(\%i7) intersection (S_1, S_2, S_3);
(\%o7) \{d\}
(\%i8) intersection (S_1, S_2, S_3, S_4);
(\%o8) \{\}

\texttt{kron_delta (x_1, x_2, \ldots, x_p)}

Represents the Kronecker delta function.

\texttt{kron_delta} simplifies to 1 when \texttt{x_i} and \texttt{y_j} are equal for all pairs of arguments, and it simplifies to 0 when \texttt{x_i} and \texttt{y_j} are not equal for some pair of arguments. Equality is
determined using \texttt{is(equal(xi,xj))} and inequality by \texttt{is(notequal(xi,xj))}. For exactly one argument, \texttt{kron_delta} signals an error.

Examples:

\begin{verbatim}
(%i1) kron_delta(a,a);
1
(%i2) kron_delta(a,b,a,b);
\texttt{kron_delta}(a, b)
(%i3) kron_delta(a,a,b,a+1);
0
(%i4) assume(equal(x,y));
\texttt{[equal(x, y)]}
(%i5) kron_delta(x,y);
1
\end{verbatim}

\textbf{listify} (a)

Returns a list containing the members of a when a is a set. Otherwise, \texttt{listify} returns a.

\texttt{full_listify} replaces all set operators in a by list operators.

Examples:

\begin{verbatim}
(%i1) listify ({a, b, c, d});
\texttt{[a, b, c, d]}
(%i2) listify (F ({a, b, c, d}));
\texttt{F({a, b, c, d})}
\end{verbatim}

\textbf{makeset} (expr, x, s)

Returns a set with members generated from the expression expr, where x is a list of variables in expr, and s is a set or list of lists. To generate each set member, expr is evaluated with the variables x bound in parallel to a member of s.

Each member of s must have the same length as x. The list of variables x must be a list of symbols, without subscripts. Even if there is only one symbol, x must be a list of one element, and each member of s must be a list of one element.

See also \texttt{makelist}.

Examples:

\begin{verbatim}
(%i11) makeset (i/j, [i, j], [[1, a], [2, b], [3, c], [4, d]]);
\texttt{[1, 2, 3, 4]}
(%i12) makeset (i, a b c d)
\end{verbatim}
(\%i4) makeset (i + j + k, [i, j, k], S3);
(\%o4) \{3 \times, 3 \times, y + \times, 3 z, z + 2 \times, z + y + \times, 
z + 2 \times, 2 z + \times, 2 z + y\}
(\%i5) makeset (\sin(x), [x], \{[1], [2], [3]\});
(\%o5) \{\sin(1), \sin(2), \sin(3)\}

\textbf{moebius (n)}

Represents the Moebius function.

When \(n\) is product of \(k\) distinct primes, \textit{moebius} \((n)\) simplifies to \((-1)^k\); when \(n = 1\), it simplifies to 1; and it simplifies to 0 for all other positive integers.

\textit{moebius} distributes over equations, lists, matrices, and sets.

Examples:

(\%i1) moebius (1);
(\%o1) 1
(\%i2) moebius (2 * 3 * 5);
(\%o2) -1
(\%i3) moebius (11 * 17 * 29 * 31);
(\%o3) 1
(\%i4) moebius (2^32);
(\%o4) 0
(\%i5) moebius (n);
(\%o5) moebius(n)
(\%i6) moebius (n = 12);
(\%o6) moebius(n) = 0
(\%i7) moebius ([11, 11 * 13, 11 * 13 * 15]);
(\%o7) [-1, 1, 1]
(\%i8) moebius (matrix ([11, 12], [13, 14]));
(\%o8) [ -1 0 ]
\hspace{1cm} [ ]
\hspace{1cm} [ -1 1 ]
(\%i9) moebius ({21, 22, 23, 24});
(\%o9) {-1, 0, 1}

\textbf{multinomial_coeff}

\textit{multinomial_coeff (a_1, ..., a_n)}
\textit{multinomial_coeff ()}

Returns the multinomial coefficient.

When each \(a_k\) is a nonnegative integer, the multinomial coefficient gives the number of ways of placing \(a_1 + \ldots + a_n\) distinct objects into \(n\) boxes with \(a_k\) elements in the \(k\)th box. In general, \textit{multinomial_coeff (a_1, ..., a_n)} evaluates to \((a_1 + \ldots + a_n)!/(a_1! \ldots a_n!)\).

\textit{multinomial_coeff()} (with no arguments) evaluates to 1.

\textit{minfactorial} may be able to simplify the value returned by \textit{multinomial_coeff}.

Examples:

(\%i1) multinomial_coeff (1, 2, x);
\[(x + 3)! \quad \frac{\text{---------}}{x!} \]

\[
(%o1) \quad \frac{(x + 1) (x + 2) (x + 3)}{2}
\]

\[
(%i2) \quad \text{minfactorial}(\%);
\]

\[
(%o2) \quad \frac{2}{(x + 1) (x + 2) (x + 3)}
\]

\[
(%i3) \quad \text{multinomial}\_\text{coeff}(-6, 2);
\]

\[
(%o3) \quad \frac{(-4)!}{2 (-6)!}
\]

\[
(%i4) \quad \text{minfactorial}(\%);
\]

\[
(%o4) \quad 10
\]

\textbf{num\_distinct\_partitions}

\textbf{num\_distinct\_partitions(n)}

\textbf{num\_distinct\_partitions(n, list)}

Returns the number of distinct integer partitions of \(n\) when \(n\) is a nonnegative integer. Otherwise, \textbf{num\_distinct\_partitions} returns a noun expression.

\textbf{num\_distinct\_partitions(n, list)} returns a list of the number of distinct partitions of 1, 2, 3, ..., \(n\).

A distinct partition of \(n\) is a list of distinct positive integers \(k_1, \ldots, k_m\) such that \(n = k_1 + \ldots + k_m\).

Examples:

\[
(%i1) \quad \text{num\_distinct\_partitions}(12);
\]

\[
(%o1) \quad 15
\]

\[
(%i2) \quad \text{num\_distinct\_partitions}(12, \text{list});
\]

\[
(%o2) \quad [1, 1, 1, 2, 2, 3, 4, 5, 6, 8, 10, 12, 15]
\]

\[
(%i3) \quad \text{num\_distinct\_partitions}(n);
\]

\[
(%o3) \quad \text{num\_distinct\_partitions}(n)
\]

\textbf{num\_partitions}

\textbf{num\_partitions(n)}

\textbf{num\_partitions(n, list)}

Returns the number of integer partitions of \(n\) when \(n\) is a nonnegative integer. Otherwise, \textbf{num\_partitions} returns a noun expression.

\textbf{num\_partitions(n, list)} returns a list of the number of integer partitions of 1, 2, 3, ..., \(n\).

For a nonnegative integer \(n\), \textbf{num\_partitions(n)} is equal to \textbf{cardinality(integer\_partitions(n))}; however, \textbf{num\_partitions} does not actually construct the set of partitions, so it is much faster.

Examples:

\[
(%i1) \quad \text{num\_partitions}(5) = \text{cardinality}(\text{integer\_partitions}(5));
\]

\[
(%o1) \quad 7 = 7
\]

\[
(%i2) \quad \text{num\_partitions}(8, \text{list});
\]

\[
(%o2) \quad [1, 1, 2, 3, 5, 7, 11, 15, 22]
\]
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(%i3) num_partitions (n);  
(%o3) num_partitions(n)

**partition_set (a, f)**  
Partitions the set a according to the predicate f.

**partition_set** returns a list of two sets. The first set comprises the elements of a for which f evaluates to false, and the second comprises any other elements of a. **partition_set** does not apply is to the return value of f.

**partition_set** complains if a is not a literal set.

See also **subset**.

Examples:

(%i1) partition_set ({2, 7, 1, 8, 2, 8}, evenp);  
(%o1) [{1, 7}, {2, 8}]

(%i2) partition_set ({x, rat(y), rat(y) + z, 1},  
lambda ([x], ratp(x)));  
(%o2) [{1, x}, {y, y + z}]

**permutations (a)**  
Returns a set of all distinct permutations of the members of the list or set a. Each permutation is a list, not a set.

When a is a list, duplicate members of a are included in the permutations.

**permutations** complains if a is not a literal list or set.

See also **random_permutation**.

Examples:

(%i1) permutations ([a, a]);  
(%o1) {[a, a]}

(%i2) permutations ([a, a, b]);  
(%o2) {[a, a, b], [a, b, a], [b, a, a]}

**powerset (a)**  
**powerset (a, n)**

Returns the set of all subsets of a, or a subset of that set.

**powerset(a)** returns the set of all subsets of the set a. **powerset(a)** has \(2^{\text{cardinality}(a)}\) members.

**powerset(a, n)** returns the set of all subsets of a that have cardinality n.

**powerset** complains if a is not a literal set, or if n is not a nonnegative integer.

Examples:

(%i1) powerset ({a, b, c});  
(%o1) {{}, {a}, {a, b}, {a, b, c}, {a, c}, {b}, {b, c}, {c}}

(%i2) powerset ({w, x, y, z}, 4);  
(%o2) {{w, x, y, z}}

(%i3) powerset ({w, x, y, z}, 3);  
(%o3) {{w, x, y}, {w, x, z}, {w, y, z}, {x, y, z}}
random_permutation (a)

Returns a random permutation of the set or list a, as constructed by the Knuth shuffle algorithm.

The return value is a new list, which is distinct from the argument even if all elements happen to be the same. However, the elements of the argument are not copied.

Examples:

```
(%i1) random_permutation ([a, b, c, 1, 2, 3]);
(%o1) [c, 1, 2, 3, a, b]
(%i2) random_permutation ([a, b, c, 1, 2, 3]);
(%o2) [b, 3, 1, c, a, 2]
(%i3) random_permutation ({x + 1, y + 2, z + 3});
(%o3) [y + 2, z + 3, x + 1]
(%i4) random_permutation ({x + 1, y + 2, z + 3});
(%o4) [x + 1, y + 2, z + 3]
```

setdifference (a, b)

Returns a set containing the elements in the set a that are not in the set b.

setdifference complains if either a or b is not a literal set.

Examples:

```
(%i1) S_1 : {a, b, c, x, y, z};
(%o1) {a, b, c, x, y, z}
(%i2) S_2 : {aa, bb, c, x, y, zz};
(%o2) {aa, bb, c, x, y, zz}
(%i3) setdifference (S_1, S_2);
(%o3) {a, b, z}
(%i4) setdifference (S_2, S_1);
(%o4) {aa, bb, zz}
(%i5) setdifference (S_1, S_1);
(%o5) {}  
(%i6) setdifference (S_1, {});
(%o6) {a, b, c, x, y, z}
(%i7) setdifference ({}, S_1);
(%o7) {}
```

setequalp (a, b)

Returns true if sets a and b have the same number of elements and is(x = y) is true for x in the elements of a and y in the elements of b, considered in the order determined by listify. Otherwise, setequalp returns false.

Examples:

```
(%i1) setequalp ({1, 2, 3}, {1, 2, 3});
```
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(%i1) true
(%i2) setequalp ({a, b, c}, {1, 2, 3});
(%o2) false
(%i3) setequalp ({x^2 - y^2}, {(x + y) * (x - y)});
(%o3) false

setify (a)

Constructs a set from the elements of the list a. Duplicate elements of the list a are deleted and the elements are sorted according to the predicate orderlessp. setify complains if a is not a literal list.

Examples:

(%i1) setify ([1, 2, 3, a, b, c]);
(%o1) {1, 2, 3, a, b, c}
(%i2) setify ([a, b, c, a, b, c]);
(%o2) {a, b, c}
(%i3) setify ([7, 13, 11, 1, 3, 9, 5]);
(%o3) {1, 3, 5, 7, 9, 11, 13}

setp (a)

Returns true if and only if a is a Maxima set.

setp returns true for unsimplified sets (that is, sets with redundant members) as well as simplified sets.

setp is equivalent to the Maxima function setp(a) := not atom(a) and op(a) = 'set.

Examples:

(%i1) simp : false;
(%o1) false
(%i2) {a, a, a};
(%o2) {a, a, a}
(%i3) setp (%);
(%o3) true

set_partitions

set_partitions (a)
set_partitions (a, n)

Returns the set of all partitions of a, or a subset of that set.

set_partitions(a, n) returns a set of all decompositions of a into n nonempty disjoint subsets.

set_partitions(a) returns the set of all partitions.

stirling2 returns the cardinality of the set of partitions of a set.

A set of sets P is a partition of a set S when
1. each member of P is a nonempty set,
2. distinct members of P are disjoint,
3. the union of the members of P equals S.
Examples:

The empty set is a partition of itself, the conditions 1 and 2 being vacuously true.

\begin{verbatim}
(%i1) set_partitions ({});
(%o1) {{}}
\end{verbatim}

The cardinality of the set of partitions of a set can be found using \texttt{stirling2}.

\begin{verbatim}
(%i1) s: {0, 1, 2, 3, 4, 5}$
(%i2) p: set_partitions (s, 3)$
(%i3) cardinality(p) = stirling2 (6, 3);
(%o3) 90 = 90
\end{verbatim}

Each member of \( p \) should have \( n = 3 \) members; let’s check.

\begin{verbatim}
(%i1) s: {0, 1, 2, 3, 4, 5}$
(%i2) p: set_partitions (s, 3)$
(%i3) map (cardinality, p);
(%o3) {3}
\end{verbatim}

Finally, for each member of \( p \), the union of its members should equal \( s \); again let’s check.

\begin{verbatim}
(%i1) s: {0, 1, 2, 3, 4, 5}$
(%i2) p: set_partitions (s, 3)$
(%i3) map (lambda ([x], apply (union, listify (x))), p);
(%o3) {{0, 1, 2, 3, 4, 5}}
\end{verbatim}

\[ \text{some} \quad \text{[Function]} \]

\[ \text{some} (f, a) \]
\[ \text{some} (f, L_1, ..., L_n) \]

Returns \texttt{true} if the predicate \( f \) is \texttt{true} for one or more given arguments.

Given one set as the second argument, \texttt{some}(f, s) returns \texttt{true} if \texttt{is}(f(a_i)) returns \texttt{true} for one or more \( a_i \) in \( s \). \texttt{some} may or may not evaluate \( f \) for all \( a_i \) in \( s \). Since sets are unordered, \texttt{some} may evaluate \( f(a_i) \) in any order.

Given one or more lists as arguments, \texttt{some}(f, L_1, ..., L_n) returns \texttt{true} if \texttt{is}(f(x_1, ..., x_n)) returns \texttt{true} for one or more \( x_1, ..., x_n \) in \( L_1, ..., L_n \), respectively. \texttt{some} may or may not evaluate \( f \) for some combinations \( x_1, ..., x_n \). \texttt{some} evaluates lists in the order of increasing index.

Given an empty set {} or empty lists [] as arguments, \texttt{some} returns \texttt{false}.

When the global flag \texttt{maperror} is \texttt{true}, all lists \( L_1, ..., L_n \) must have equal lengths. When \texttt{maperror} is \texttt{false}, list arguments are effectively truncated to the length of the shortest list.

Return values of the predicate \( f \) which evaluate (via \texttt{is}) to something other than \texttt{true} or \texttt{false} are governed by the global flag \texttt{prederror}. When \texttt{prederror} is \texttt{true}, such values are treated as \texttt{false}. When \texttt{prederror} is \texttt{false}, such values are treated as \texttt{unknown}.

Examples:

\texttt{some} applied to a single set. The predicate is a function of one argument.

\begin{verbatim}
(%i1) some (integerp, {1, 2, 3, 4, 5, 6});
(%o1) true
\end{verbatim}
some applied to two lists. The predicate is a function of two arguments.

Return values of the predicate $f$ which evaluate to something other than true or false are governed by the global flag prederror.

stirling1 ($n$, $m$)

Represents the Stirling number of the first kind.

When $n$ and $m$ are nonnegative integers, the magnitude of stirling1 ($n$, $m$) is the number of permutations of a set with $n$ members that have $m$ cycles.

stirling1 is a simplifying function. Maxima knows the following identities:

1. $\text{stirling1}(1, k) = \text{kron}\_\delta(1, k), k >= 0$ (see http://dlmf.nist.gov/26.8.E2)
2. $\text{stirling1}(n, n) = 1, n >= 0$ (see http://dlmf.nist.gov/26.8.E1)
3. $\text{stirling1}(n, n - 1) = -\text{binomial}(n, 2), n >= 1$, (see http://dlmf.nist.gov/26.8.E16)
4. $\text{stirling1}(n, 0) = \text{kron}\_\delta(n, 0), n >= 0$ (see http://dlmf.nist.gov/26.8.E14 and http://dlmf.nist.gov/26.8.E1)
5. $\text{stirling1}(n, 1) = (-1)^{n - 1}(n - 1)!, n >= 1$ (see http://dlmf.nist.gov/26.8.E14)
6. $\text{stirling1}(n, k) = 0, n >= 0$ and $k > n$.

These identities are applied when the arguments are literal integers or symbols declared as integers, and the first argument is nonnegative. stirling1 does not simplify for non-integer arguments.

Examples:

(!i12) some (atom, {1, 2, sin(3), 4, 5 + y, 6});
(%o12) true

some applied to two lists. The predicate is a function of two arguments.

(Return values of the predicate $f$ which evaluate to something other than true or false are governed by the global flag prederror.)

stirling1 ($n$, $m$)

Represents the Stirling number of the first kind.

When $n$ and $m$ are nonnegative integers, the magnitude of stirling1 ($n$, $m$) is the number of permutations of a set with $n$ members that have $m$ cycles.

stirling1 is a simplifying function. Maxima knows the following identities:

1. $\text{stirling1}(1, k) = \text{kron}\_\delta(1, k), k >= 0$ (see http://dlmf.nist.gov/26.8.E2)
2. $\text{stirling1}(n, n) = 1, n >= 0$ (see http://dlmf.nist.gov/26.8.E1)
3. $\text{stirling1}(n, n - 1) = -\text{binomial}(n, 2), n >= 1$, (see http://dlmf.nist.gov/26.8.E16)
4. $\text{stirling1}(n, 0) = \text{kron}\_\delta(n, 0), n >= 0$ (see http://dlmf.nist.gov/26.8.E14 and http://dlmf.nist.gov/26.8.E1)
5. $\text{stirling1}(n, 1) = (-1)^{n - 1}(n - 1)!, n >= 1$ (see http://dlmf.nist.gov/26.8.E14)
6. $\text{stirling1}(n, k) = 0, n >= 0$ and $k > n$.

These identities are applied when the arguments are literal integers or symbols declared as integers, and the first argument is nonnegative. stirling1 does not simplify for non-integer arguments.

Examples:

(!i11) declare (n, integer)$
(%i2) assume (n >= 0)$
(%i3) stirling1 (n, n);
(%o3) 1

stirling2 (n, m)

Represents the Stirling number of the second kind.

When \( n \) and \( m \) are nonnegative integers, \( \text{stirling2} \ (n, m) \) is the number of ways a set with cardinality \( n \) can be partitioned into \( m \) disjoint subsets.

\( \text{stirling2} \) is a simplifying function. Maxima knows the following identities.

1. \( \text{stirling2}(n, 0) = 1, n > 0 \) (see \texttt{http://dlmf.nist.gov/26.8.E17} and \( \text{stirling2}(0,0) = 1 \))
2. \( \text{stirling2}(n, n) = 1, n > 0, \) (see \texttt{http://dlmf.nist.gov/26.8.E4})
3. \( \text{stirling2}(n, 1) = 1, n > 0, \) (see \texttt{http://dlmf.nist.gov/26.8.E17} and \( \text{stirling2}(0,1) = 0 \))
4. \( \text{stirling2}(n, 2) = (n - 1) - 1, n > 1, \) (see \texttt{http://dlmf.nist.gov/26.8.E17})
5. \( \text{stirling2}(n, n - 1) = \text{binomial}(n, 2), n > 1 \) (see \texttt{http://dlmf.nist.gov/26.8.E16})
6. \( \text{stirling2}(n, k) = 0, n > 0 \) and \( k > n. \)

These identities are applied when the arguments are literal integers or symbols declared as integers, and the first argument is nonnegative. \( \text{stirling2} \) does not simplify for non-integer arguments.

Examples:

(%i1) declare (n, integer)$
(%i2) assume (n >= 0)$
(%i3) stirling2 (n, n);
(%o3) 1

\( \text{stirling2} \) does not simplify for non-integer arguments.

(%i1) stirling2 (%pi, %pi);
(%o1) \text{stirling2}(%pi, %pi)

subset (a, f)

Returns the subset of the set \( a \) that satisfies the predicate \( f. \)

\( \text{subset} \) returns a set which comprises the elements of \( a \) for which \( f \) returns anything other than \texttt{false}. \( \text{subset} \) does not apply \texttt{is} to the return value of \( f. \)

\( \text{subset} \) complains if \( a \) is not a literal set.

See also \texttt{partition_set}.

Examples:

(%i1) subset ({1, 2, x, x + y, z, x + y + z}, atom);
(%o1) {1, 2, x, z}
(%i2) subset ({1, 2, 7, 8, 9, 14}, evenp);
(%o2) {2, 8, 14}
**Chapter 35: Sets**

**subsetp (a, b)**

Returns true if and only if the set a is a subset of b.

subsetp complains if either a or b is not a literal set.

Examples:

```lisp
(%i1) subsetp ({1, 2, 3}, {a, 1, b, 2, c, 3});
(%o1) true
(%i2) subsetp ({a, 1, b, 2, c, 3}, {1, 2, 3});
(%o2) false
```

**symmdifference (a_1, ..., a_n)**

Returns the symmetric difference of sets a_1, ..., a_n.

Given two arguments, symmdifference (a, b) is the same as union (setdifference (a, b), setdifference (b, a)).

symmdifference complains if any argument is not a literal set.

Examples:

```lisp
(%i1) S_1 : {a, b, c};
(%o1) {a, b, c}
(%i2) S_2 : {1, b, c};
(%o2) {1, b, c}
(%i3) S_3 : {a, b, z};
(%o3) {a, b, z}
(%i4) symmdifference ();
(%o4) {}
(%i5) symmdifference (S_1);
(%o5) {a, b, c}
(%i6) symmdifference (S_1, S_2);
(%o6) {1, a}
(%i7) symmdifference (S_1, S_2, S_3);
(%o7) {1, b, z}
(%i8) symmdifference ({}, S_1, S_2, S_3);
(%o8) {1,b, z}
```

**union (a_1, ..., a_n)**

Returns the union of the sets a_1 through a_n.

union() (with no arguments) returns the empty set.

union complains if any argument is not a literal set.

Examples:

```lisp
(%i1) S_1 : {a, b, c + d, %e};
(%o1) {e, a, b, d + c}
(%i2) S_2 : {pi, i, e, c + d};
(%o2) {e, i, pi, d + c}
(%i3) S_3 : {17, 29, 1729, pi, i};
(%o3) {17, 29, 1729, pi, i}
(%i4) union ();
(%o4) {}
```
(%i5) union (S_1);
(%o5) \{e, a, b, d + c\}
(%i6) union (S_1, S_2);
(%o6) \{e, i, \pi, a, b, d + c\}
(%i7) union (S_1, S_2, S_3);
(%o7) \{17, 29, 1729, e, i, \pi, a, b, d + c\}
(%i8) union ({}, S_1, S_2, S_3);
(%o8) \{17, 29, 1729, e, i, \pi, a, b, d + c\}
36 Function Definition

36.1 Introduction to Function Definition

36.2 Function

36.2.1 Ordinary functions

To define a function in Maxima you use the := operator. E.g.

\[ f(x) := \sin(x) \]

defines a function \( f \). Anonymous functions may also be created using \texttt{lambda}. For example

\[ \texttt{lambda ([i, j], \ldots)} \]

can be used instead of \( f \) where

\[
\begin{align*}
& f(i,j) := \text{block ([], \ldots);} \\
& \text{map (lambda ([i], i+1), l)}
\end{align*}
\]

would return a list with 1 added to each term.

You may also define a function with a variable number of arguments, by having a final argument which is assigned to a list of the extra arguments:

\[
\begin{align*}
& (\%i1) \quad f ([u]) := u; \\
& (\%o1) \quad f([u]) := u \\
& (\%i2) \quad f (1, 2, 3, 4); \\
& (\%o2) \quad [1, 2, 3, 4] \\
& (\%i3) \quad f (a, b, [u]) := [a, b, u]; \\
& (\%o3) \quad f(a, b, [u]) := [a, b, u] \\
& (\%i4) \quad f (1, 2, 3, 4, 5, 6); \\
& (\%o4) \quad [1, 2, [3, 4, 5, 6]]
\end{align*}
\]

The right hand side of a function is an expression. Thus if you want a sequence of expressions, you do

\[
\begin{align*}
& f(x) := (\text{expr1, expr2, \ldots, exprn}); \\
& \text{and the value of exprn is what is returned by the function.}
\end{align*}
\]

If you wish to make a \texttt{return} from some expression inside the function then you must use \texttt{block} and \texttt{return}.

\[
\begin{align*}
& \text{block ([], expr1, \ldots, if (a > 10) then return(a), \ldots, exprn)} \\
& \quad \text{is itself an expression, and so could take the place of the right hand side of a function definition. Here it may happen that the return happens earlier than the last expression.}
\end{align*}
\]

The first \([\ldots]\) in the block, may contain a list of variables and variable assignments, such as \([a: 3, b, c: []]\), which would cause the three variables \(a,b,\)and \(c\) to not refer to their global values, but rather have these special values for as long as the code executes inside the \texttt{block}, or inside functions called from inside the \texttt{block}. This is called \texttt{dynamic} binding, since the variables last from the start of the block to the time it exits. Once you return from the \texttt{block}, or throw out of it, the old values (if any) of the variables will be restored. It is certainly a good idea to protect your variables in this way. Note that the assignments in the block variables, are done in parallel. This means, that if you had used \(c: a\) in the
above, the value of c would have been the value of a at the time you just entered the block, but before a was bound. Thus doing something like

```
block ([a: a], expr1, ... a: a+3, ..., exprn)
```

will protect the external value of a from being altered, but would let you access what that value was. Thus the right hand side of the assignments, is evaluated in the entering context, before any binding occurs. Using just `block ([x], ...)` would cause the x to have itself as value, just as if it would have if you entered a fresh Maxima session.

The actual arguments to a function are treated in exactly same way as the variables in a block. Thus in

```
f(x) := (expr1, ..., exprn);
```

and

```
f(1);
```

we would have a similar context for evaluation of the expressions as if we had done

```
block ([x: 1], expr1, ..., exprn)
```

Inside functions, when the right hand side of a definition, may be computed at runtime, it is useful to use `define` and possibly `buildq`.

### 36.2.2 Array functions

An array function stores the function value the first time it is called with a given argument, and returns the stored value, without recomputing it, when that same argument is given. Such a function is often called a memoizing function.

Array function names are appended to the global list `arrays` (not the global list `functions`). `arrayinfo` returns the list of arguments for which there are stored values, and `listarray` returns the stored values. `dispfun` and `fundef` return the array function definition.

`arraymake` constructs an array function call, analogous to `funmake` for ordinary functions. `arrayapply` applies an array function to its arguments, analogous to `apply` for ordinary functions. There is nothing exactly analogous to `map` for array functions, although `map(lambda([x], a[x]), L)` or `makelist(a[x], x, L)`, where L is a list, are not too far off the mark.

`remarray` removes an array function definition (including any stored function values), analogous to `remfunction` for ordinary functions.

`kill(a[x])` removes the value of the array function a stored for the argument x; the next time a is called with argument x, the function value is recomputed. However, there is no way to remove all of the stored values at once, except for `kill(a)` or `remarray(a)`, which also remove the function definition.

### 36.3 Macros

`buildq (L, expr)`

Substitutes variables named by the list L into the expression expr, in parallel, without evaluating expr. The resulting expression is simplified, but not evaluated, after `buildq` carries out the substitution.
The elements of $L$ are symbols or assignment expressions $\text{symbol}: \text{value}$, evaluated in parallel. That is, the binding of a variable on the right-hand side of an assignment is the binding of that variable in the context from which $\text{buildq}$ was called, not the binding of that variable in the variable list $L$. If some variable in $L$ is not given an explicit assignment, its binding in $\text{buildq}$ is the same as in the context from which $\text{buildq}$ was called.

Then the variables named by $L$ are substituted into $\text{expr}$ in parallel. That is, the substitution for every variable is determined before any substitution is made, so the substitution for one variable has no effect on any other.

If any variable $x$ appears as $\text{splice}(x)$ in $\text{expr}$, then $x$ must be bound to a list, and the list is spliced (interpolated) into $\text{expr}$ instead of substituted.

Any variables in $\text{expr}$ not appearing in $L$ are carried into the result verbatim, even if they have bindings in the context from which $\text{buildq}$ was called.

Examples

$\text{a}$ is explicitly bound to $x$, while $\text{b}$ has the same binding (namely 29) as in the calling context, and $\text{c}$ is carried through verbatim. The resulting expression is not evaluated until the explicit evaluation `'%'.

\begin{verbatim}
(%i1) (a: 17, b: 29, c: 1729)$
(%i2) buildq ([a: x, b], a + b + c);
(%o2) x + c + 29
(%i3) '%%;
(%o3) x + 1758
\end{verbatim}

$\text{e}$ is bound to a list, which appears as such in the arguments of $\text{foo}$, and interpolated into the arguments of $\text{bar}$.

\begin{verbatim}
(%i1) buildq ([e: [a, b, c]], foo (x, e, y));
(%o1) foo(x, [a, b, c], y)
(%i2) buildq ([e: [a, b, c]], bar (x, splice (e), y));
(%o2) bar(x, a, b, c, y)
\end{verbatim}

The result is simplified after substitution. If simplification were applied before substitution, these two results would be the same.

\begin{verbatim}
(%i1) buildq ([e: [a, b, c]], splice (e) + splice (e));
(%o1) 2 c + 2 b + 2 a
(%i2) buildq ([e: [a, b, c]], 2 * splice (e));
(%o2) 2 a b c
\end{verbatim}

The variables in $L$ are bound in parallel; if bound sequentially, the first result would be $\text{foo}(b, b)$. Substitutions are carried out in parallel; compare the second result with the result of $\text{subst}$, which carries out substitutions sequentially.

\begin{verbatim}
(%i1) buildq ([a: b, b: a], foo (a, b));
(%o1) foo(b, a)
(%i2) buildq ([u: v, v: w, w: x, x: y, y: z, z: u],
   bar (u, v, w, x, y, z));
(%o2) bar(v, w, x, y, z, u)
(%i3) subst ([u=v, v=w, w=x, x=y, y=z, z=u],
   bar (u, v, w, x, y, z));
(%o3) bar(u, u, u, u, u, u)
\end{verbatim}
Construct a list of equations with some variables or expressions on the left-hand side and their values on the right-hand side. `macroexpand` shows the expression returned by `show_values`.

```lisp
(%i1) show_values ([L]) ::= buildq ([L], map ("=", 'L, L));
(%o1) show_values([L]) ::= buildq([L], map("=", 'L, L))
(%i2) (a: 17, b: 29, c: 1729)$
(%i3) show_values (a, b, c - a - b);
(%o3) [a = 17, b = 29, c - b - a = 1683]
(%i4) macroexpand (show_values (a, b, c - a - b));
(%o4) map(=, '([a, b, c - b - a]), [a, b, c - b - a])
```

Given a function of several arguments, create another function for which some of the arguments are fixed.

```lisp
(%i1) curry (f, [a]) :=
   buildq ([f, a], lambda ([[x]], apply (f, append (a, x))))$
(%i2) by3 : curry ("*", 3);
(%o2) lambda([[x]], apply(*, append([3], x)))
(%i3) by3 (a + b);
(%o3) 3 (b + a)
```

`macroexpand (expr)`

Returns the macro expansion of `expr` without evaluating it, when `expr` is a macro function call. Otherwise, `macroexpand` returns `expr`. If the expansion of `expr` yields another macro function call, that macro function call is also expanded.

`macroexpand` quotes its argument. However, if the expansion of a macro function call has side effects, those side effects are executed.

See also `::=`, `macros`, and `macroexpand1`.

Examples

```lisp
(%i1) g (x) ::= x / 99;
 (%o1)                            x
                g(x) ::= --
         99
(%i2) h (x) ::= buildq ([x], g (x - a));
(%o2)                              h(x) ::= buildq([x], g(x - a))
(%i3) a: 1234;
(%o3) 1234
(%i4) macroexpand (h (y));
 y - a
----
 99
(%i5) h (y);
 y - 1234
------
 99
```
macroexpand1 (expr)  
 Returns the macro expansion of expr without evaluating it, when expr is a macro function call. Otherwise, macroexpand1 returns expr.

macroexpand1 quotes its argument. However, if the expansion of a macro function call has side effects, those side effects are executed.

If the expansion of expr yields another macro function call, that macro function call is not expanded.

See also ::=, macros, and macroexpand.

Examples

(%i1) g (x) ::= x / 99;
   \( x \)
   (%o1) g(x) ::= \( \frac{x}{99} \)

(%i2) h (x) ::= buildq ([x], g (x - a));
   \( x - a \)
   (%o2) h(x) ::= buildq([x], g((x - a))

(%i3) a: 1234;
   (%o3) 1234

(%i4) macroexpand1 (h (y));
   \( y - a \)
   (%o4) g(y - a)

(%i5) h (y);
   \( y - a \)
   (%o5) \( \frac{y - 1234}{99} \)

macros  
 Default value: []

macros is the list of user-defined macro functions. The macro function definition operator ::= puts a new macro function onto this list, and kill, remove, and remfunction remove macro functions from the list.

See also infolists.

splice (a)  
 Splices (interpolates) the list named by the atom a into an expression, but only if splice appears within buildq; otherwise, splice is treated as an undefined function. If appearing within buildq as a alone (without splice), a is substituted (not interpolated) as a list into the result. The argument of splice can only be an atom; it cannot be a literal list or an expression which yields a list.

Typically splice supplies the arguments for a function or operator. For a function f, the expression f (splice (a)) within buildq expands to f (a[1], a[2], a[3], ...). For an operator o, the expression "o" (splice (a)) within buildq expands to "o" (a[1], a[2], a[3], ...), where o may be any type of operator (typically one which takes multiple arguments). Note that the operator must be enclosed in double quotes ".

Examples
apply \((F, [x_1, \ldots, x_n])\) [Function]

Constructs and evaluates an expression \(F(arg_1, \ldots, arg_n)\).

\(apply\) does not attempt to distinguish array functions from ordinary functions; when \(F\) is the name of an array function, \(apply\) evaluates \(F(\ldots)\) (that is, a function call with parentheses instead of square brackets). \(arrayapply\) evaluates a function call with square brackets in this case.

See also \texttt{funmake} and \texttt{args}.

Examples:

\(apply\) evaluates its arguments. In this example, \(\texttt{min}\) is applied to the value of \(L\).

\begin{verbatim}
(\%i1) L : [1, 5, -10.2, 4, 3];
(\%o1) [1, 5, -10.2, 4, 3]
(\%i2) apply (min, L);
(\%o2) -10.2
\end{verbatim}

\(apply\) evaluates arguments, even if the function \(F\) quotes them.

\begin{verbatim}
(\%i1) F (x) := x / 1729;
(\%o1) F(x) := ----
\end{verbatim}

\begin{verbatim}
(\%o2) 1729
(\%i2) fname : F;
(\%o2) F
(\%i3) dispfun (F);
(\%t3) F(x) := ----
(\%t3) 1729
(\%o3) [\%t3]
(\%i4) dispfun (fname);
fundef: no such function: fname
-- an error. To debug this try: debugmode(true);
\end{verbatim}
apply (dispfun, [fname]);

x

F(x) := ----
1729

apply evaluates the function name F. Single quote ' defeats evaluation. demoivre is
the name of a global variable and also a function.

demoivre;
false
demoivre (exp (%i * x));
%i sin(x) + cos(x)
demoivre (exp (%i * x));
-- an error. To debug this try: debugmode(true);
applying a function where a function was expected.

How to convert a nested list into a matrix:

a:([[1,2],[3,4]]);
[[1, 2], [3, 4]]
apply(matrix,a);
[ 1  2 ]
[ 3  4 ]

block ([Function]

block (\[v_1, \ldots, v_m\], expr_1, \ldots, expr_n)
block (expr_1, \ldots, expr_n)

The function block allows to make the variables \(v_1, \ldots, v_m\) to be local for a
sequence of commands. If these variables are already bound block saves the current
values of the variables \(v_1, \ldots, v_m\) (if any) upon entry to the block, then unbinds
the variables so that they evaluate to themselves; The local variables may be bound
to arbitrary values within the block but when the block is exited the saved values are
restored, and the values assigned within the block are lost.

If there is no need to define local variables then the list at the beginning of the block
command may be omitted. In this case if neither return nor go are used block
behaves similar to the following construct:

( expr_1, expr_2, \ldots, expr_n );

expr_1, \ldots, expr_n will be evaluated in sequence and the value of the last expression
will be returned. The sequence can be modified by the go, throw, and return func-
tions. The last expression is expr_n unless return or an expression containing throw
is evaluated.

The declaration local(v_1, \ldots, v_m) within block saves the properties associated
with the symbols v_1, \ldots, v_m, removes any properties before evaluating other ex-
pressions, and restores any saved properties on exit from the block. Some declara-
tions are implemented as properties of a symbol, including :=, array, dependencies,
atvalue, matchdeclare, atomgrad, constant, nonscalar, assume, and some others. The effect of local is to make such declarations effective only within the block; otherwise declarations within a block are actually global declarations.

block may appear within another block. Local variables are established each time a new block is evaluated. Local variables appear to be global to any enclosed blocks. If a variable is non-local in a block, its value is the value most recently assigned by an enclosing block, if any, otherwise, it is the value of the variable in the global environment. This policy may coincide with the usual understanding of "dynamic scope".

The value of the block is the value of the last statement or the value of the argument to the function return which may be used to exit explicitly from the block. The function go may be used to transfer control to the statement of the block that is tagged with the argument to go. To tag a statement, precede it by an atomic argument as another statement in the block. For example: block ([x], x:1, loop, x: x+1, ..., go(loop), ...). The argument to go must be the name of a tag appearing within the block. One cannot use go to transfer to a tag in a block other than the one containing the go.

Blocks typically appear on the right side of a function definition but can be used in other places as well.

See also return and go.

break (expr_1, ..., expr_n) [Function]
Evaluates and prints expr_1, ..., expr_n and then causes a Maxima break at which point the user can examine and change his environment. Upon typing exit; the computation resumes.

catch (expr_1, ..., expr_n) [Function]
Evaluates expr_1, ..., expr_n one by one; if any leads to the evaluation of an expression of the form throw (arg), then the value of the catch is the value of throw (arg), and no further expressions are evaluated. This "non-local return" thus goes through any depth of nesting to the nearest enclosing catch. If there is no catch enclosing a throw, an error message is printed.

If the evaluation of the arguments does not lead to the evaluation of any throw then the value of catch is the value of expr_n.

(%i1) lambda ([x], if x < 0 then throw(x) else f(x))$
(%i2) g(1) := catch (map ('%, 1))$
(%i3) g ([1, 2, 3, 7]);
(%o3) [f(1), f(2), f(3), f(7)]
(%i4) g ([1, 2, -3, 7]);
(%o4) -3

The function g returns a list of f of each element of l if l consists only of non-negative numbers; otherwise, g "catches" the first negative element of l and "throws" it up.
 compilfile

    compilfile(filename, f_1, ..., f_n)
    compilfile(filename, functions)
    compilfile(filename, all)

Translates Maxima functions into Lisp and writes the translated code into the file
filename.

compilfile(filename, f_1, ..., f_n) translates the specified functions. compilfile
(filename, functions) and compilfile (filename, all) translate all user-defined
functions.

The Lisp translations are not evaluated, nor is the output file processed by the Lisp
compiler. translate creates and evaluates Lisp translations. compilfile translates
Maxima into Lisp, and then executes the Lisp compiler.

See also translate, translate_file, and compilfile_file.

 compile

    compile(f_1, ..., f_n)
    compile(functions)
    compile(all)

Translates Maxima functions f_1, ..., f_n into Lisp, evaluates the Lisp translations,
and calls the Lisp function COMPIL on each translated function. compile returns a
list of the names of the compiled functions.

compile (all) or compile (functions) compiles all user-defined functions.

compile quotes its arguments; the quote-quote operator ' ' defeats quotation.

Compiling a function to native code can mean a big increase in speed and might cause
the memory footprint to reduce drastically. Code tends to be especially effective when
the flexibility it needs to provide is limited. If compilation doesn’t provide the speed
that is needed a few ways to limit the code’s functionality are the following:

- If the function accesses global variables the complexity of the function can be
drastically be reduced by limiting these variables to one data type, for example
  using mode_declare or a statement like the following one: put(x_1, bigfloat,
  numerical_type)
- The compiler might warn about undeclared variables if text could either be a
  named option to a command or (if they are assigned a value to) the name of a
  variable. Prepending the option with a single quote ' tells the compiler that the
text is meant as an option.

define

    define (f(x_1, ..., x_n), expr)
    define (f[x_1, ..., x_n], expr)
    define (f[x_1, ..., x_n](y_1, ..., y_m), expr)
    define (funmake (f, [x_1, ..., x_n]), expr)
    define (arraymake (f, [x_1, ..., x_n]), expr)
    define (ev(expr_1, expr_2))

Defines a function named f with arguments x_1, ..., x_n and function body expr.
define always evaluates its second argument (unless explicitly quoted). The func-
tion so defined may be an ordinary Maxima function (with arguments enclosed in parentheses) or an array function (with arguments enclosed in square brackets).

When the last or only function argument \( x_n \) is a list of one element, the function defined by \texttt{define} accepts a variable number of arguments. Actual arguments are assigned one-to-one to formal arguments \( x_1, \ldots, x_{n-1} \), and any further actual arguments, if present, are assigned to \( x_n \) as a list.

When the first argument of \texttt{define} is an expression of the form \( f(x_1, \ldots, x_n) \) or \( f[x_1, \ldots, x_n] \), the function arguments are evaluated but \( f \) is not evaluated, even if there is already a function or variable by that name.

When the first argument is an expression with operator \texttt{funmake}, \texttt{arraymake}, or \texttt{ev}, the first argument is evaluated; this allows for the function name to be computed, as well as the body.

All function definitions appear in the same namespace; defining a function \( f \) within another function \( g \) does not automatically limit the scope of \( f \) to \( g \). However, \texttt{local}(\( f \)) makes the definition of function \( f \) effective only within the block or other compound expression in which \texttt{local} appears.

If some formal argument \( x_k \) is a quoted symbol (after evaluation), the function defined by \texttt{define} does not evaluate the corresponding actual argument. Otherwise all actual arguments are evaluated.

See also \( := \) and \( ::= \).

Examples:

\texttt{define} always evaluates its second argument (unless explicitly quoted).

\begin{verbatim}
(%i1) expr : cos(y) - sin(x);
   cos(y) - sin(x)
(%i2) define (F1 (x, y), expr);
   F1(x, y) := cos(y) - sin(x)
(%i3) F1 (a, b);
   cos(b) - sin(a)
(%i4) F2 (x, y) := expr;
   F2(x, y) := expr
(%i5) F2 (a, b);
   cos(y) - sin(x)
\end{verbatim}

The function defined by \texttt{define} may be an ordinary Maxima function or an array function.

\begin{verbatim}
(%i1) define (G1 (x, y), x.y - y.x);
   G1(x, y) := x . y - y . x
(%i2) define (G2 [x, y], x.y - y.x);
   G2 := x . y - y . x
       x, y
\end{verbatim}

When the last or only function argument \( x_n \) is a list of one element, the function defined by \texttt{define} accepts a variable number of arguments.

\begin{verbatim}
(%i1) define (H ([L]), '(apply ("+", L)));
   H([L]) := apply("+", L)
\end{verbatim}
When the first argument is an expression with operator `funmake`, `arraymake`, or `ev`, the first argument is evaluated.

```
(%i1) [F : I, u : x];
(%o1) [I, x]
(%i2) funmake (F, [u]);
(%o2) I(x)
(%i3) define (funmake (F, [u]), cos(u) + 1);
(%o3) I(x) := cos(x) + 1
(%i4) define (arraymake (F, [u]), cos(u) + 1);
(%o4) I := cos(x) + 1
```

```
(%i5) define (foo (x, y), bar (y, x));
(%o5) foo(x, y) := bar(y, x)
(%i6) define (ev (foo (x, y)), sin(x) - cos(y));
(%o6) bar(y, x) := sin(x) - cos(y)
```

define_variable (name, default_value, mode)  [Function]
Introduces a global variable into the Maxima environment. `define_variable` is useful in user-written packages, which are often translated or compiled.

`define_variable` carries out the following steps:

1. `mode_declare (name, mode)` declares the mode of `name` to the translator. See `mode_declare` for a list of the possible modes.
2. If the variable is unbound, `default_value` is assigned to `name`.
3. Associates `name` with a test function to ensure that `name` is only assigned values of the declared mode.

The `value_check` property can be assigned to any variable which has been defined via `define_variable` with a mode other than any. The `value_check` property is a lambda expression or the name of a function of one variable, which is called when an attempt is made to assign a value to the variable. The argument of the `value_check` function is the would-be assigned value.

`define_variable` evaluates `default_value`, and quotes `name` and `mode`. `define_variable` returns the current value of `name`, which is `default_value` if `name` was unbound before, and otherwise it is the previous value of `name`.

Examples:

`foo` is a Boolean variable, with the initial value `true`.

```
(%i1) define_variable (foo, true, boolean);
(%o1) true
(%i2) foo;
(%o2) true
(%i3) foo: false;
(%o3) false
```
(%i4) foo: %pi;
translator: foo was declared with mode boolean
   , but it has value: %pi
   -- an error. To debug this try: debugmode(true);
(%i5) foo;
   false

bar is an integer variable, which must be prime.
(%i1) define_variable (bar, 2, integer);
   2
(%i2) qput (bar, prime_test, value_check);
   prime_test
(%i3) prime_test (y) := if not primep(y) then
    error(y, "is not prime.");
   prime_test(y) := if not primep(y)
    then error(y, "is not prime.")
(%i4) bar: 1439;
   1439
(%i5) bar: 1440;
   1440
   is not prime.
   -- an error. To debug this try: debugmode(true);
(%i6) bar;
   1439

baz_quux is a variable which cannot be assigned a value. The mode any_check is like any, but any_check enables the value_check mechanism, and any does not.
(%i1) define_variable (baz_quux, 'baz_quux, any_check);
   baz_quux
(%i2) F: lambda ([y], if y # 'baz_quux then
    error("Cannot assign to ‘baz_quux’.");
   lambda([y], if y # 'baz_quux
    then error(Cannot assign to ‘baz_quux’.))
(%i3) qput (baz_quux, 'F, value_check);
(%i4) baz_quux: 'baz_quux;
   baz_quux
(%i5) baz_quux: sqrt(2);
   Cannot assign to ‘baz_quux’.
   -- an error. To debug this try: debugmode(true);
(%i6) baz_quux;
   baz_quux

\textbf{dispfun} \quad [\textbf{Function}]

\texttt{dispfun (f_1, \ldots, f_n)}
\texttt{dispfun (all)}
Displays the definition of the user-defined functions \(f_1, \ldots, f_n\). Each argument may be the name of a macro (defined with ::=), an ordinary function (defined with := or
define), an array function (defined with := or define, but enclosing arguments in square brackets []), a subscripted function (defined with := or define, but enclosing some arguments in square brackets and others in parentheses ( )), one of a family of subscripted functions selected by a particular subscript value, or a subscripted function defined with a constant subscript.

dispfun (all) displays all user-defined functions as given by the functions, arrays, and macros lists, omitting subscripted functions defined with constant subscripts.

dispfun creates an intermediate expression label (%t1, %t2, etc.) for each displayed function, and assigns the function definition to the label. In contrast, fundef returns the function definition.

dispfun quotes its arguments; the quote-quote operator '' defeats quotation. dispfun returns the list of intermediate expression labels corresponding to the displayed functions.

Examples:

```
(%i1) m(x, y) ::= x^(-y);
   - y
(%o1) m(x, y) ::= x

(%i2) f(x, y) := x^(-y);
   - y
(%o2) f(x, y) := x

(%i3) g[x, y] := x^(-y);
   - y
   x, y
(g) := x

(%i4) h[x](y) := x^(-y);
   - y
   x
h (y) := x

(%i5) i[8](y) := 8^(-y);
   - y
   8
i (y) := 8
```
\(\text{(%i6)}\) dispfun \((m, f, g, h, h[5], h[10], i[8])\);

\(\text{(%t6)}\) 
m(x, y) ::= x

\(\text{(%t7)}\) 
f(x, y) ::= x

\(\text{(%t8)}\) 
g := x

\(\text{(%t9)}\) 
h (y) := x

\(\text{(%t10)}\) 
\begin{align*}
& h (y) := \\
& \frac{1}{5} y
\end{align*}

\(\text{(%t11)}\) 
\begin{align*}
& h (y) := \\
& \frac{1}{10} y
\end{align*}

\(\text{(%t12)}\) 
i (y) := 8

\(\text{(%o12)}\) 
\[\text{(%t6, %t7, %t8, %t9, %t10, %t11, %t12)}\]

\(\text{(%i13)}\) 
\text{"\%\";}

\(\text{(%o13)}\) 
\[\text{[m(x, y) ::= x , f(x, y) ::= x , g := x , h (y) ::= x , h (y) ::= -- , h (y) ::= --, i (y) := 8 ]}\]

\textbf{fullmap \((f, \text{expr}_1, \ldots)\)}

Similar to map, but \texttt{fullmap} keeps mapping down all subexpressions until the main operators are no longer the same.

\texttt{fullmap} is used by the Maxima simplifier for certain matrix manipulations; thus, Maxima sometimes generates an error message concerning \texttt{fullmap} even though \texttt{fullmap} was not explicitly called by the user.

Examples:
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(%i1) a + b * c;
(%o1) b c + a

(%i2) fullmap (g, %);
(%o2) g(b) g(c) + g(a)

(%i3) map (g, %th(2));
(%o3) g(b c) + g(a)

fullmapl (f, list_1, ...)

[Function]

Similar to fullmap, but fullmapl only maps onto lists and matrices.

Example:

(%i1) fullmapl ("+", [3, [4, 5]], [[a, 1], [0, -1.5]]);
(%o1) [[a + 3, 4], [4, 3.5]]

functions

[System variable]

Default value: []

functions is the list of ordinary Maxima functions in the current session. An ordinary function is a function constructed by define or := and called with parentheses (). A function may be defined at the Maxima prompt or in a Maxima file loaded by load or batch.

Array functions (called with square brackets, e.g., F[x]) and subscripted functions (called with square brackets and parentheses, e.g., F[x](y)) are listed by the global variable arrays, and not by functions.

Lisp functions are not kept on any list.

Examples:

(%i1) F_1 (x) := x - 100;
(%o1) F_1(x) := x - 100

(%i2) F_2 (x, y) := x / y;

(%i3) define (F_3 (x), sqrt (x));

(%i4) G_1 [x] := x - 100;

(%i5) G_2 [x, y] := x / y;

(%i6) define (G_3 [x], sqrt (x));

(%i7) H_1 [x] (y) := x^y;

(%o7) H_1 (y) := x
(%i8) functions;
(%o8)       [F_1(x), F_2(x, y), F_3(x)]
(%i9) arrays;
(%o9)       [G_1, G_2, G_3, H_1]

funcdef (f)

Returns the definition of the function f.

The argument may be

•  the name of a macro (defined with ::=),
•  an ordinary function (defined with := or define),
•  an array function (defined with := or define, but enclosing arguments in square
  brackets []),
•  a subscripted function (defined with := or define, but enclosing some arguments
  in square brackets and others in parentheses ()),
•  one of a family of subscripted functions selected by a particular subscript value,
•  or a subscripted function defined with a constant subscript.

funcdef quotes its argument; the quote-quote operator '' defeats quotation.
funcdef (f) returns the definition of f. In contrast, dispfun (f) creates an interme-

dia
tate expression label and assigns the definition to the label.

funmake (F, [arg_1, ..., arg_n])

Returns an expression F(arg_1, ..., arg_n). The return value is simplified, but
not evaluated, so the function F is not called, even if it exists.

funmake does not attempt to distinguish array functions from ordinary functions;
when F is the name of an array function, funmake returns F(...) (that is, a function
call with parentheses instead of square brackets). arraymake returns a function call
with square brackets in this case.

funmake evaluates its arguments.

See also apply and args.

Examples:

funmake applied to an ordinary Maxima function.

(%i11) F (x, y) := y^2 - x^2;
2 2
(%o11)       F(x, y) := y - x
(%i12) funmake (F, [a + 1, b + 1]);
(%o12)       F(a + 1, b + 1)
(%i13) ''%;
2 2
(%o13)       (b + 1) - (a + 1)

funmake applied to a macro.

(%i11) G (x) ::= (x - 1)/2;
     x - 1
(%o11)       G(x) ::= ------
         2
funmake applied to a subscripted function.

\[
H[a](x) := (x - 1)^a
\]

\[
H(x) := (x - 1)^a
\]

\[
\text{funmake}(H[n], [\%e])
\]

\[
\lambda([x], (x - 1)^n)(\%e)
\]

\[
\frac{n}{2}
\]

funmake applied to a symbol which is not a defined function of any kind.

\[
\text{funmake}(A, [u])
\]

funmake evaluates its arguments, but not the return value.

\[
det(a, b, c) := b^2 - 4*a*c
\]

\[
det(a, b, c) := b - 4 a c
\]

\[
(x : 8, y : 10, z : 12)
\]

\[
f : det
\]

\[
\text{funmake}(f, [x, y, z])
\]

Maxima simplifies funmake's return value.

\[
\text{funmake}(\sin, [\frac{\pi}{2}])
\]

\[
1
\]
lambda

lambda ([x_1, ..., x_m], expr_1, ..., expr_n)
lambda ([L]), expr_1, ..., expr_n
lambda (x_1, ..., x_m, [L]), expr_1, ..., expr_n)

Defines and returns a lambda expression (that is, an anonymous function). The function may have required arguments x_1, ..., x_m and/or optional arguments L, which appear within the function body as a list. The return value of the function is expr_n. A lambda expression can be assigned to a variable and evaluated like an ordinary function. A lambda expression may appear in some contexts in which a function name is expected.

When the function is evaluated, unbound local variables x_1, ..., x_m are created. lambda may appear within block or another lambda; local variables are established each time another block or lambda is evaluated. Local variables appear to be global to any enclosed block or lambda. If a variable is not local, its value is the value most recently assigned in an enclosing block or lambda, if any, otherwise, it is the value of the variable in the global environment. This policy may coincide with the usual understanding of "dynamic scope".

After local variables are established, expr_1 through expr_n are evaluated in turn. The special variable %%, representing the value of the preceding expression, is recognized. throw and catch may also appear in the list of expressions.

return cannot appear in a lambda expression unless enclosed by block, in which case return defines the return value of the block and not of the lambda expression, unless the block happens to be expr_n. Likewise, go cannot appear in a lambda expression unless enclosed by block.

lambda quotes its arguments; the quote-quote operator ''' defeats quotation.

Examples:

- A lambda expression can be assigned to a variable and evaluated like an ordinary function.

  (%i1) f: lambda ([x], x^2);

  (%o1) lambda([x], x^2)

  (%i2) f(a);

  (%o2) a^2

- A lambda expression may appear in contexts in which a function evaluation is expected.

  (%i1) lambda ([x], x^2) (a);

  (%o1) a

  (%i2) apply (lambda ([x], x^2), [a]);

  (%o2) a

  (%i3) map (lambda ([x], x^2), [a, b, c, d, e]);

  (%o3) [a^2, b^2, c^2, d^2, e^2]
• Argument variables are local variables. Other variables appear to be global variables. Global variables are evaluated at the time the lambda expression is evaluated, unless some special evaluation is forced by some means, such as `'`

```
(%i1) a: %pi$
(%i2) b: %e$
(%i3) g: lambda ([a], a*b);
   lambda([a], a b)
(%i4) b: %gamma$
(%i5) g(1/2);
   %gamma
   -----
     2
(%i6) g2: lambda ([a], a*'b);
   lambda([a], a %gamma)
(%i7) b: %e$
(%i8) g2(1/2);
   %gamma
   -----
     2
```

• Lambda expressions may be nested. Local variables within the outer lambda expression appear to be global to the inner expression unless masked by local variables of the same names.

```
(%i1) h: lambda ([a, b], h2: lambda ([a], a*b), h2(1/2));
   lambda([a, b], h2 : lambda([a], a b), h2(-))
(%i2) h(%pi, %gamma);
   %gamma
   -----
     2
```

• Since `lambda` quotes its arguments, lambda expression i below does not define a "multiply by a" function. Such a function can be defined via `buildq`, as in lambda expression ii below.

```
(%i1) i: lambda ([a], lambda ([x], a*x));
   lambda([a], lambda([x], a x))
(%i2) i(1/2);
   lambda([x], a x)
(%i3) ii: lambda([a], buildq([a: a], lambda([x], a*x)));
   lambda([a], buildq([a : a], lambda([x], a x)))
(%i4) ii(1/2);
   lambda([x], (-) x)
```
(%i5) i2(1/2)(%pi);
(%o5) %pi
---
2

- A lambda expression may take a variable number of arguments, which are indicated by \([L]\) as the sole or final argument. The arguments appear within the function body as a list.

(%i1) f : lambda ([aa, bb, [cc]], aa * cc + bb);
(%o1) lambda([aa, bb, [cc]], aa cc + bb)
(%i2) f (foo, %i, 17, 29, 256);
(%o2) [17 foo + %i, 29 foo + %i, 256 foo + %i]
(%i3) g : lambda ([[aal]], apply ("+", aa));
(%o3) lambda([[aal]], apply(+, aa))
(%i4) g (17, 29, x, y, z, %e);
(%o4) z + y + x + %e + 46

local (v_1, \ldots, v_n)

Saves the properties associated with the symbols \(v_1, \ldots, v_n\), removes any properties before evaluating other expressions, and restores any saved properties on exit from the block or other compound expression in which local appears.

Some declarations are implemented as properties of a symbol, including \(:=\), \texttt{array}, \texttt{dependencies}, \texttt{atvalue}, \texttt{matchdeclare}, \texttt{atomgrad}, \texttt{constant}, \texttt{nonscalar}, \texttt{assume}, and some others. The effect of local is to make such declarations effective only within the block or other compound expression in which local appears; otherwise such declarations are global declarations.

local can only appear in block or in the body of a function definition or lambda expression, and only one occurrence is permitted in each.

local quotes its arguments. local returns \texttt{done}.

Example:

A local function definition.

(%i1) foo (x) := 1 - x;
(%o1) foo(x) := 1 - x
(%i2) foo (100);
(%o2) - 99
(%i3) block (local (foo), foo (x) := 2 * x, foo (100));
(%o3) 200
(%o4) - 99

macroexpansion

[Option variable]

Default value: \texttt{false}

macroexpansion controls whether the expansion (that is, the return value) of a macro function is substituted for the macro function call. A substitution may speed up subsequent expression evaluations, at the cost of storing the expansion.

false The expansion of a macro function is not substituted for the macro function call.
The first time a macro function call is evaluated, the expansion is stored. The expansion is not recomputed on subsequent calls; any side effects (such as `print` or assignment to global variables) happen only when the macro function call is first evaluated. Expansion in an expression does not affect other expressions which have the same macro function call.

**displace**

The first time a macro function call is evaluated, the expansion is substituted for the call, thus modifying the expression from which the macro function was called. The expansion is not recomputed on subsequent calls; any side effects happen only when the macro function call is first evaluated. Expansion in an expression does not affect other expressions which have the same macro function call.

**Examples**

When `macroexpansion` is `false`, a macro function is called every time the calling expression is evaluated, and the calling expression is not modified.

```lisp
(%i1) f (x) ::= h (x) / g (x);
   h(x)
(%o1) f(x) ::= ----
   g(x)
(%i2) g (x) ::= block (print("x + 99 is equal to", x),
   return (x + 99));
   return(x + 99)
(%o2) g(x) ::= block(print("x + 99 is equal to", x),
   return(x + 99))
(%i3) h (x) ::= block (print("x - 99 is equal to", x),
   return (x - 99));
   return(x - 99)
(%o3) h(x) ::= block(print("x - 99 is equal to", x),
   return(x - 99))
(%i4) macroexpansion: false;
   false
(%i5) f (a * b);
x - 99 is equal to x
x + 99 is equal to x
   a b - 99
   a b + 99
(%o5)
(%i6) dispfun (f);
   h(x)
(%o6) [%t6]
(%i7) f (a * b);
x - 99 is equal to x
x + 99 is equal to x
   a b - 99
   a b + 99
(%o7)
```
When `macroexpansion` is `expand`, a macro function is called once, and the calling expression is not modified.

```cpp
(%i1) f (x) := h (x) / g (x);
  h(x)  
(%o1) f(x) := ----
  g(x)
(%i2) g (x) ::= block (print ("x + 99 is equal to", x),
  return (x + 99));
  g(x)
(%o2) g(x) ::= block(print("x + 99 is equal to", x),
  return(x + 99))
(%i3) h (x) ::= block (print ("x - 99 is equal to", x),
  return (x - 99));
  h(x)
(%o3) h(x) ::= block(print("x - 99 is equal to", x),
  return(x - 99))
(%i4) macroexpansion: expand;
(%o4) expand
(%i5) f (a * b);
  x - 99 is equal to x
  x + 99 is equal to x
  a b - 99
  --------
  a b + 99
(%o5) dispfun (f);
  mmacroexpanded(x - 99, h(x))
  mmacroexpanded(x + 99, g(x))
(%t6) f(x) := ----------------------------
  mmacroexpanded(x + 99, g(x))
(%o6) [%t6]
(%i7) f (a * b);
  a b - 99
  --------
  a b + 99
(%o7)
```

When `macroexpansion` is `displace`, a macro function is called once, and the calling expression is modified.

```cpp
(%i1) f (x) := h (x) / g (x);
  h(x)  
(%o1) f(x) := ----
  g(x)
(%i2) g (x) ::= block (print ("x + 99 is equal to", x),
  return (x + 99));
  g(x)
(%o2) g(x) ::= block(print("x + 99 is equal to", x),
  return(x + 99))
(%i3) h (x) ::= block (print ("x - 99 is equal to", x),
  return (x - 99));
  h(x)
(%o3) h(x) ::= block(print("x - 99 is equal to", x),
  return(x - 99))
```
Chapter 36: Function Definition

(%i4) macroexpansion: displace;
(%o4) displace
(%i5) f (a * b);
  x - 99 is equal to x
  x + 99 is equal to x
  a b - 99
  -------
  a b + 99
(%i6) dispfun (f);
  x - 99
(%t6) f(x) := -------
       x + 99
(%o6) [%t6]
(%i7) f (a * b);
  a b - 99
  -------
  a b + 99

mode_checkp
[Option variable]
Default value: true
When mode_checkp is true, mode_declare checks the modes of bound variables.

mode_check_errorp
[Option variable]
Default value: false
When mode_check_errorp is true, mode_declare calls error.

mode_check_warnp
[Option variable]
Default value: true
When mode_check_warnp is true, mode errors are described.

mode_declare (y_1, mode_1, ..., y_n, mode_n)
[Function]
mode_declare is used to declare the modes of variables and functions for subsequent translation or compilation of functions. mode_declare is typically placed at the beginning of a function definition, at the beginning of a Maxima script, or executed at the interactive prompt.

The arguments of mode_declare are pairs consisting of a variable and a mode which is one of boolean, fixnum, number, rational, or float. Each variable may also be a list of variables all of which are declared to have the same mode.

If a variable is an array, and if every element of the array which is referenced has a value then array (yi, complete, dim1, dim2, ...) rather than array(yi, dim1, dim2, ...) should be used when first declaring the bounds of the array. If all the elements of the array are of mode fixnum (float), use fixnum (float) instead of complete. Also if every element of the array is of the same mode, say m, then

mode_declare (completearray (yi), m)
should be used for efficient translation.

Numeric code using arrays might run faster by declaring the expected size of the array, as in:

\[
\text{mode\_declare (complete\_array (a [10, 10]), float)}
\]

for a floating point number array which is 10 x 10.

One may declare the mode of the result of a function by using \textit{function (f_1, f_2, ...)} as an argument; here \textit{f_1, f_2, ...} are the names of functions. For example the expression,

\[
\text{mode\_declare ([function (f_1, f_2, ...)], fixnum)}
\]

declares that the values returned by \textit{f_1, f_2, ...} are single-word integers.

\textit{mode\_declare} is a synonym for \textit{mode\_declare}.

\textbf{mode\_identity (arg_1, arg_2)}

A special form used with \textit{mode\_declare} and \textit{macros} to declare, e.g., a list of lists of flonums, or other compound data object. The first argument to \textit{mode\_identity} is a primitive value mode name as given to \textit{mode\_declare} (i.e., one of \textit{float}, \textit{fixnum}, \textit{number}, \textit{list}, or \textit{any}), and the second argument is an expression which is evaluated and returned as the value of \textit{mode\_identity}. However, if the return value is not allowed by the mode declared in the first argument, an error or warning is signalled.

The important thing is that the mode of the expression as determined by the Maxima to Lisp translator, will be that given as the first argument, independent of anything that goes on in the second argument. E.g., \texttt{x: 3.3; mode\_identity (fixnum, x)}; yields an error. \texttt{mode\_identity (flonum, x)} returns 3.3 . This has a number of uses, e.g., if you knew that \texttt{first (l)} returned a number then you might write \texttt{mode\_identity (number, first (l))}. However, a more efficient way to do it would be to define a new primitive,

\[
\texttt{firstnumb (x) ::= buildq ([x], mode\_identity (number, first(x)));
}
\]

and use \texttt{firstnumb} every time you take the first of a list of numbers.

\textbf{remfunction}

\texttt{remfunction (f_1, \ldots, f_n)}

\texttt{remfunction (all)}

Unbinds the function definitions of the symbols \textit{f_1, f_2, ...}. The arguments may be the names of ordinary functions (created by \texttt{:=} or \texttt{define}) or macro functions (created by \texttt{::=}).

\texttt{remfunction (all)} unbinds all function definitions.

\texttt{remfunction} quotes its arguments.

\texttt{remfunction} returns a list of the symbols for which the function definition was unbound. \texttt{false} is returned in place of any symbol for which there is no function definition.

\texttt{remfunction} does not apply to array functions or subscripted functions. \texttt{remarray} applies to those types of functions.

\textbf{savedef}

Default value: \texttt{true}
When `savedef` is `true`, the Maxima version of a user function is preserved when the function is translated. This permits the definition to be displayed by `dispfun` and allows the function to be edited.

When `savedef` is `false`, the names of translated functions are removed from the `functions` list.

`transcompile`  
[Option variable]  
Default value: `true`  
When `transcompile` is `true`, `translate` and `translate_file` generate declarations to make the translated code more suitable for compilation. `compfile` sets `transcompile: true` for the duration.

`translate`  
[Function]  
`translate (f_1, . . ., f_n)`  
`translate (functions)`  
`translate (all)`  
Translates the user-defined functions $f_1, \ldots, f_n$ from the Maxima language into Lisp and evaluates the Lisp translations. Typically the translated functions run faster than the originals.

`translate (all)` or `translate (functions)` translates all user-defined functions.

Functions to be translated should include a call to `mode_declare` at the beginning when possible in order to produce more efficient code. For example:

```plaintext
f (x_1, x_2, ...) := block ([v_1, v_2, ...],
    mode_declare (v_1, mode_1, v_2, mode_2, ...), ...)
```

where the $x_1, x_2, \ldots$ are the parameters to the function and the $v_1, v_2, \ldots$ are the local variables.

The names of translated functions are removed from the `functions` list if `savedef` is `false` (see below) and are added to the `props` lists.

Functions should not be translated unless they are fully debugged.

Expressions are assumed simplified; if they are not, correct but non-optimal code gets generated. Thus, the user should not set the `simp` switch to `false` which inhibits simplification of the expressions to be translated.

The switch `translate`, if `true`, causes automatic translation of a user's function to Lisp.

Note that translated functions may not run identically to the way they did before translation as certain incompatibilities may exist between the Lisp and Maxima versions. Principally, the `rat` function with more than one argument and the `ratvars` function should not be used if any variables are `mode_declare`'d canonical rational expressions (CRE). Also the `prederror: false` setting will not translate.

`savedef` - if `true` will cause the Maxima version of a user function to remain when the function is `translate`'d. This permits the definition to be displayed by `dispfun` and allows the function to be edited.

`transrun` - if `false` will cause the interpreted version of all functions to be run (provided they are still around) rather than the translated version.

The result returned by `translate` is a list of the names of the functions translated.
translate_file

translate_file(maxima_filename)
translate_file(maxima_filename, lisp_filename)

Translates a file of Maxima code into a file of Lisp code. translate_file returns a list of three filenames: the name of the Maxima file, the name of the Lisp file, and the name of file containing additional information about the translation. translate_file evaluates its arguments.

translate_file("foo.mac"); load("foo.LISP") is the same as the command
batch("foo.mac") except for certain restrictions, the use of '' and %, for example.

translate_file(maxima_filename) translates a Maxima file maxima_filename into
a similarly-named Lisp file. For example, foo.mac is translated into foo.LISP. The
Maxima filename may include a directory name or names, in which case the Lisp
output file is written to the same directory from which the Maxima input comes.

translate_file(maxima_filename, lisp_filename) translates a Maxima file
maxima_filename into a Lisp file lisp_filename. translate_file ignores the filename
extension, if any, of lisp_filename; the filename extension of the Lisp output file is
always LISP. The Lisp filename may include a directory name or names, in which
case the Lisp output file is written to the specified directory.

translate_file also writes a file of translator warning messages of various degrees
of severity. The filename extension of this file is UNLISP. This file may contain
valuable information, though possibly obscure, for tracking down bugs in translated
code. The UNLISP file is always written to the same directory from which the Maxima
input comes.

translate_file emits Lisp code which causes some declarations and definitions to
take effect as soon as the Lisp code is compiled. See compile_file for more on this
topic.

See also

tr_array_as_ref
tr_bound_function_applyp,
tr_exponent
tr_file tty messagesp,
tr_float can branch complex,
tr_function_call_default,
tr_numer,
tr_optimize max loop,
tr_semicompile,
tr_state vars,
tr_warnings get,
tr_warn bad function calls
tr_warn fexpr,
tr_warn meval,
tr_warn mode,
tr_warn undeclared,
and tr_warn undefined variable.
transrun
[Option variable]
Default value: true
When transrun is false will cause the interpreted version of all functions to be run
(provided they are still around) rather than the translated version.

tr_array_as_ref
[Option variable]
Default value: true
If translate_fast_arrays is false, array references in Lisp code emitted by
translate_file are affected by tr_array_as_ref. When tr_array_as_ref is
true, array names are evaluated, otherwise array names appear as literal symbols in
translated code.
tr_array_as_ref has no effect if translate_fast_arrays is true.

tr_bound_function_applyp
[Option variable]
Default value: true
When tr_bound_function_applyp is true, Maxima gives a warning if a bound vari-
able (such as a function argument) is found being used as a function. tr_bound
function_applyp does not affect the code generated in such cases.
For example, an expression such as g (f, x) := f (x+1) will trigger the warning
message.

tr_file_tty_messagesp
[Option variable]
Default value: false
When tr_file_tty_messagesp is true, messages generated by translate_file dur-
during translation of a file are displayed on the console and inserted into the UNLISP
file. When false, messages about translation of the file are only inserted into the
UNLISP file.

tr_float_can_branch_complex
[Option variable]
Default value: true
Tells the Maxima-to-Lisp translator to assume that the functions acos, asin, asec,
and acsc can return complex results.
The ostensible effect of tr_float_can_branch_complex is the following. However, it
appears that this flag has no effect on the translator output.
When it is true then acos(x) is of mode any even if x is of mode float (as set by
mode_declare). When false then acos(x) is of mode float if and only if x is of
mode float.

tr_function_call_default
[Option variable]
Default value: general
false means give up and call meval, expr means assume Lisp fixed arg function.
genereal, the default gives code good for mexprs and mlexprs but not macros.
genereal assures variable bindings are correct in compiled code. In general mode,
when translating F(X), if F is a bound variable, then it assumes that apply (f, [x])
is meant, and translates a such, with appropriate warning. There is no need to turn
this off. With the default settings, no warning messages implies full compatibility of
translated and compiled code with the Maxima interpreter.
The list of the switches that affect the form of the translated output. This information is useful to system people when trying to debug the translator. By comparing the translated product to what should have been produced for a given state, it is possible to track down bugs.

**tr_warnings_get ()**
Prints a list of warnings which have been given by the translator during the current translation.

**tr_warn_bad_function_calls**
Default value: true
- Gives a warning when when function calls are being made which may not be correct due to improper declarations that were made at translate time.

**tr_warn_fexpr**
Default value: compfile
- Gives a warning if any FEXPRs are encountered. FEXPRs should not normally be output in translated code, all legitimate special program forms are translated.

**tr_warn_meval**
Default value: compfile
- Gives a warning if the function meval gets called. If meval is called that indicates problems in the translation.

**tr_warn_mode**
Default value: all
- Gives a warning when variables are assigned values inappropriate for their mode.
tr_warn_undeclared  
[Option variable]
Default value: compile
- Determines when to send warnings about undeclared variables to the TTY.

tr_warn_undefined_variable  
[Option variable]
Default value: all
- Gives a warning when undefined global variables are seen.

compile_file  
[Function]
compiler_file (filename)
compiler_file (filename, compiled_filename)
compiler_file (filename, compiled_filename, lisp_filename)
Translates the Maxima file filename into Lisp, executes the Lisp compiler, and, if the translation and compilation succeed, loads the compiled code into Maxima.

compile_file returns a list of the names of four files: the original Maxima file, the Lisp translation, notes on translation, and the compiled code. If the compilation fails, the fourth item is false.

Some declarations and definitions take effect as soon as the Lisp code is compiled (without loading the compiled code). These include functions defined with the := operator, macros define with the ::= operator, alias, declare, define_variable, mode_declare, and infix, matchfix, nofix, postfix, prefix, and compfile.

Assignments and function calls are not evaluated until the compiled code is loaded. In particular, within the Maxima file, assignments to the translation flags (tr_numer, etc.) have no effect on the translation.

filename may not contain :lisp statements.

compile_file evaluates its arguments.

declare_translated (f_1, f_2, ...)
[Function]
When translating a file of Maxima code to Lisp, it is important for the translator to know which functions it sees in the file are to be called as translated or compiled functions, and which ones are just Maxima functions or undefined. Putting this declaration at the top of the file, lets it know that although a symbol does which does not yet have a Lisp function value, will have one at call time. (MFUNCTION-CALL fn arg1 arg2 ...) is generated when the translator does not know fn is going to be a Lisp function.
37 Program Flow

37.1 Lisp and Maxima

Maxima is written in Lisp, and it is easy to access Lisp functions and variables from Maxima and vice versa. Lisp and Maxima symbols are distinguished by a naming convention. A Lisp symbol which begins with a dollar sign $ corresponds to a Maxima symbol without the dollar sign.

A Maxima symbol which begins with a question mark ? corresponds to a Lisp symbol without the question mark. For example, the Maxima symbol foo corresponds to the Lisp symbol $FOO$, while the Maxima symbol ?foo corresponds to the Lisp symbol FOO. Note that ?foo is written without a space between ? and foo; otherwise it might be mistaken for describe ("foo").

Hyphen -, asterisk *, or other special characters in Lisp symbols must be escaped by backslash \ where they appear in Maxima code. For example, the Lisp identifier *foo-bar* is written ?\*foo\-bar\* in Maxima.

Lisp code may be executed from within a Maxima session. A single line of Lisp (containing one or more forms) may be executed by the special command :lisp. For example,

(\%ii) :lisp (foo $x $y)
calls the Lisp function foo with Maxima variables x and y as arguments. The :lisp construct can appear at the interactive prompt or in a file processed by batch or demo, but not in a file processed by load, batchload, translate_file, or compile_file.

The function to_lisp opens an interactive Lisp session. Entering (to-maxima) closes the Lisp session and returns to Maxima.

Lisp functions and variables which are to be visible in Maxima as functions and variables with ordinary names (no special punctuation) must have Lisp names beginning with the dollar sign $.

Maxima is case-sensitive, distinguishing between lowercase and uppercase letters in identifiers. There are some rules governing the translation of names between Lisp and Maxima.

1. A Lisp identifier not enclosed in vertical bars corresponds to a Maxima identifier in lowercase. Whether the Lisp identifier is uppercase, lowercase, or mixed case, is ignored. E.g., Lisp $foo$, $FOO$, and $Foo$ all correspond to Maxima foo. But this is because $foo$, $FOO$ and $Foo$ are converted by the Lisp reader by default to the Lisp symbol $FOO$.

2. A Lisp identifier which is all uppercase or all lowercase and enclosed in vertical bars corresponds to a Maxima identifier with case reversed. That is, uppercase is changed to lowercase and lowercase to uppercase. E.g., Lisp |$FOO| and |$foo| correspond to Maxima foo and FOO, respectively.

3. A Lisp identifier which is mixed uppercase and lowercase and enclosed in vertical bars corresponds to a Maxima identifier with the same case. E.g., Lisp |$Foo| corresponds to Maxima Foo.

The #\$ Lisp macro allows the use of Maxima expressions in Lisp code. #\$expr\$ expands to a Lisp expression equivalent to the Maxima expression expr.

(msetq $foo #\$[x, y]$)
This has the same effect as entering

```
(%i1) foo: [x, y];
```

The Lisp function `displa` prints an expression in Maxima format.

```
(%i1) :lisp #\$[x, y, z]\$
((MLIST SIMP) $X \ Y \ Z$)
(%i1) :lisp (displa '((MLIST SIMP) $X \ Y \ Z$))
[x, y, z]
NIL
```

Functions defined in Maxima are not ordinary Lisp functions. The Lisp function `mfuncall` calls a Maxima function. For example:

```
(%i1) foo(x,y) := x*y$
(%i2) :lisp (mfuncall '$foo 'a 'b)
((MTIMES SIMP) A B)
```

Some Lisp functions are shadowed in the Maxima package, namely the following.

```
complement continue //
float functionp array
exp listen signum
atan asin acos
asinh acosh atanh
tanh cosh sinh
tan break gcd
```

### 37.2 Garbage Collection

Symbolic computation tends to create a good deal of garbage (temporary or intermediate results that are eventually not used), and effective handling of this can be crucial to successful completion of some programs.

Under GCL, on UNIX systems where the mprotect system call is available (including SUN OS 4.0 and some variants of BSD) a stratified garbage collection is available. This limits the collection to pages which have been recently written to. See the GCL documentation under ALLOCATE and GBC. At the Lisp level doing `(setq si::*notify-gbc* t)` will help you determine which areas might need more space.

For other Lisps that run Maxima, we refer the reader to the documentation for that Lisp on how to control GC.

### 37.3 Introduction to Program Flow

Maxima provides a `do` loop for iteration, as well as more primitive constructs such as `go`.

### 37.4 Functions and Variables for Program Flow

`backtrace` [Function]

```
backtrace ()
backtrace (n)
```

Prints the call stack, that is, the list of functions which called the currently active function.
backtrace() prints the entire call stack.

Backtrace (n) prints the n most recent functions, including the currently active function.

Backtrace can be called from a script, a function, or the interactive prompt (not only in a debugging context).

Examples:

- **backtrace()** prints the entire call stack.

  ```maxima
  (%i1) h(x) := g(x/7)$
  (%i2) g(x) := f(x-11)$
  (%i3) f(x) := e(x^2)$
  (%i4) e(x) := (backtrace(), 2*x + 13)$
  (%i5) h(10);
  #0: e(x=4489/49)
  #1: f(x=-67/7)
  #2: g(x=10/7)
  #3: h(x=10)
  (%o5) 9615/49
  (%o5) ----
  (%o5) 49
  ```

- **backtrace (n)** prints the n most recent functions, including the currently active function.

  ```maxima
  (%i1) h(x) := (backtrace(1), g(x/7))$
  (%i2) g(x) := (backtrace(1), f(x-11))$
  (%i3) f(x) := (backtrace(1), e(x^2))$
  (%i4) e(x) := (backtrace(1), 2*x + 13)$
  (%i5) h(10);
  #0: h(x=10)
  #0: g(x=10/7)
  #0: f(x=-67/7)
  #0: e(x=4489/49)
  (%o5) 9615/49
  (%o5) ----
  (%o5) 49
  ```

The `do` statement is used for performing iteration. The general form of the `do` statements Maxima supports is:

- **for variable: initial_value step increment thru limit do body**
• for variable: initial_value step increment while condition do body
• for variable: initial_value step increment unless condition do body
• for variable in list do body

If the loop is expected to generate a list as output the command `makelist` may be the appropriate command to use instead, See Section 5.4.3 [Performance considerations for Lists], page 66.

`initial_value`, `increment`, `limit`, and `body` can be any expression. `list` is a list. If the increment is 1 then "step 1" may be omitted; As always, if `body` needs to contain more than one command these commands can be specified as a comma-separated list surrounded by parenthesis or as a `block`. Due to its great generality the `do` statement will be described in two parts. The first form of the `do` statement (which is shown in the first three items above) is analogous to that used in several other programming languages (Fortran, Algol, PL/I, etc.); then the other features will be mentioned.

The execution of the `do` statement proceeds by first assigning the `initial_value` to the `variable` (henceforth called the control-variable). Then: (1) If the control-variable has exceeded the limit of a `thru` specification, or if the condition of the `unless` is `true`, or if the condition of the `while` is `false` then the `do` terminates. (2) The `body` is evaluated. (3) The increment is added to the control-variable. The process from (1) to (3) is performed repeatedly until the termination condition is satisfied. One may also give several termination conditions in which case the `do` terminates when any of them is satisfied.

In general the `thru` test is satisfied when the control-variable is greater than the `limit` if the `increment` was non-negative, or when the control-variable is less than the `limit` if the `increment` was negative. The `increment` and `limit` may be non-numeric expressions as long as this inequality can be determined. However, unless the `increment` is syntactically negative (e.g. is a negative number) at the time the `do` statement is input, Maxima assumes it will be positive when the `do` is executed. If it is not positive, then the `do` may not terminate properly.

Note that the `limit`, `increment`, and termination condition are evaluated each time through the loop. Thus if any of these involve much computation, and yield a result that does not change during all the executions of the `body`, then it is more efficient to set a variable to their value prior to the `do` and use this variable in the `do` form.

The value normally returned by a `do` statement is the atom `done`. However, the function `return` may be used inside the `body` to exit the `do` prematurely and give it any desired value. Note however that a `return` within a `do` that occurs in a `block` will exit only the `do` and not the `block`. Note also that the `go` function may not be used to exit from a `do` into a surrounding `block`.

The control-variable is always local to the `do` and thus any variable may be used without affecting the value of a variable with the same name outside of the `do`. The control-variable is unbound after the `do` terminates.

(%i1) for a:-3 thru 26 step 7 do display(a)$
   a = - 3
         a = 4
Chapter 37: Program Flow

a = 11

a = 18

a = 25

(%i1) s: 0$
(%i2) for i: 1 while i <= 10 do s: s+i;
(%o2) done
(%i3) s;
(%o3) 55

Note that the condition while i <= 10 is equivalent to unless i > 10 and also thru 10.

(%i1) series: 1$
(%i2) term: exp (sin (x))$
(%i3) for p: 1 unless p > 7 do
(term: diff (term, x)/p,  
   series: series + subst (x=0, term)*x^p)$
(%i4) series;

3 2
7 x - 6 x + 5 x - 4 x + 2 x - x + x + 1
90 240 15 8 2

which gives 8 terms of the Taylor series for e^-sin(x).

(%i1) poly: 0$
(%i2) for i: 1 thru 5 do
   for j: i step -1 thru 1 do
      poly: poly + i*x^j$
(%i3) poly;

5 4 3 2
5 x + 9 x + 12 x + 14 x + 15 x

(%i4) guess: -3.0$
(%i5) for i: 1 thru 10 do
   (guess: subst (guess, x, 0.5*(x + 10/x)),  
      if abs (guess^2 - 10) < 0.00005 then return (guess));
(%o5) - 3.162280701754386

This example computes the negative square root of 10 using the Newton- Raphson iteration a maximum of 10 times. Had the convergence criterion not been met the value returned would have been done.

Instead of always adding a quantity to the control-variable one may sometimes wish to change it in some other way for each iteration. In this case one may use next expression instead of step increment. This will cause the control-variable to be set to the result of evaluating expression each time through the loop.

(%i6) for count: 2 next 3*count thru 20 do display (count)$

count = 2
count = 6

count = 18

As an alternative to `for` `variable` `value` `do...` the syntax `for` `variable` `from` `value` `do...` may be used. This permits the `from` `value` to be placed after the `step` or `next` `value` or after the termination condition. If `from` `value` is omitted then 1 is used as the initial value.

Sometimes one may be interested in performing an iteration where the control-variable is never actually used. It is thus permissible to give only the termination conditions omitting the initialization and updating information as in the following example to compute the square-root of 5 using a poor initial guess.

```maxima
(%i1) x: 1000$
(%i2) thru 20 do x: 0.5*(x + 5.0/x)$
(%i3) x;
(%o3) 2.23606797749979
(%i4) sqrt(5), numer;
(%o4) 2.23606797749979
```

If it is desired one may even omit the termination conditions entirely and just give `do` `body` which will continue to evaluate the `body` indefinitely. In this case the function `return` should be used to terminate execution of the `do`.

```maxima
(%i1) newton (f, x):= ([y, df, dfx], df: diff (f ('x), 'x),
    do (y: ev(df), x: x - f(x)/y,
      if abs (f (x)) < 5e-6 then return (x))$%)
(%i2) sqr (x) := x^2 - 5.0$
(%i3) newton (sqr, 1000)
(%o3) 2.236068027062195
```

(Note that `return`, when executed, causes the current value of `x` to be returned as the value of the `do`. The `block` is exited and this value of the `do` is returned as the value of the `block` because the `do` is the last statement in the block.)

One other form of the `do` is available in Maxima. The syntax is:

```
for `variable` in `list` `end_tests` `do` `body`
```

The elements of `list` are any expressions which will successively be assigned to the `variable` on each iteration of the `body`. The optional termination tests `end_tests` can be used to terminate execution of the `do`; otherwise it will terminate when the `list` is exhausted or when a `return` is executed in the `body`. (In fact, `list` may be any non-atomic expression, and successive parts are taken.)

```maxima
(%i1) for f in [log, rho, atan] do ldisp(f(1))$
(%t1) 0
(%t2) rho(1)
(%t3) %-pi
(%t3) 4
(%i4) ev(%t3, numer);
(%o4) 0.78539816
Function errcatch

Evaluates expr_1, ..., expr_n one by one and returns [expr_n] (a list) if no error occurs. If an error occurs in the evaluation of any argument, errcatch prevents the error from propagating and returns the empty list [] without evaluating any more arguments.

errcatch is useful in batch files where one suspects an error might occur which would terminate the batch if the error weren’t caught.

Function error

Evaluates and prints expr_1, ..., expr_n, and then causes an error return to top level Maxima or to the nearest enclosing errcatch.

The variable error is set to a list describing the error. The first element of error is a format string, which merges all the strings among the arguments expr_1, ..., expr_n, and the remaining elements are the values of any non-string arguments.

erroormsg() formats and prints error. This is effectively reprinting the most recent error message.

Option variable error_size

Default value: 10
error_size modifies error messages according to the size of expressions which appear in them. If the size of an expression (as determined by the Lisp function ERROR-SIZE) is greater than error_size, the expression is replaced in the message by a symbol, and the symbol is assigned the expression. The symbols are taken from the list error_syms.
Otherwise, the expression is smaller than error_size, and the expression is displayed in the message.

See also error and error_syms.

Example:
The size of U, as determined by ERROR-SIZE, is 24.

(%i1) U: (C^D^E + B + A)/(cos(X-1) + 1)$

(%i2) error_size: 20$

(%i3) error ("Example expression is", U);
Example expression is errexp1 -- an error. Quitting. To debug this try debugmode(true);

(%i4) errexp1;

E
D
C  + B + A
--------------
cos(X - 1) + 1

(%i5) error_size: 30$
(%i6) error ("Example expression is", U);

E
D
C + B + A

Example expression is --------------
cos(X - 1) + 1
-- an error. Quitting. To debug this try debugmode(true);

error_syms

[Option variable]
Default value: [errexp1, errexp2, errexp3]
In error messages, expressions larger than error_size are replaced by symbols, and
the symbols are set to the expressions. The symbols are taken from the list error_syms. The first too-large expression is replaced by error_syms[1], the second by error_syms[2], and so on.
If there are more too-large expressions than there are elements of error_syms, symbols are constructed automatically, with the n-th symbol equivalent to concat (‘errexp, n).
See also error and error_size.

errormsg ()

[Function]
Reprints the most recent error message. The variable error holds the message, and errormsg formats and prints it.

errormsg

[Option variable]
Default value: true
When false the output of error messages is suppressed.
The option variable errormsg can not be set in a block to a local value. The global value of errormsg is always present.

(%i1) errormsg;
(%o1) true
(%i2) sin(a,b);
sin: wrong number of arguments.
-- an error. To debug this try: debugmode(true);
(%i3) errormsg:false;
(%o3) false
(%i4) sin(a,b);
-- an error. To debug this try: debugmode(true);

The option variable errormsg can not be set in a block to a local value.

(%i1) f(bool):=block([errormsg:bool], print ("value of errormsg is", errormsg))$
(%i2) errormsg: true;
(%o2) true
(%i3) f(false);
value of errormsg is true
(%o3) true
(\%i4) errormsg: false;
(\%o4) false
(\%i5) f(true);
value of errormsg is false
(\%o5) false

\textbf{go (tag)}

is used within a block to transfer control to the statement of the block which is tagged with the argument to go. To tag a statement, precede it by an atomic argument as another statement in the block. For example:

\begin{verbatim}
block ([x], x:1, loop, x+1, ..., go(loop), ...)
\end{verbatim}

The argument to go must be the name of a tag appearing in the same block. One cannot use go to transfer to tag in a block other than the one containing the go.

\textbf{if}

Represents conditional evaluation. Various forms of if expressions are recognized.

\texttt{if cond\_1 then expr\_1 else expr\_0} evaluates to \texttt{expr\_1} if \texttt{cond\_1} evaluates to \texttt{true}, otherwise the expression evaluates to \texttt{expr\_0}.

The command \texttt{if cond\_1 then expr\_1 elseif cond\_2 then expr\_2 elseif ... else expr\_0} evaluates to \texttt{expr\_k} if \texttt{cond\_k} is \texttt{true} and all preceding conditions are \texttt{false}. If none of the conditions are \texttt{true}, the expression evaluates to \texttt{expr\_0}.

A trailing \texttt{else false} is assumed if \texttt{else} is missing. That is, the command \texttt{if cond\_1 then expr\_1 elseif cond\_2 then expr\_2 elseif ... elseif cond\_n then expr\_n} is equivalent to \texttt{if cond\_1 then expr\_1 elseif cond\_2 then expr\_2 elseif ... elseif cond\_n then expr\_n else false}.

The alternatives \texttt{expr\_0}, ..., \texttt{expr\_n} may be any Maxima expressions, including nested if expressions. The alternatives are neither simplified nor evaluated unless the corresponding condition is \texttt{true}.

The conditions \texttt{cond\_1}, ..., \texttt{cond\_n} are expressions which potentially or actually evaluate to \texttt{true} or \texttt{false}. When a condition does not actually evaluate to \texttt{true} or \texttt{false}, the behavior of if is governed by the global flag \texttt{prederror}. When \texttt{prederror} is \texttt{true}, it is an error if any evaluated condition does not evaluate to \texttt{true} or \texttt{false}. Otherwise, conditions which do not evaluate to \texttt{true} or \texttt{false} are accepted, and the result is a conditional expression.

Among other elements, conditions may comprise relational and logical operators as follows.

<table>
<thead>
<tr>
<th>Operation</th>
<th>Symbol</th>
<th>Type</th>
</tr>
</thead>
<tbody>
<tr>
<td>less than</td>
<td>&lt;</td>
<td>relational infix</td>
</tr>
<tr>
<td>less than or equal to</td>
<td>&lt;=</td>
<td>relational infix</td>
</tr>
<tr>
<td>equality (syntactic)</td>
<td>=</td>
<td>relational infix</td>
</tr>
<tr>
<td>negation of equality</td>
<td>#</td>
<td>relational function</td>
</tr>
<tr>
<td>negation of equal</td>
<td>notequal</td>
<td>relational function</td>
</tr>
</tbody>
</table>
greater than       =>       relational infix
or equal to        >       relational infix
and greater than    >       relational infix
or and             >       logical infix
or not             >       logical prefix

map (f, expr_1, ..., expr_n)  [Function]

Returns an expression whose leading operator is the same as that of the expressions
expr_1, ..., expr_n but whose subparts are the results of applying f to the corresponding
subparts of the expressions. f is either the name of a function of n arguments
or is a lambda form of n arguments.

maperror - if false will cause all of the mapping functions to (1) stop when they
finish going down the shortest expr_i if not all of the expr_i are of the same length
and (2) apply f to [expr_1, expr_2, ...] if the expr_i are not all the same type of
object. If maperror is true then an error message will be given in the above two
instances.

One of the uses of this function is to map a function (e.g. partfrac) onto each term
of a very large expression where it ordinarily wouldn’t be possible to use the function
on the entire expression due to an exhaustion of list storage space in the course of
the computation.

See also scanmap, maplist, outermap, matrixmap and apply.

(%i1) map(f,x+a*y+b*z);        (%o1) f(b z) + f(a y) + f(x)
(%i2) map(lambda([u],partfrac(u,x)),x+1/(x^3+4*x^2+5*x+2));
     1 1 1
(%o2) ----- - ----- + ----------- + x
     x + 2 x + 1 2
                     (x + 1)
(%i3) map(ratsimp, x/(x^2+x)+(y^2+y)/y);
                   1
(%o3) y + ---- + 1
                   x + 1
(%i4) map("=",[a,b],[-0.5,3]);
(%o4) [a = - 0.5, b = 3]

mapatom (expr)  [Function]

Returns true if and only if expr is treated by the mapping routines as an atom. "Map-
atoms" are atoms, numbers (including rational numbers), and subscripted variables.

maperror            [Option variable]

Default value: true

When maperror is false, causes all of the mapping functions, for example
map (f, expr_1, expr_2, ...)

to (1) stop when they finish going down the shortest \( expr_i \) if not all of the \( expr_i \) are of the same length and (2) apply \( f \) to \([ expr_1, expr_2, \ldots] \) if the \( expr_i \) are not all the same type of object.

If \texttt{maperror} is \texttt{true} then an error message is displayed in the above two instances.

\textbf{mapprint} \hspace{1cm} \textbf{[Option variable]}

Default value: \texttt{true}

When \texttt{mapprint} is \texttt{true}, various information messages from \texttt{map}, \texttt{maplist}, and \texttt{fullmap} are produced in certain situations. These include situations where \texttt{map} would use \texttt{apply}, or \texttt{map} is truncating on the shortest list.

If \texttt{mapprint} is \texttt{false}, these messages are suppressed.

\textbf{maplist} \hspace{1cm} \textbf{[Function]}

Returns a list of the applications of \( f \) to the parts of the expressions \( expr_1, \ldots, expr_n \). \( f \) is the name of a function, or a lambda expression.

\texttt{maplist} differs from \texttt{map}(\( f \), \( expr_1, \ldots, expr_n \)) which returns an expression with the same main operator as \( expr_i \) has (except for simplifications and the case where \texttt{map} does an \texttt{apply}).

\textbf{prederror} \hspace{1cm} \textbf{[Option variable]}

Default value: \texttt{false}

When \texttt{prederror} is \texttt{true}, an error message is displayed whenever the predicate of an \texttt{if} statement or an \texttt{is} function fails to evaluate to either \texttt{true} or \texttt{false}.

If \texttt{false}, \texttt{unknown} is returned instead in this case. The \texttt{prederror: false} mode is not supported in translated code; however, \texttt{maybe} is supported in translated code.

See also \texttt{is} and \texttt{maybe}.

\textbf{return} \hspace{1cm} \textbf{[Function]}

May be used to exit explicitly from the current \texttt{block}, \texttt{while}, \texttt{for} or \texttt{do} loop bringing its argument. It therefore can be compared with the \texttt{return} statement found in other programming languages but it yields one difference: In maxima only returns from the current block, not from the entire function it was called in. In this aspect it more closely resembles the \texttt{break} statement from C.

\begin{verbatim}
(%i1) for i:1 thru 10 do o:i;
(%o1) done
(%i2) for i:1 thru 10 do if i=3 then return(i);
(%o2) 3
(%i3) for i:1 thru 10 do
  (block([i],
    i:3,
    return(i)
  ),
    return(8)
);  
(%o3) 8
\end{verbatim}
(%i4) block([i],
    i:4,
    block([o],
        o:5,
        return(o)
    ),
    return(i),
    return(10)
);
(%o4) 4

See also for, while, do and block.

\textbf{scanmap}

\texttt{scanmap}\,(f, expr)

\texttt{scanmap}\,(f, expr, \texttt{bottomup})

Recursively applies \(f\) to \(expr\), in a top down manner. This is most useful when complete factorization is desired, for example:

\begin{verbatim}
(%i1) exp:(a^2+2*a+1)*y + x^2$
(%i2) scanmap(factor,exp);

2 2
(a + 1) y + x
\end{verbatim}

Note the way in which \texttt{scanmap} applies the given function \texttt{factor} to the constituent subexpressions of \(expr\); if another form of \(expr\) is presented to \texttt{scanmap} then the result may be different. Thus, \(%o2\) is not recovered when \texttt{scanmap} is applied to the expanded form of \(exp\):

\begin{verbatim}
(%i3) scanmap(factor,expand(exp));

2 2
a y + 2 a y + y + x
\end{verbatim}

Here is another example of the way in which \texttt{scanmap} recursively applies a given function to all subexpressions, including exponents:

\begin{verbatim}
(%i4) expr : u*v^(a*x+b) + c$
(%i5) scanmap('f, expr);

f(f(f(a) f(x)) + f(b))
\end{verbatim}

\texttt{scanmap}\,(f, expr, \texttt{bottomup}) applies \(f\) to \(expr\) in a bottom-up manner. E.g., for undefined \(f\),

\begin{verbatim}
scanmap(f,a*x+b) ->
  f(a*x+b) -> f(f(a*x)+f(b)) -> f(f(a)*f(x)+f(b))
scanmap(f,a*x+b,bottomup) -> f(a)*f(x)+f(b)
  -> f(f(a)*f(x))+f(b) ->
  f(f(f(a)*f(x)))+f(b)
\end{verbatim}

In this case, you get the same answer both ways.

\textbf{throw (expr)}

\texttt{throw}\,(expr)

Evaluates \(expr\) and throws the value back to the most recent \texttt{catch}. \texttt{throw} is used with \texttt{catch} as a nonlocal return mechanism.
outermap \( (f, a_1, \ldots, a_n) \)  

Applies the function \( f \) to each one of the elements of the outer product \( a_1 \) cross \( a_2 \) \[ \ldots \] cross \( a_n \).

\( f \) is the name of a function of \( n \) arguments or a lambda expression of \( n \) arguments. Each argument \( a_k \) may be a list or nested list, or a matrix, or any other kind of expression.

The outermap return value is a nested structure. Let \( x \) be the return value. Then \( x \) has the same structure as the first list, nested list, or matrix argument, \( x[i_1] \ldots [i_m] \) has the same structure as the second list, nested list, or matrix argument, \( x[i_1] \ldots [i_m][j_1] \ldots [j_n] \) has the same structure as the third list, nested list, or matrix argument, and so on, where \( m, n, \ldots \) are the numbers of indices required to access the elements of each argument (one for a list, two for a matrix, one or more for a nested list). Arguments which are not lists or matrices have no effect on the structure of the return value.

Note that the effect of outermap is different from that of applying \( f \) to each one of the elements of the outer product returned by cartesian_product. outermap preserves the structure of the arguments in the return value, while cartesian_product does not.

outermap evaluates its arguments.

See also map, maplist, and apply.

Examples:

Elementary examples of outermap. To show the argument combinations more clearly, \( F \) is left undefined.

\[(\%i1) \text{outermap} \ (F, [a, b, c], [1, 2, 3]); \]
\%(i1) \[[F(a, 1), F(a, 2), F(a, 3)], [F(b, 1), F(b, 2), F(b, 3)], [F(c, 1), F(c, 2), F(c, 3)]\]

\%(i2) \text{outermap} \ (F, \text{matrix} ([a, b], [c, d]), \text{matrix} ([1, 2], [3, 4]));
\%(i2) \[[[F(a, 1), F(a, 2)], [F(b, 1), F(b, 2)], [F(c, 1), F(c, 2)], [F(d, 1), F(d, 2)]],[[F(a, 3), F(a, 4)], [F(b, 3), F(b, 4)], [F(c, 3), F(c, 4)], [F(d, 3), F(d, 4)]],

\%(i3) \text{outermap} \ (F, [a, b], x, \text{matrix} ([1, 2], [3, 4]));
\%(i3) \[[F(a, x, 1), F(a, x, 2)], [F(b, x, 1), F(b, x, 2)],

\%(i4) \text{outermap} \ (F, [a, b], \text{matrix} ([1, 2]), \text{matrix} ([x], [y]));
\%(i4) \[[[F(a, 1, x)], [F(a, 2, x)]], [[F(a, 1, y)], [F(a, 2, y)]],

\%(i4) \text{outermap} \ (F, [a, b], \text{matrix} ([1, 2]), \text{matrix} ([x], [y]));
\%(i4) \[[F(a, 1, x)], [F(a, 2, x)], [F(a, 1, y)], [F(a, 2, y)]\]
A closer examination of the `outermap` return value. The first, second, and third arguments are a matrix, a list, and a matrix, respectively. The return value is a matrix. Each element of that matrix is a list, and each element of each list is a matrix.

```
(%i11) arg_1 : matrix ([a, b], [c, d]);
          [ a  b ]
     (%o11) [     ]
          [ c  d ]

(%i12) arg_2 : [11, 22];
     (%o12) [11, 22]

(%i13) arg_3 : matrix ([xx, yy]);
          [ xx yy ]
     (%o13) [     ]

(%i14) xx_0 : outermap (lambda ([x, y, z], x / y + z), arg_1, arg_2, arg_3);

Col 1 = [
     [ a a ]
     [ xx + -- yy + -- ],
     [ 11 11 ]
     [ 11 11 ]
     [ b b ]
     [ xx + -- yy + -- ],
     [ 11 11 ]
     [ 11 11 ]
     [ c c ]
     [ xx + -- yy + -- ],
     [ 11 11 ]
     [ 11 11 ]
     [ d d ]
     [ xx + -- yy + -- ]
     [ 11 11 ]
     [ 11 11 ]

Col 2 = [
     [ a a ]
     [ xx + -- yy + -- ],
     [ 11 11 ]
     [ 11 11 ]
     [ b b ]
     [ xx + -- yy + -- ],
     [ 11 11 ]
     [ 11 11 ]
     [ c c ]
     [ xx + -- yy + -- ],
     [ 11 11 ]
     [ 11 11 ]
     [ d d ]
     [ xx + -- yy + -- ]
     [ 11 11 ]
     [ 11 11 ]

(%i15) xx_1 : xx_0 [1][1];
     [ a a ]
     [ xx + -- yy + -- ]
     [ 11 11 ]
     [ 11 11 ]

(%i16) xx_2 : xx_0 [1][1][1];
     [ a a ]
     [ xx + -- yy + -- ]
     [ 11 11 ]

(%i17) xx_3 : xx_0 [1][1][1][1];
     a
     xx + --
     11

(%i18) [op (arg_1), op (arg_2), op (arg_3)];
     [matrix, [, matrix]
outermap preserves the structure of the arguments in the return value, while cartesian_product does not.

(%i11) outermap (F, [a, b, c], [1, 2, 3]);
(%o11) [[F(a, 1), F(a, 2), F(a, 3)], [F(b, 1), F(b, 2), F(b, 3)],
[F(c, 1), F(c, 2), F(c, 3)]
(%i12) setify (flatten (%));
(%o12) {F(a, 1), F(a, 2), F(a, 3), F(b, 1), F(b, 2), F(b, 3),
F(c, 1), F(c, 2), F(c, 3)}
(%i13) map (lambda ([L], apply (F, L)),
cartesian_product ({a, b, c}, {1, 2, 3}));
(%o13) {F(a, 1), F(a, 2), F(a, 3), F(b, 1), F(b, 2), F(b, 3),
F(c, 1), F(c, 2), F(c, 3)}
(%i14) is (equal (%, %th (2)));
(%o14) true
38 Debugging

38.1 Source Level Debugging

Maxima has a built-in source level debugger. The user can set a breakpoint at a function, and then step line by line from there. The call stack may be examined, together with the variables bound at that level.

The command \texttt{:help} or \texttt{:h} shows the list of debugger commands. (In general, commands may be abbreviated if the abbreviation is unique. If not unique, the alternatives will be listed.) Within the debugger, the user can also use any ordinary Maxima functions to examine, define, and manipulate variables and expressions.

A breakpoint is set by the \texttt{:br} command at the Maxima prompt. Within the debugger, the user can advance one line at a time using the \texttt{:n} ("next") command. The \texttt{:bt} ("backtrace") command shows a list of stack frames. The \texttt{:r} ("resume") command exits the debugger and continues with execution. These commands are demonstrated in the example below.

\begin{verbatim}
(%i1) load ("/tmp/foobar.mac");

(%o1) /tmp/foobar.mac

(%i2) :br foo
Turning on debugging debugmode(true)
Bkpt 0 for foo (in /tmp/foobar.mac line 1)

(%i2) bar (2,3);
Bkpt 0:(foobar.mac 1)
/tmp/foobar.mac:1::

(dbm:1) :bt <-- :bt typed here gives a backtrace
#0: foo(y=5)(foobar.mac line 1)
#1: bar(x=2,y=3)(foobar.mac line 9)

(dbm:1) :n <-- Here type :n to advance line
(foobar.mac 2)
/tmp/foobar.mac:2::

(dbm:1) :n <-- Here type :n to advance line
(foobar.mac 3)
/tmp/foobar.mac:3::

(dbm:1) u; <-- Investigate value of u
28

(dbm:1) u: 33; <-- Change u to be 33
33
\end{verbatim}
The file /tmp/foobar.mac is the following:

```lisp
foo(y) := block ([u:y^2],
    u: u+3,
    u: u^2,
    u);
```

```lisp
bar(x,y) := (
    x: x+2,
    y: y+2,
    x: foo(y),
    x+y);
```

**USE OF THE DEBUGGER THROUGH EMACS**

If the user is running the code under GNU emacs in a shell window (dbl shell), or is running the graphical interface version, Xmaxima, then if he stops at a break point, he will see his current position in the source file which will be displayed in the other half of the window, either highlighted in red, or with a little arrow pointing at the right line. He can advance single lines at a time by typing M-n (Alt-n).

Under Emacs you should run in a dbl shell, which requires the dbl.el file in the elisp directory. Make sure you install the elisp files or add the Maxima elisp directory to your path: e.g., add the following to your .emacs file or the site-init.el

```lisp
(setq load-path (cons "/usr/share/maxima/5.9.1/emacs" load-path))
(autoload 'dbl "dbl")
```

then in emacs

M-x dbl

should start a shell window in which you can run programs, for example Maxima, gcl, gdb etc. This shell window also knows about source level debugging, and display of source code in the other window.

The user may set a break point at a certain line of the file by typing C-x space. This figures out which function the cursor is in, and then it sees which line of that function the cursor is on. If the cursor is on, say, line 2 of foo, then it will insert in the other window the command, “:br foo 2”, to break foo at its second line. To have this enabled, the user must have maxima-mode.el turned on in the window in which the file foo.bar.mac is visiting. There are additional commands available in that file window, such as evaluating the function into the Maxima, by typing Alt-Control-x.

### 38.2 Keyword Commands

Keyword commands are special keywords which are not interpreted as Maxima expressions. A keyword command can be entered at the Maxima prompt or the debugger prompt, although not at the break prompt. Keyword commands start with a colon, ':'. For example, to evaluate a Lisp form you may type :lisp followed by the form to be evaluated.

```
(%i1) :lisp (+ 2 3)
```

```
(1094)
```

```
(dbm:1) :r
```

```
(%o2)
```

```
<-- Type :r to resume the computation
```
The number of arguments taken depends on the particular command. Also, you need not type the whole command, just enough to be unique among the break keywords. Thus :br would suffice for :break.

The keyword commands are listed below.

/:break F n/:break F n
Set a breakpoint in function F at line offset n from the beginning of the function. If F is given as a string, then it is assumed to be a file, and n is the offset from the beginning of the file. The offset is optional. If not given, it is assumed to be zero (first line of the function or file).

/:bt/:bt
Print a backtrace of the stack frames

/:continue/:continue
Continue the computation

/:delete/:delete
Delete the specified breakpoints, or all if none are specified

/:disable/:disable
Disable the specified breakpoints, or all if none are specified

/:enable/:enable
Enable the specified breakpoints, or all if none are specified

/:frame n/:frame n
Print stack frame n, or the current frame if none is specified

/:help/:help
Print help on a debugger command, or all commands if none is specified

/:info/:info
Print information about item

/:lisp some-form/:lisp some-form
Evaluate some-form as a Lisp form

/:lisp-quiet some-form/:lisp-quiet some-form
Evaluate Lisp form some-form without any output

/:next/:next
Like :step, except :next steps over function calls

/:quit/:quit
Quit the current debugger level without completing the computation

/:resume/:resume
Continue the computation

/:step/:step
Continue the computation until it reaches a new source line

/:top/:top
Return to the Maxima prompt (from any debugger level) without completing the computation

### 38.3 Functions and Variables for Debugging

ddebugmode

[Option variable]

Default value: \texttt{false}

When a Maxima error occurs, Maxima will start the debugger if debugmode is \texttt{true}. The user may enter commands to examine the call stack, set breakpoints, step through Maxima code, and so on. See debugging for a list of debugger commands.

Enabling debugmode will not catch Lisp errors.
refcheck
[Option variable]
Default value: false
When refcheck is true, Maxima prints a message each time a bound variable is used for the first time in a computation.

setcheck
[Option variable]
Default value: false
If setcheck is set to a list of variables (which can be subscripted), Maxima prints a message whenever the variables, or subscripted occurrences of them, are bound with the ordinary assignment operator : , the :: assignment operator, or function argument binding, but not the function assignment := nor the macro assignment ::= operators. The message comprises the name of the variable and the value it is bound to.
setcheck may be set to all or true thereby including all variables.
Each new assignment of setcheck establishes a new list of variables to check, and any variables previously assigned to setcheck are forgotten.
The names assigned to setcheck must be quoted if they would otherwise evaluate to something other than themselves. For example, if x, y, and z are already bound, then enter

\[
\text{setcheck: ['x, 'y, 'z]}
\]
to put them on the list of variables to check.
No printout is generated when a variable on the setcheck list is assigned to itself, e.g., X: 'X.

setcheckbreak
[Option variable]
Default value: false
When setcheckbreak is true, Maxima will present a break prompt whenever a variable on the setcheck list is assigned a new value. The break occurs before the assignment is carried out. At this point, setval holds the value to which the variable is about to be assigned. Hence, one may assign a different value by assigning to setval.
See also setcheck and setval.

setval
[System variable]
Holds the value to which a variable is about to be set when a setcheckbreak occurs. Hence, one may assign a different value by assigning to setval.
See also setcheck and setcheckbreak.
timer (f_1, ..., f_n)
timer (all)
timer ()
Given functions f_1, ..., f_n, timer puts each one on the list of functions for which timing statistics are collected. timer(f)$ timer(g)$ puts f and then g onto the list; the list accumulates from one call to the next.
timer(all) puts all user-defined functions (as named by the global variable functions) on the list of timed functions.
With no arguments, \texttt{timer} returns the list of timed functions. Maxima records how much time is spent executing each function on the list of timed functions. \texttt{timer\_info} returns the timing statistics, including the average time elapsed per function call, the number of calls, and the total time elapsed. \texttt{untimer} removes functions from the list of timed functions. \texttt{timer} quotes its arguments. \texttt{f(x) := x^2}$ $g: f$ $\texttt{timer(g)}$ does not put $f$ on the timer list.

If \texttt{trace(f)} is in effect, then \texttt{timer(f)} has no effect; \texttt{trace} and \texttt{timer} cannot both be in effect at the same time.

See also \texttt{timer\_devalue}.

\texttt{untimer(f_1, \ldots, f_n)} \hfill [Function]
\texttt{untimer()} \hfill [Function]

Given functions $f_1, \ldots, f_n$, \texttt{untimer} removes each function from the timer list. With no arguments, \texttt{untimer} removes all functions currently on the timer list. After \texttt{untimer}($f$) is executed, \texttt{timer\_info}($f$) still returns previously collected timing statistics, although \texttt{timer\_info()} (with no arguments) does not return information about any function not currently on the timer list. \texttt{timer}($f$) resets all timing statistics to zero and puts $f$ on the timer list again.

\texttt{timer\_devalue} \hfill [Option variable]

Default value: \texttt{false}

When \texttt{timer\_devalue} is \texttt{true}, Maxima subtracts from each timed function the time spent in other timed functions. Otherwise, the time reported for each function includes the time spent in other functions. Note that time spent in untimed functions is not subtracted from the total time.

See also \texttt{timer} and \texttt{timer\_info}.

\texttt{timer\_info(f_1, \ldots, f_n)} \hfill [Function]
\texttt{timer\_info()} \hfill [Function]

Given functions $f_1, \ldots, f_n$, \texttt{timer\_info} returns a matrix containing timing information for each function. With no arguments, \texttt{timer\_info} returns timing information for all functions currently on the timer list.

The matrix returned by \texttt{timer\_info} contains the function name, time per function call, number of function calls, total time, and \texttt{gctime}, which meant "garbage collection time" in the original Macsyma but is now always zero.

The data from which \texttt{timer\_info} constructs its return value can also be obtained by the \texttt{get} function:

\texttt{get(f, 'calls)}; \texttt{get(f, 'runtime)}; \texttt{get(f, 'gctime)};

See also \texttt{timer}.

\texttt{trace(f_1, \ldots, f_n)} \hfill [Function]
\texttt{trace(all)} \hfill [Function]
\texttt{trace()} \hfill [Function]

Given functions $f_1, \ldots, f_n$, \texttt{trace} instructs Maxima to print out debugging information whenever those functions are called. \texttt{trace(f)}$ \texttt{trace(g)}$ puts $f$ and then $g$ onto the list of functions to be traced; the list accumulates from one call to the next.
trace(all) puts all user-defined functions (as named by the global variable functions) on the list of functions to be traced.

With no arguments, trace returns a list of all the functions currently being traced.

The untrace function disables tracing. See also trace_options.

trace quotes its arguments. Thus, f(x) := x^2$ g: f$ trace(g)$ does not put f on the trace list.

When a function is redefined, it is removed from the timer list. Thus after timer(f)$ f(x) := x^2$, function f is no longer on the timer list.

If timer (f) is in effect, then trace (f) has no effect; trace and timer can't both be in effect for the same function.

trace_options (f, option_1, ..., option_n) [Function]

trace_options (f)

Sets the trace options for function f. Any previous options are superseded. trace_options (f, ...) has no effect unless trace (f) is also called (either before or after trace_options).

trace_options (f) resets all options to their default values.

The option keywords are:

- **nprint** Do not print a message at function entry and exit.
- **break** Put a breakpoint before the function is entered, and after the function is exited. See break.
- **lisp_print** Display arguments and return values as Lisp objects.
- **info** Print -> true at function entry and exit.
- **errorcatch** Catch errors, giving the option to signal an error, retry the function call, or specify a return value.

Trace options are specified in two forms. The presence of the option keyword alone puts the option into effect unconditionally. (Note that option foo is not put into effect by specifying foo: true or a similar form; note also that keywords need not be quoted.) Specifying the option keyword with a predicate function makes the option conditional on the predicate.

The argument list to the predicate function is always [level, direction, function, item] where level is the recursion level for the function, direction is either enter or exit, function is the name of the function, and item is the argument list (on entering) or the return value (on exiting).

Here is an example of unconditional trace options:

(%i1) ff(n) := if equal(n, 0) then 1 else n * ff(n - 1)$

(%i2) trace (ff)$

(%i3) trace_options (ff, lisp_print, break)$

(%i4) ff(3);
Here is the same function, with the break option conditional on a predicate:

(%i5) trace_options (ff, break(pp))$

(%i6) pp (level, direction, function, item) := block (print (item),
    return (function = 'ff and level = 3 and direction = exit))$

(%i7) ff(6);

untrace

untrace (f_1, . . ., f_n)
untrace ()

Given functions f_1, . . ., f_n, untrace disables tracing enabled by the trace function. With no arguments, untrace disables tracing for all functions.

untrace returns a list of the functions for which it disabled tracing.
39 alt-display

39.1 Introduction to alt-display

The *alt-display* package provides a means to change the way that Maxima displays its output. The *alt-display1d* and *alt-display2d* Lisp hooks were introduced to Maxima in 2002, but were not easily accessible from the Maxima REPL until the introduction of this package.

The package provides a general purpose function to define alternative display functions, and a separate function to set the display function. The package also provides customized display functions to produce output in \TeX, Texinfo, XML and all three output formats within Texinfo.

Here is a sample session:

```plaintext
(%i1) load("alt-display.mac")$
(%i2) set_alt_display(2,tex_display)$

(%i3) x/(x^2+y^2) = 1;
\mbox{\tt\red({\it \%o_3}) \black}}\{x\over{y^2+x^2}\}=1$

(%i4) set_alt_display(2,mathml_display)$

(%i5) x/(x^2+y^2) = 1;
<math xmlns="http://www.w3.org/1998/Math/MathML"> <mi>mlabel</mi> <mfenced separators=""> <msub><mi>%o</mi>5</msub> <mo>,</mo> <mfrac><mi>x</mi> <msup><mi>y</mi>2</msup> <mo>+</mo> <msup><mi>x</mi>2</msup> </mfrac> <mo>=</mo> <mn>1</mn> </mfenced> </math>

(%i6) set_alt_display(2,multi_display_for_texinfo)$

(%i7) x/(x^2+y^2) = 1;
@iftex
@tex
\mbox{\tt\red({\it \%o_7}) \black}}\{x\over{y^2+x^2}\}=1$
@end tex
@end iftex
@ifhtml
@html
<math xmlns="http://www.w3.org/1998/Math/MathML"> <mi>mlabel</mi> <mfenced separators=""> <msub><mi>%o</mi>7</msub> <mo>,</mo> <mfrac><mi>x</mi> <msup><mi>y</mi>2</msup> <mo>+</mo> <msup><mi>x</mi>2</msup> </mfrac> <mo>=</mo> </mfenced> </math>
@end html
@end ifhtml
```

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If the alternative display function causes an error, the error is trapped and the display function is reset to the default display. In the following example, the error function is set to display the output. This throws an error, which is handled by resetting the 2d-display to the default.

```maxima
(%i8) set_alt_display(2,?error)$

(%i9) x;

Error in *alt-display2d*.
Message: Condition designator ((MLABEL) $%O9 $X) is not of type (OR SYMBOL STRING FUNCTION).

*alt-display2d* reset to nil.
-- an error. To debug this try: debugmode(true);

(%i10) x;
(%o10) x
```

### 39.2 Functions and Variables for alt-display

**define_alt_display (function(input), expr)**

This function is similar to `define`: it evaluates its arguments and expands into a function definition. The function is a function of a single input `input`. For convenience, a substitution is applied to `expr` after evaluation, to provide easy access to Lisp variable names.

Set a time-stamp on each prompt:

```maxima
(%i1) load("alt-display.mac")$

(%i2) display2d:false$

(%i3) define_alt_display(time_stamp(x),
   block([alt_display1d:false,alt_display2d:false],
      simp:true,
      prompt_prefix:printf(false,"~a~%",timedate()),
      displa(x)));

(%o3) time_stamp(x):=block([simp:false],
   block([?\*alt\-display1d\*:false,?\*alt\-display2d\*:false],
      simp:true,
      ...)
```
Chapter 39: alt-display

```
?\*prompt\-prefix\*:printf(false,"~a~%",timedate()),
?displa(x))
(%i4) set_alt_display(1,time_stamp);

(%o4) done
2016-10-03 09:49:27-05:00
(%i5)
The input line %i3 defines time_stamp using define_alt_display. The output line
 %o3 shows that the Maxima variable names alt_display1d, alt_display2d and
 prompt_prefix have been replaced by their Lisp translations, as has displa been
 replaced by ?displa (the display function).

The display variables alt_display1d and alt_display2d are both bound to false
in the body of time_stamp to prevent an infinite recursion in displa.

info_display (form) [Function]
This is an alias for the default 1-d display function. It may be used as an alternative
1-d or 2-d display function.

(%i1) load("alt-display.mac")$

(%i2) set_alt_display(2,info_display);

(%o2) done
(%i3) x/y;

(%o3) x/y

mathml_display (form) [Function]
Produces MathML output.

(%i1) load("alt-display.mac")$

(%i2) set_alt_display(2,mathml_display);
<math xmlns="http://www.w3.org/1998/Math/MathML"> <mi>mlabel</mi> 
<mfenced separators=""> <msub><mi>%o</mi> <mn>2</mn></msub> 
<mo>,</mo><mi>done</mi> </mfenced> </math>

\[ \text{multi_display_for_texinfo (form)} \] [Function]
Produces TeXinfo output using all three display functions.

(%i2) set_alt_display(2,\text{multi_display_for_texinfo})$

(%i3) x/(x^2+y^2);
reset_displays ()
    [Functions]
    Resets the prompt prefix and suffix to the empty string, and sets both 1-d and 2-d display functions to the default.

set_alt_display (num, display-function)
    [Function]
    The input num is the display to set; it may be either 1 or 2. The second input display-function is the display function to use. The display function may be either a Maxima function or a lambda expression.
    Here is an example where the display function is a lambda expression; it just displays the result as \( \text{\TeX} \).
    \[
    (\%i1) \text{load("alt-display.mac")}$
    \]
    \[
    (\%i2) \text{set_alt_display(2, lambda([form], tex(?caddr(form))))}$
    \]
    \[
    (\%i3) \integrate(\exp(-t^2),t,0,inf);
    $${\sqrt{\pi}}/2$$
    \]
    A user-defined display function should take care that it prints its output. A display function that returns a string will appear to display nothing, nor cause any errors.

set_prompt (fix, expr)
    [Function]
    Set the prompt prefix or suffix to expr. The input fix must evaluate to one of prefix, suffix, general, prolog or epilog. The input expr must evaluate to either a string or false; if false, the fix is reset to the default value.
    \[
    (\%i1) \text{load("alt-display.mac")}$
    \]
    \[
    (\%i2) \text{set_prompt('prefix,printf(false,"It is now: \%a\%","\%\%",timedate()))}$
    \]
It is now: 2014-01-07 15:23:23-05:00

The following example shows the effect of each option, except prolog. Note that the epilog prompt is printed as Maxima closes down. The general is printed between the end of input and the output, unless the input line ends in $.

Here is an example to show where the prompt strings are placed.

```
(%i1) load("alt-display.mac")$
(%i2) set_prompt(prefix,"<<prefix>> ",suffix,"<<suffix>> ",general,
    printf(false,"<<general>>"%),epilog,printf(false,"<<epilog>>"%));
(%o2) done

<<prefix>> (%i3) <<suffix>> x/y;
<<general>>
x
(%o3) -
y
<<prefix>> (%i4) <<suffix>> quit();
<<general>>
<<epilog>>
```

Here is an example that shows how to colorize the input and output when Maxima is running in a terminal or terminal emulator like Emacs1.

```
(%i12) set_prompt(prefix,"[1;31m",suffix,"[0;32m",general,"[1;34m",epilog,"[00;m");
(%i12) done
(%i13) integrate(exp(-x^2),x,0,inf);
(%o13) \[\frac{\sqrt{\pi}}{2}\]
(%i14) quit();
```

Each prompt string starts with the ASCII escape character (27) followed by an open square bracket (91); each string ends with a lower-case m (109). The webpages http://misc.flogisoft.com/bash/tip_colors_and_formatting and http://www.tldp.org/HOWTO/Bash-Prompt-HOWTO/x329.html provide information on how to use control strings to set the terminal colors.

1 Readers using the info reader in Emacs will see the actual prompt strings; other readers will see the colorized output.
40 asympa

40.1 Introduction to asympa

asympa is a package for asymptotic analysis. The package contains simplification functions for asymptotic analysis, including the “big O” and “little o” functions that are widely used in complexity analysis and numerical analysis.

load ("asympa") loads this package.

40.2 Functions and variables for asympa
41 augmented_lagrangian

41.1 Functions and Variables for augmented_lagrangian

augmented_lagrangian_method [Function]

augmented_lagrangian_method (FOM, xx, C, yy)
augmented_lagrangian_method (FOM, xx, C, yy, optional_args)
augmented_lagrangian_method ([FOM, grad], xx, C, yy)
augmented_lagrangian_method ([FOM, grad], xx, C, yy, optional_args)

Returns an approximate minimum of the expression FOM with respect to the variables xx, holding the constraints C equal to zero. yy is a list of initial guesses for xx. The method employed is the augmented Lagrangian method (see Refs [1] and [2]).

grad, if present, is the gradient of FOM with respect to xx, represented as a list of expressions, one for each variable in xx. If not present, the gradient is constructed automatically.

FOM and each element of grad, if present, must be ordinary expressions, not names of functions or lambda expressions.

optional_args represents additional arguments, specified as symbol = value. The optional arguments recognized are:

niter Number of iterations of the augmented Lagrangian algorithm

lbfgs_tolerance Tolerance supplied to LBFGS

iprint IPRINT parameter (a list of two integers which controls verbosity) supplied to LBFGS

%lambda Initial value of %lambda to be used for calculating the augmented Lagrangian

This implementation minimizes the augmented Lagrangian by applying the limited-memory BFGS (LBFGS) algorithm, which is a quasi-Newton algorithm.

load("augmented_lagrangian") loads this function.

See also Chapter 66 [lbfgs-pkg], page 969.

References:

Examples:
(%i1) load ("lbfgs");
(%o1) /home/gunter/src/maxima-code/share/lbfgs/lbfgs.mac
(%i2) load ("augmented_lagrangian");
(%o2) /home/gunter/src/maxima-code/share/contrib/augmented_lagrangian.mac
(%i3) FOM: x^2 + 2*y^2;
    2  2
(%o3)       2 y + x
(%i4) xx: [x, y];
(%o4)       [x, y]
(%i5) C: [x + y - 1];
(%o5)       [y + x - 1]
(%i6) yy: [1, 1];
(%o6)       [1, 1]
(%i7) augmented_lagrangian_method(FOM, xx, C, yy, iprint=[-1,0]);
(%o7) [[[x = 0.666659841080023, y = 0.333340272455448],
    %lambda = [- 1.333337940892518]]

Same example as before, but this time the gradient is supplied as an argument.

(%i11) load ("lbfgs")$
(%i12) load ("augmented_lagrangian")$
(%i13) FOM: x^2 + 2*y^2;
    2  2
(%o13)       2 y + x
(%i14) xx: [x, y];
(%o14)       [x, y]
(%i15) grad : [2*x, 4*y];
(%o15)       [2 x, 4 y]
(%i16) C: [x + y - 1];
(%o16)       [y + x - 1]
(%i17) yy: [1, 1];
(%o17)       [1, 1]
(%i18) augmented_lagrangian_method ([FOM, grad], xx, C, yy, iprint = [-1, 0]);
(%o18) [[[x = 0.6666598410800247, y = 0.3333402724554464],
    %lambda = [- 1.333337940892525]]
42 Bernstein

42.1 Functions and Variables for Bernstein

\texttt{bernstein\_poly} \texttt{(k, n, x)}

Provided \(k\) is not a negative integer, the Bernstein polynomials are defined by \(\text{bernstein\_poly}(k,n,x) = \binom{n}{k} x^k (1-x)^{n-k}\); for a negative integer \(k\), the Bernstein polynomial \(\text{bernstein\_poly}(k,n,x)\) vanishes. When either \(k\) or \(n\) are non integers, the option variable \texttt{bernstein\_explicit} controls the expansion of the Bernstein polynomials into its explicit form; example:

\begin{verbatim}
(%i1) load(bernstein)$
(%i2) bernstein_poly(k,n,x);
(%o2) bernstein_poly(k, n, x)
(%i3) bernstein_poly(k,n,x), bernstein_explicit : true;
    n - k   k
(%o3) binomial(n, k) (1 - x) x
\end{verbatim}

The Bernstein polynomials have both a \texttt{gradef} property and an \texttt{integrate} property:

\begin{verbatim}
(%i4) diff(bernstein_poly(k,n,x),x);
(%o4) (bernstein_poly(k - 1, n - 1, x) - bernstein_poly(k, n - 1, x)) n
(%i5) integrate(bernstein_poly(k,n,x),x);
(%o5)
\frac{k + 1}{---------------------------}
\end{verbatim}

For numeric inputs, both real and complex, the Bernstein polynomials evaluate to a numeric result:

\begin{verbatim}
(%i6) bernstein_poly(5,9, 1/2 + %i);
  39375 %i  39375
(%o6) 128 + 256
(%i7) bernstein_poly(5,9, 0.5b0 + %i);
(%o7) 3.076171875b2 %i + 1.5380859375b2
\end{verbatim}

To use \texttt{bernstein\_poly}, first \texttt{load("bernstein").}

\texttt{bernstein\_explicit}

Default value: \texttt{false}

When either \(k\) or \(n\) are non integers, the option variable \texttt{bernstein\_explicit} controls the expansion of \(\text{bernstein}(k,n,x)\) into its explicit form; example:

\begin{verbatim}
(%i1) bernstein_poly(k,n,x);
(%o1) bernstein_poly(k, n, x)
(%i2) bernstein_poly(k,n,x), bernstein_explicit : true;
\end{verbatim}
\[
\text{When both } k \text{ and } n \text{ are explicitly integers, } \text{bernstein}(k,n,x) \text{ always expands to its explicit form.}
\]

\[
\text{multibernstein_poly ([}k1,k2,\ldots, kp\text{], [}n1,n2,\ldots, np\text{], [}x1,x2,\ldots, xp\text{])}
\]

The multibernstein polynomial \( \text{multibernstein_poly ([}k1,k2,\ldots, kp\text{], [}n1,n2,\ldots, np\text{], [}x1,x2,\ldots, xp\text{])} \) is the product of bernstein polynomials \( \text{bernstein_poly}(k1, n1, x1) \text{ bernstein_poly}(k2, n2, x2) \ldots \text{bernstein_poly}(kp, np, xp) \).

To use \( \text{multibernstein_poly} \), first \text{load("bernstein")}.  

\[
\text{bernstein_approx (f, [}x1, x1, \ldots, xn\text{], n)}
\]

Return the \( n \)-th order uniform Bernstein polynomial approximation for the function \((x1, x2, \ldots, xn) \mapsto f\). Examples

\[
(\%i1) \ \text{bernstein_approx}(f(x),[x], 2);  
\]

\[
(\%o1) \quad f(1) \ x + 2 \ f(-) (1 - x) \ x + f(0) (1 - x)
\]

\[
(\%i2) \ \text{bernstein_approx}(f(x,y),[x,y], 2);  
\]

\[
(\%o2) \quad f(1, 1) \ x \ y + 2 \ f(-, 1) (1 - x) \ x \ y
\]

\[
+ f(0, 1) (1 - x) \ y + 2 \ f(1, -) x (1 - y) \ y
\]

\[
+ 4 \ f(-, -) (1 - x) \ y (1 - y) + 2 \ f(0, -) (1 - x) (1 - y)
\]

\[
+ \ f(1, 0) x (1 - y) + 2 \ f(-, 0) (1 - x) x (1 - y)
\]

\[
+ 2 \ f(0, 0) (1 - x) (1 - y)
\]

To use \( \text{bernstein_approx} \), first \text{load("bernstein")}.  

\[
\text{bernstein_expand (e, [}x1, x1, \ldots, xn\text{])}
\]

Express the \text{polynomial} \( e \) exactly as a linear combination of multi-variable Bernstein polynomials.

\[
(\%i1) \ \text{bernstein_expand}(x*y+1,[x,y]);  
\]

\[
(\%o1) \quad 2 \ y + (1 - x) \ y + x (1 - y) + (1 - x) (1 - y)
\]

\[
(\%i2) \ \text{expand(%)};  
\]

\[
(\%o2) \quad x \ y + 1
\]

Maxima signals an error when the first argument isn’t a polynomial.

To use \( \text{bernstein_expand} \), first \text{load("bernstein")}.  

43 bitwise

The package `bitwise` provides functions that allow to manipulate bits of integer constants. As always maxima attempts to simplify the result of the operation if the actual value of a constant isn’t known considering attributes that might be known for the variables, see the `declare` mechanism.

43.1 Functions and Variables for bitwise

**bit_not (int)**

Inverts all bits of a signed integer. The result of this action reads $-\text{int} - 1$.

```
(%i1) load("bitwise")
(%i2) bit_not(i);
(%o2) bit_not(i)
(%i3) bit_not(bit_not(i));
(%o3) i
(%i4) bit_not(3);
(%o4) - 4
(%i5) bit_not(100);
(%o5) - 101
(%i6) bit_not(-101);
(%o6) 100
```

**bit_and (int1, ...)**

This function calculates a bitwise and of two or more signed integers.

```
(%i1) load("bitwise")
(%i2) bit_and(i,i);
(%o2) i
(%i3) bit_and(i,i,i);
(%o3) i
(%i4) bit_and(1,3);
(%o4) 1
(%i5) bit_and(-7,7);
(%o5) 1
```

If it is known if one of the parameters to `bit_and` is even this information is taken into consideration by the function.

```
(%i1) load("bitwise")
(%i2) declare(e,even,o,odd);
(%o2) done
(%i3) bit_and(1,e);
(%o3) 0
(%i4) bit_and(1,o);
(%o4) 1
```

**bit_or (int1, ...)**

This function calculates a bitwise or of two or more signed integers.

```
(%i1) load("bitwise")
```
(%i2) bit_or(i,i);
  
  (%o2) i

(%i3) bit_or(i,i,i);
  
  (%o3) i

(%i4) bit_or(1,3);
  
  (%o4) 3

(%i5) bit_or(-7,7);
  
  (%o5) -1

If it is known if one of the parameters to bit_or is even this information is taken into
consideration by the function.

(%i1) load("bitwise")$
  
  (%i2) declare(e,even,o,odd);
  
  (%o2) done

(%i3) bit_or(1,e);
  
  (%o3) e + 1

(%i4) bit_or(1,o);
  
  (%o4) o

[Function] bit_xor (int1, ...)

This function calculates a bitwise or of two or more signed integers.

(%i1) load("bitwise")$
  
  (%i2) bit_xor(i,i);
  
  (%o2) 0

(%i3) bit_xor(i,i,i);
  
  (%o3) i

(%i4) bit_xor(1,3);
  
  (%o4) 2

(%i5) bit_xor(-7,7);
  
  (%o5) -2

If it is known if one of the parameters to bit_xor is even this information is taken
into consideration by the function.

(%i1) load("bitwise")$
  
  (%i2) declare(e,even,o,odd);
  
  (%o2) done

(%i3) bit_xor(1,e);
  
  (%o3) e + 1

(%i4) bit_xor(1,o);
  
  (%o4) o - 1

[Function] bit_lsh (int, nBits)

This function shifts all bits of the signed integer int to the left by nBits bits. The
width of the integer is extended by nBits for this process. The result of bit_lsh
therefore is int * 2.

(%i1) load("bitwise")$
  
  (%i2) bit_lsh(0,1);
  
  (%o2) 0
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\[
\begin{align*}
(\%i3) & \text{ bit\_lsh}(1,0); & 1 \\
(\%o3) & \text{ bit\_lsh}(1,1); & 2 \\
(\%i4) & \text{ bit\_lsh}(1,i); & \text{ bit\_lsh}(1, i) \\
(\%o5) & \text{ bit\_lsh}(-3,1); & -6 \\
(\%i6) & \text{ bit\_lsh}(-2,1); & -4 \\
\end{align*}
\]

**bit\_rsh** (\textit{int}, \textit{nBits})

This function shifts all bits of the signed integer \textit{int} to the right by \textit{nBits} bits. The width of the integer is reduced by \textit{nBits} for this process.

\[
\begin{align*}
(\%i1) & \text{ load("bitwise")$} \\
(\%i2) & \text{ bit\_rsh}(0,1); & 0 \\
(\%o2) & \text{ bit\_rsh}(2,0); & 2 \\
(\%i3) & \text{ bit\_rsh}(2,1); & 1 \\
(\%o4) & \text{ bit\_rsh}(2,2); & 0 \\
(\%i5) & \text{ bit\_rsh}(-3,1); & -2 \\
(\%o5) & \text{ bit\_rsh}(-2,1); & -1 \\
(\%i6) & \text{ bit\_rsh}(-2,2); & -1 \\
\end{align*}
\]

**bit\_length** (\textit{int})

determines how many bits a variable needs to be long in order to store the number \textit{int}. This function only operates on positive numbers.

\[
\begin{align*}
(\%i1) & \text{ load("bitwise")$} \\
(\%i2) & \text{ bit\_length}(0); & 0 \\
(\%o2) & \text{ bit\_length}(1); & 1 \\
(\%i3) & \text{ bit\_length}(7); & 3 \\
(\%o3) & \text{ bit\_length}(8); & 4 \\
\end{align*}
\]

**bit\_onep** (\textit{int}, \textit{nBit})

determines if bits \textit{nBit} is set in the signed integer \textit{int}.

\[
\begin{align*}
(\%i1) & \text{ load("bitwise")$} \\
(\%i2) & \text{ bit\_onep}(85,0); & \text{ true}
\end{align*}
\]
(\%i3) bit_onep(85,1);  
(\%o3) false
(\%i4) bit_onep(85,2);  
(\%o4) true
(\%i5) bit_onep(85,3);  
(\%o5) false
(\%i6) bit_onep(85,100);  
(\%o6) false
(\%i7) bit_onep(i,100);  
(\%o7) bit_onep(i, 100)

For signed numbers the sign bit is interpreted to be more than \texttt{nBit} to the left of the leftmost bit of \texttt{int} that reads 1.

(\%i11) load("bitwise")$
(\%i12) bit_onep(-2,0);  
(\%o12) false
(\%i13) bit_onep(-2,1);  
(\%o13) true
(\%i14) bit_onep(-2,2);  
(\%o14) true
(\%i15) bit_onep(-2,3);  
(\%o15) true
(\%i16) bit_onep(-2,4);  
(\%o16) true

If it is known if the number to be tested is even this information is taken into consideration by the function.

(\%i11) load("bitwise")$
(\%i12) declare(e,even,o,odd);  
(\%o12) done
(\%i13) bit_onep(e,0);  
(\%o13) false
(\%i14) bit_onep(o,0);  
(\%o14) true
44 bode

44.1 Functions and Variables for bode

\texttt{bode\_gain (H, range, ...plot\_opts...)}

[Function]

Function to draw Bode gain plots.

Examples (1 through 7 from http://www.swarthmore.edu/NatSci/echeeve1/Ref/Bode/BodeHow.html, 8 from Ron Crummett):

(\%i1) load("bode")$

(\%i2) H1 (s) := 100 * (1 + s) / ((s + 10) * (s + 100))$

(\%i3) bode\_gain (H1 (s), [w, 1/1000, 1000])$

(\%i4) H2 (s) := 1 / (1 + s/omega0)$

(\%i5) bode\_gain (H2 (s), [w, 1/1000, 1000]), omega0 = 10$

(\%i6) H3 (s) := 1 / (1 + s/omega0)^2$

(\%i7) bode\_gain (H3 (s), [w, 1/1000, 1000]), omega0 = 10$

(\%i8) H4 (s) := 1 + s/omega0$

(\%i9) bode\_gain (H4 (s), [w, 1/1000, 1000]), omega0 = 10$

(\%i10) H5 (s) := 1/s$

(\%i11) bode\_gain (H5 (s), [w, 1/1000, 1000])$

(\%i12) H6 (s) := 1/((s/omega0)^2 + 2 * zeta * (s/omega0) + 1)$

(\%i13) bode\_gain (H6 (s), [w, 1/1000, 1000]),
         omega0 = 10, zeta = 1/10$

(\%i14) H7 (s) := (s/omega0)^2 + 2 * zeta * (s/omega0) + 1$

(\%i15) bode\_gain (H7 (s), [w, 1/1000, 1000]),
         omega0 = 10, zeta = 1/10$

(\%i16) H8 (s) := 0.5 / (0.0001 * s^3 + 0.002 * s^2 + 0.01 * s)$

(\%i17) bode\_gain (H8 (s), [w, 1/1000, 1000])$

To use this function write first load("bode"). See also \texttt{bode\_phase}. 
**bode_phase** (**H**, **range**, ...**plot_opts**...)  
[Function]  
Function to draw Bode phase plots.  
Examples (1 through 7 from  
http://www.swarthmore.edu/NatSci/echeeve1/Ref/Bode/BodeHow.html,  
8 from Ron Crummett):  

```
(%i1) load("bode")$

(%i2) H1 (s) := 100 * (1 + s) / ((s + 10) * (s + 100))$

(%i3) bode_phase (H1 (s), [w, 1/1000, 1000])$

(%i4) H2 (s) := 1 / (1 + s/omega0)$

(%i5) bode_phase (H2 (s), [w, 1/1000, 1000]), omega0 = 10$

(%i6) H3 (s) := 1 / (1 + s/omega0)^2$

(%i7) bode_phase (H3 (s), [w, 1/1000, 1000]), omega0 = 10$

(%i8) H4 (s) := 1 + s/omega0$

(%i9) bode_phase (H4 (s), [w, 1/1000, 1000]), omega0 = 10$

(%i10) H5 (s) := 1/s$

(%i11) bode_phase (H5 (s), [w, 1/1000, 1000])$

(%i12) H6 (s) := 1/((s/omega0)^2 + 2 * zeta * (s/omega0) + 1)$

(%i13) bode_phase (H6 (s), [w, 1/1000, 1000]),  
    omega0 = 10, zeta = 1/10$

(%i14) H7 (s) := (s/omega0)^2 + 2 * zeta * (s/omega0) + 1$

(%i15) bode_phase (H7 (s), [w, 1/1000, 1000]),  
    omega0 = 10, zeta = 1/10$

(%i16) H8 (s) := 0.5 / (0.0001 * s^3 + 0.002 * s^2 + 0.01 * s)$

(%i17) bode_phase (H8 (s), [w, 1/1000, 1000])$

(%i18) block ([bode_phase_unwrap : false],  
    bode_phase (H8 (s), [w, 1/1000, 1000])));

(%i19) block ([bode_phase_unwrap : true],  
    bode_phase (H8 (s), [w, 1/1000, 1000]));
```
To use this function write first `load("bode")`. See also `bode_gain`. 
45  celine

45.1 Introduction to celine

Maxima implementation of Sister Celine’s method. Barton Willis wrote this code. It is released under the Creative Commons CC0 license (https://creativecommons.org/about/cc0).


Let \( f = F(n,k) \). The function celine returns a set of recursion relations for \( F \) of the form

\[
p_{0}(n) * f(n,k) + p_{1}(n) * f(n+1,k) + ... + p_{p}(n) * f(n+p,k+q),
\]

where \( p_{0} \) through \( p_{p} \) are polynomials. If Maxima is unable to determine that \( \frac{\sum(\sum(a(i,j) * F(n+i,k+j),i,0,p),j,0,q)}{F(n,k)} \) is a rational function of \( n \) and \( k \), celine returns the empty set. When \( f \) involves parameters (variables other than \( n \) or \( k \)), celine might make assumptions about these parameters. Using 'put' with a key of 'proviso,' Maxima saves these assumptions on the input label.

To use this function, first load the package integer_sequence, opsubst, and to_poly_solve.

Examples:

```maxima
(%i1) load("integer_sequence")$
(%i2) load("opsubst")$
(%i3) load("to_poly_solve")$
(%i4) load("celine")$
(%i5) celine(n!,n,k,1,0);
(%o5) \{fff(n + 1, k) - n fff(n, k) - fff(n, k)}
```

Verification that this result is correct:

```maxima
(%i1) load("integer_sequence")$
(%i2) load("opsubst")$
(%i3) load("to_poly_solve")$
(%i4) load("celine")$
(%i5) g1: {fff(n+1,k)-n*fff(n,k)-fff(n,k)};
(%o5) \{fff(n + 1, k) - n fff(n, k) - fff(n, k)}
(%i6) ratsimp(minfactorial(first(g1))),fff(n,k) := n!;
(%o6) 0
```

An example with parameters including the test that the result of the example is correct:

```maxima
(%i1) load("integer_sequence")$
(%i2) load("opsubst")$
(%i3) load("to_poly_solve")$
(%i4) load("celine")$
(%i5) e : pochhammer(a,k) * pochhammer(-k,n) / (pochhammer(b,k));
    (a)   (a)   (n)
    (k)   (k)   (k)
(%o5)
    (b)  --------
        k
```
(\%i6) recur : celine(e,n,k,2,1);
(\%o6) \{fff(n + 2, k + 1) - fff(n + 2, k) - b fff(n + 1, k + 1)
+ n ((- fff(n + 1, k + 1)) + 2 fff(n + 1, k) - a fff(n, k)
- fff(n, k)) + a (fff(n + 1, k) - fff(n, k)) + 2 fff(n + 1, k)
2
- n fff(n, k)\}
(\%i7) /* Test this result for correctness */
(\%i8) first(%), fff(n,k) := ''(e)$
(\%i9) makefact(makegamma(%))$
(\%o9) 0
(\%i10) minfactorial(factor(minfactorial(factor(%))))

The proviso data suggests that setting a = b may result in a lower order recursion which
is shown by the following example:

(\%i1) load("integer_sequence")$
(\%i2) load("opsubst")$
(\%i3) load("to_poly_solve")$
(\%i4) load("celine")$
(\%i5) e : pochhammer(a,k) * pochhammer(-k,n) / (pochhammer(b,k));
   (a) (- k)
   k n
   -----------
   (b)
   k

(\%o5)
(\%i6) recur : celine(e,n,k,2,1);
(\%o6) \{fff(n + 2, k + 1) - fff(n + 2, k) - b fff(n + 1, k + 1)
+ n ((- fff(n + 1, k + 1)) + 2 fff(n + 1, k) - a fff(n, k)
- fff(n, k)) + a (fff(n + 1, k) - fff(n, k)) + 2 fff(n + 1, k)
2
- n fff(n, k)\}
(\%i7) get('%,'proviso);
(\%o7) false
(\%i8) celine(subst(b=a,e),n,k,1,1);
(\%o8) \{fff(n + 1, k + 1) - fff(n + 1, k) + n fff(n, k)
+ fff(n, k)\}
46 clebsch_gordan

46.1 Functions and Variables for clebsch_gordan

clebsch_gordan \((j_1, j_2, m_1, m_2, j, m)\)  
[Function]  
Compute the Clebsch-Gordan coefficient \(<j_1, j_2, m_1, m_2 \mid j, m>\).

racah_v \((a, b, c, a_1, b_1, c_1)\)  
[Function]  
Compute Racah's V coefficient (computed in terms of a related Clebsch-Gordan coefficient).

racah_w \((j_1, j_2, j_5, j_4, j_3, j_6)\)  
[Function]  
Compute Racah's W coefficient (computed in terms of a Wigner 6j symbol).

wigner_3j \((j_1, j_2, j_3, m_1, m_2, m_3)\)  
[Function]  
Compute Wigner's 3j symbol (computed in terms of a related Clebsch-Gordan coefficient).

wigner_6j \((j_1, j_2, j_3, j_4, j_5, j_6)\)  
[Function]  
Compute Wigner's 6j symbol.

wigner_9j \((a, b, c, d, e, f, g, h, i, j)\)  
[Function]  
Compute Wigner's 9j symbol.
47 cobyla

47.1 Introduction to cobyla

fmin_cobyla is a Common Lisp translation (via f2cl) of the Fortran constrained optimization routine COBYLA by Powell[1][2][3].

COBYLA minimizes an objective function $F(X)$ subject to M inequality constraints of the form $g(X) \geq 0$ on $X$, where $X$ is a vector of variables that has $N$ components.

Equality constraints $g(X)=0$ can often be implemented by a pair of inequality constraints $g(X)\geq 0$ and $-g(X)\geq 0$. Maxima’s interface to COBYLA allows equality constraints and internally converts the equality constraints to a pair of inequality constraints.

The algorithm employs linear approximations to the objective and constraint functions, the approximations being formed by linear interpolation at $N+1$ points in the space of the variables. The interpolation points are regarded as vertices of a simplex. The parameter RHO controls the size of the simplex and it is reduced automatically from RHOBEG to RHOEND. For each RHO the subroutine tries to achieve a good vector of variables for the current size, and then RHO is reduced until the value RHOEND is reached. Therefore RHOBEG and RHOEND should be set to reasonable initial changes to and the required accuracy in the variables respectively, but this accuracy should be viewed as a subject for experimentation because it is not guaranteed. The routine treats each constraint individually when calculating a change to the variables, rather than lumping the constraints together into a single penalty function. The name of the subroutine is derived from the phrase Constrained Optimization BY Linear Approximations.

References:
[1] Fortran Code is from http://plato.asu.edu/sub/nlores.html#general

47.2 Functions and Variables for cobyla

fmin_cobyla

fmin_cobyla (F, X, Y)
fmin_cobyla (F, X, Y, optional_args)

Returns an approximate minimum of the expression $F$ with respect to the variables $X$, subject to an optional set of constraints. $Y$ is a list of initial guesses for $X$.

$F$ must be an ordinary expressions, not names of functions or lambda expressions.

optional_args represents additional arguments, specified as symbol = value. The optional arguments recognized are:
constraints
List of inequality and equality constraints that must be satisfied by \( X \). The inequality constraints must be actual inequalities of the form \( g(X) \geq h(X) \) or \( g(X) \leq h(X) \). The equality constraints must be of the form \( g(X) = h(X) \).

\( \text{rhobeg} \)
Initial value of the internal RHO variable which controls the size of simplex. (Defaults to 1.0)

\( \text{rhoend} \)
The desired final value rho parameter. It is approximately the accuracy in the variables. (Defaults to 1d-6.)

\( \text{iprint} \)
Verbose output level. (Defaults to 0)

- 0 - No output
- 1 - Summary at the end of the calculation
- 2 - Each new value of RHO and SIGMA is printed, including the vector of variables, some function information when RHO is reduced.
- 3 - Like 2, but information is printed when \( F(X) \) is computed.

\( \text{maxfun} \)
The maximum number of function evaluations. (Defaults to 1000).

On return, a vector is given:
1. The value of the variables giving the minimum. This is a list of elements of the form \( \text{var} = \text{value} \) for each of the variables listed in \( X \).
2. The minimized function value
3. The number of function evaluations.
4. Return code with the following meanings
   - 0 - No errors.
   - 1 - Limit on maximum number of function evaluations reached.
   - 2 - Rounding errors inhibiting progress.

\( \text{load(fmin_cobyla)} \)
loads this function.

\( \text{bf_fmin_cobyla} \)
This function is identical to \( \text{fmin_cobyla} \), except that bigfloat operations are used, and the default value for \( \text{rhoend} \) is \( 10^{-(\text{fpprec}/2)} \).

See \( \text{fmin_cobyla} \) for more information.
\( \text{load(bf_fmin_cobyla)} \) loads this function.

### 47.3 Examples for cobyla
Minimize \( x_1 x_2 \) with \( 1-x_1^2-x_2^2 \geq 0 \). The theoretical solution is \( x_1 = 1/\sqrt{2} \), \( x_2 = -1/\sqrt{2} \).

\[(%i1) \text{load(fmin_cobyla)}\]
\[ (%i2) \text{fmin_cobyla}(x1*x2, [x1, x2], [1,1],
    constraints = [x1^2+x2^2<=1], iprint=1); \]
Normal return from subroutine COBYLA

NFVALS =  66  F =-5.000000E-01  MAXCV = 1.999845E-12
X =  7.071058E-01  -7.071077E-01
(%o2) [[x1 = 0.70710584934848, x2 = -0.7071077130248],
    -0.4999999999926, [[-1.999955756559757e-12],[]], 66]

There are additional examples in the share/cobyla/ex directory.
48 contrib_ode

48.1 Introduction to contrib_ode

Maxima's ordinary differential equation (ODE) solver ode2 solves elementary linear ODEs of first and second order. The function contrib_ode extends ode2 with additional methods for linear and non-linear first order ODEs and linear homogeneous second order ODEs. The code is still under development and the calling sequence may change in future releases. Once the code has stabilized it may be moved from the contrib directory and integrated into Maxima.

This package must be loaded with the command load('contrib_ode) before use.

The calling convention for contrib_ode is identical to ode2. It takes three arguments: an ODE (only the left hand side need be given if the right hand side is 0), the dependent variable, and the independent variable. When successful, it returns a list of solutions.

The form of the solution differs from ode2. As non-linear equations can have multiple solutions, contrib_ode returns a list of solutions. Each solution can have a number of forms:

- an explicit solution for the dependent variable,
- an implicit solution for the dependent variable,
- a parametric solution in terms of variable %t, or
- a transformation into another ODE in variable %u.

%c is used to represent the constant of integration for first order equations. %k1 and %k2 are the constants for second order equations. If contrib_ode cannot obtain a solution for whatever reason, it returns false, after perhaps printing out an error message.

It is necessary to return a list of solutions, as even first order non-linear ODEs can have multiple solutions. For example:

```maxima
(%i1) load('contrib_ode)$
(%i2) eqn:x*'diff(y,x)^2-(1+x*y)*'diff(y,x)+y=0;
   dy 2   dy
   x (--) - (1 + x y) -- + y = 0
   dx    dx
(%o2) first order equation not linear in y'
(%i3) contrib_ode(eqn,y,x);
   dy 2   dy
   x (--) - (1 + x y) -- + y = 0
   dx    dx
(%o3) [y = log(x) + %c, y = %c %e ]
(%i4) method;
(%o4) factor
```

Nonlinear ODEs can have singular solutions without constants of integration, as in the second solution of the following example:
(%i1) load('contrib_ode)$
(%i2) eqn:'diff(y,x)^2+x*'diff(y,x)-y=0;
   dy     dy
   --  + x -- - y = 0
   dx     dx
(%o2) (--) + x -- - y = 0
     dx     dx
(%i3) contrib_ode(eqn,y,x);
   dy     dy
   --  + x -- - y = 0
   dx     dx
(%i4) method;
(%o4) clairault

The following ODE has two parametric solutions in terms of the dummy variable %t. In this case the parametric solutions can be manipulated to give explicit solutions.

(%i1) load('contrib_ode)$
(%i2) eqn:'diff(y,x)=(x+y)^2;
   dy
   --
   dx
(%o2) -- = (x + y)
   dx
(%i3) contrib_ode(eqn,y,x);
[[x = %c - atan(sqrt(%t)), y = (- x) - sqrt(%t)],
  [x = atan(sqrt(%t)) + %c, y = sqrt(%t) - x]]
(%i4) method;
(%o4) lagrange

The following example (Kamke 1.112) demonstrates an implicit solution.

(%i1) load('contrib_ode)$
(%i2) assume(x>0,y>0);
(%o2) [x > 0, y > 0]
(%i3) eqn:x*'diff(y,x)-x*sqrt(y^2+x^2)-y;
   dy     2  2
   -- - x sqrt(y + x ) - y
   dx
(%o3) x -- - x sqrt(y + x ) - y
     dx
(%i4) contrib_ode(eqn,y,x);
   y
   [x - asinh(--) = %c]
     x
(%i5) method;
(%o5) lie

The following Riccati equation is transformed into a linear second order ODE in the variable %u. Maxima is unable to solve the new ODE, so it is returned unevaluated.
Chapter 48: contrib_ode

For first order ODEs contrib_ode calls ode2. It then tries the following methods: factorization, Clairault, Lagrange, Riccati, Abel and Lie symmetry methods. The Lie method is not attempted on Abel equations if the Abel method fails, but it is tried if the Riccati method returns an unsolved second order ODE.

For second order ODEs contrib_ode calls ode2 then odelin.

Extensive debugging traces and messages are displayed if the command put('contrib_ode,true,'verbose) is executed.

48.2 Functions and Variables for contrib_ode

contrib_ode (eqn, y, x) [Function]
Returns a list of solutions of the ODE eqn with independent variable x and dependent variable y.

odelin (eqn, y, x) [Function]
odelin solves linear homogeneous ODEs of first and second order with independent variable x and dependent variable y. It returns a fundamental solution set of the ODE.

For second order ODEs, odelin uses a method, due to Bronstein and Lafaille, that searches for solutions in terms of given special functions.

ode_check (eqn, soln) [Function]
Returns the value of ODE eqn after substituting a possible solution soln. The value is equivalent to zero if soln is a solution of eqn.
(%i2) eqn:'diff(y,x,2)+(a*x+b)*y;
 2
   d y
--- + (b + a x) y
   dx
(%o2)

(%i3) ans: [y = bessel_y(1/3,2*(a*x+b)^(3/2)/(3*a))*%k2*sqrt(a*x+b)
    +bessel_j(1/3,2*(a*x+b)^(3/2)/(3*a))*%k1*sqrt(a*x+b)];
    3/2
    1  2 (b + a x)
(%o3) [y = bessel_y(-, ----------------) %k2 sqrt(a x + b)
                3  3 a
    3/2
    1  2 (b + a x)
    + bessel_j(-, ----------------) %k1 sqrt(a x + b)]
                3  3 a
(%i4) ode_check(eqn, ans[1]);
(%o4) 0

method

The variable method is set to the successful solution method.

%c

%c is the integration constant for first order ODEs.

%k1

%k1 is the first integration constant for second order ODEs.

%k2

%k2 is the second integration constant for second order ODEs.

gauss_a (a, b, c, x) and gauss_b (a, b, c, x) are 2F1 geometric functions. They represent any two independent solutions of the hypergeometric differential equation $x(1-x)$ diff(y,x,2) + [c-(a+b+1)x] diff(y,x) - aby = 0 (A&S 15.5.1).

The only use of these functions is in solutions of ODEs returned by odelin and contrib_ode. The definition and use of these functions may change in future releases of Maxima.

See also gauss_b, dgauss_a and gauss_b.

gauss_b (a, b, c, x) 
See gauss_a.

dgauss_a (a, b, c, x) 
The derivative with respect to x of gauss_a(a, b, c, x).

dgauss_b (a, b, c, x) 
The derivative with respect to x of gauss_b(a, b, c, x).
Chapter 48: contrib_ode

[Function] kummer_m (a, b, x)
Kummer’s M function, as defined in Abramowitz and Stegun, *Handbook of Mathematical Functions*, Section 13.1.2.

The only use of this function is in solutions of ODEs returned by `odelin` and `contrib_ode`. The definition and use of this function may change in future releases of Maxima.

See also `kummer_u`, `dkummer_m`, and `dkummer_u`.

[Function] kummer_u (a, b, x)
Kummer’s U function, as defined in Abramowitz and Stegun, *Handbook of Mathematical Functions*, Section 13.1.3.

See `kummer_m`.

[Function] dkummer_m (a, b, x)
The derivative with respect to x of `kummer_m(a, b, x)`.

[Function] dkummer_u (a, b, x)
The derivative with respect to x of `kummer_u(a, b, x)`.

[Function] bessel_simplify (expr)
Simplifies expressions containing Bessel functions `bessel_j`, `bessel_y`, `bessel_i`, `bessel_k`, `hankel_1`, `hankel_2`, `struve_h` and `struve_l`. Recurrence relations (given in Abramowitz and Stegun, *Handbook of Mathematical Functions*, Section 9.1.27) are used to replace functions of highest order n by functions of order n-1 and n-2. This process repeated until all the orders differ by less than 2.

(??) load('contrib_ode)$
(??) bessel_simplify(4*bessel_j(n,x^2)*(x^2-n^2/x^2)
 +x*((bessel_j(n-2,x^2)-bessel_j(n,x^2))*x
 -2*bessel_j(n+1,x^2)-2*bessel_j(n-1,x^2));
(??02) 0
(??) bessel_simplify(-2*bessel_j(1,z)*z^3-10*bessel_j(2,z)*z^2
 +15*%pi*bessel_j(1,z)*struve_h(3,z)*z-15*%pi*bessel_j(3,z)*z
 -15*%pi*bessel_j(0,z)*struve_h(2,z)*z+15*%pi*bessel_j(2,z)*z
 -30*%pi*bessel_j(1,z)*struve_h(2,z)+30*%pi*bessel_j(1,z)*bessel_j(2,z));
(??03) 0

[Function] expintegral_e_simplify (expr)
Simplify expressions containing exponential integral `expintegral_e` using the recurrence (A&S 5.1.14).

`expintegral_e(n+1,z) = (1/n) * (exp(-z)-z*expintegral_e(n,z))` n = 1,2,3 ....

48.3 Possible improvements to contrib_ode

These routines are work in progress. I still need to:

- Extend the FACTOR method `ode1_factor` to work for multiple roots.
- Extend the FACTOR method `ode1_factor` to attempt to solve higher order factors. At present it only attempts to solve linear factors.
• Fix the LAGRANGE routine \texttt{ode1\_lagrange} to prefer real roots over complex roots.
• Add additional methods for Riccati equations.
• Improve the detection of Abel equations of second kind. The existing pattern matching is weak.
• Work on the Lie symmetry group routine \texttt{ode1\_lie}. There are quite a few problems with it: some parts are unimplemented; some test cases seem to run forever; other test cases crash; yet others return very complex "solutions". I wonder if it really ready for release yet.
• Add more test cases.

### 48.4 Test cases for contrib\_ode
The routines have been tested on a approximately one thousand test cases from Murphy, Kamke, Zwillinger and elsewhere. These are included in the tests subdirectory.

• The Clairault routine \texttt{ode1\_clairault} finds all known solutions, including singular solutions, of the Clairault equations in Murphy and Kamke.
• The other routines often return a single solution when multiple solutions exist.
• Some of the "solutions" from \texttt{ode1\_lie} are overly complex and impossible to check.
• There are some crashes.

### 48.5 References for contrib\_ode
2. G. M. Murphy, Ordinary Differential Equations and Their Solutions, Van Nostrand, New York, 1960
49 descriptive

49.1 Introduction to descriptive

Package descriptive contains a set of functions for making descriptive statistical computations and graphing. Together with the source code there are three data sets in your Maxima tree: pidigits.data, wind.data and biomed.data.

Any statistics manual can be used as a reference to the functions in package descriptive.

For comments, bugs or suggestions, please contact me at 'riotorto AT yahoo DOT com'.

Here is a simple example on how the descriptive functions in descriptive do they work, depending on the nature of their arguments, lists or matrices,

```maxima
(%i1) load (descriptive)$
(%i2) /* univariate sample */ mean ([a, b, c]);
   c + b + a
(%o2) ----------
       3
(%i3) matrix ([a, b], [c, d], [e, f]);
    [ a b ]
    [    ]
    [ c d ]
    [    ]
    [ e f ]
(%o3)    
(%i4) /* multivariate sample */ mean (%);
   e + c + a f + d + b
(%o4) [----------, ----------]
       3         3

Note that in multivariate samples the mean is calculated for each column.

In case of several samples with possible different sizes, the Maxima function map can be used to get the desired results for each sample,

```maxima
(%i1) load (descriptive)$
(%i2) map (mean, [[a, b, c], [d, e]]);
   c + b + a e + d
(%o2) [----------, -----]
       3         2

In this case, two samples of sizes 3 and 2 were stored into a list.

Univariate samples must be stored in lists like

```maxima
(%i1) s1 : [3, 1, 4, 1, 5, 9, 2, 6, 5, 3, 5];
(%o1) [3, 1, 4, 1, 5, 9, 2, 6, 5, 3, 5]
```

and multivariate samples in matrices as in
(%i1) s2 : matrix ([13.17, 9.29], [14.71, 16.88], [10.58, 6.63], [13.33, 13.25], [13.21, 8.12]);

(%o1)
[ 13.17 9.29 ]
[       ]
[ 14.71 16.88 ]
[       ]
[ 18.5 16.88 ]
[       ]
[ 10.58 6.63 ]
[       ]
[ 13.33 13.25 ]
[       ]
[ 13.21 8.12 ]

In this case, the number of columns equals the random variable dimension and the number of rows is the sample size.

Data can be introduced by hand, but big samples are usually stored in plain text files. For example, file \texttt{pidigits.data} contains the first 100 digits of number $\pi$:

\[
3 \\
1 \\
4 \\
1 \\
5 \\
9 \\
2 \\
6 \\
5 \\
3 \\
\ldots
\]

In order to load these digits in Maxima,

(%i1) s1 : read_list (file_search ("pidigits.data"));
(%i2) length (s1);
(%o2) 100

On the other hand, file \texttt{wind.data} contains daily average wind speeds at 5 meteorological stations in the Republic of Ireland (This is part of a data set taken at 12 meteorological stations. The original file is freely downloadable from the StatLib Data Repository and its analysis is discussed in Haslett, J., Raftery, A. E. (1989) \textit{Space-time Modelling with Long-memory Dependence: Assessing Ireland's Wind Power Resource, with Discussion}. Applied Statistics 38, 1-50). This loads the data:

(%i1) s2 : read_matrix (file_search ("wind.data"));
(%i2) length (s2);
(%o2) 100
(%i3) s2 [%]; /* last record */
(%o3) [3.58, 6.0, 4.58, 7.62, 11.25]

Some samples contain non numeric data. As an example, file \texttt{biomed.data} (which is part of another bigger one downloaded from the StatLib Data Repository) contains four blood measures taken from two groups of patients, A and B, of different ages,
The first individual belongs to group A, is 30 years old and his/her blood measures were 167.0, 89.0, 25.6 and 364.

One must take care when working with categorical data. In the next example, symbol a is assigned a value in some previous moment and then a sample with categorical value a is taken,

```
(%i1) a : 1$
(%i2) matrix ([a, 3], [b, 5]);
   [ 1 3 ]
   [ ]
   [ b 5 ]
```

### 49.2 Functions and Variables for data manipulation

**build_sample**

- build_sample (list)
- build_sample (matrix)

Builds a sample from a table of absolute frequencies. The input table can be a matrix or a list of lists, all of them of equal size. The number of columns or the length of the lists must be greater than 1. The last element of each row or list is interpreted as the absolute frequency. The output is always a sample in matrix form.

Examples:

**Univariate frequency table.**

```
(%i1) load (descriptive)$
(%i2) sam1: build_sample([[6,1], [j,2], [2,1]]);
   [ 6 ]
   [ ]
   [ j ]
   [ ]
   [ 2 ]
(%o2)
(%i3) mean(sam1);
   2 j + 8
   [-----]
   4
(%o3)
(%i4) barsplot(sam1) $
```

**Multivariate frequency table.**

```
(%i1) load (descriptive)$
(%i2) sam2: build_sample([[6,3,1], [5,6,2], [u,2,1],[6,8,2]]);  
   [ 6 3 ]
(%o2)
```

continuous_freq
continuous_freq(data)
continuous_freq(data, m)

The first argument of continuous_freq must be a list or 1-dimensional array (as created by make_array) of numbers. Divides the range in intervals and counts how many values are inside them. The second argument is optional and either equals the number of classes we want, 10 by default, or equals a list containing the class limits and the number of classes we want, or a list containing only the limits.

If sample values are all equal, this function returns only one class of amplitude 2.

Examples:

Optional argument indicates the number of classes we want. The first list in the output contains the interval limits, and the second the corresponding counts: there are 16 digits inside the interval [0, 1.8], 24 digits in (1.8, 3.6], and so on.

(%i1) load (descriptive)$
(%i2) s1 : read_list (file_search ("pidigits.data"))$
(%i3) continuous_freq (s1, 5);  
(%o3) [[0, 1.8, 3.6, 5.4, 7.2, 9.0], [16, 24, 18, 17, 25]]

Optional argument indicates we want 7 classes with limits -2 and 12:

(%i1) load (descriptive)$
(%i2) s1 : read_list (file_search ("pidigits.data"))$
(%i3) continuous_freq (s1, [-2,12,7]);
(%o3) [[- 2, 0, 2, 4, 6, 8, 10, 12], [8, 20, 22, 17, 20, 13, 0]]

Optional argument indicates we want the default number of classes with limits -2 and 12:

(%i1) load (descriptive)$
(%i2) s1 : read_list (file_search ("pidigits.data"))$
(%i3) continuous_freq (s1, [-2,12]);
      3  4  11  18  32  39  46  53
(%o3)  \([-2, --, --, --, -5, --, --, --, -2, 12],
       5  5  5  5  5  5  5  5
       [0, 8, 20, 12, 18, 9, 8, 25, 0, 0]]

The first argument may be an array.

(%i1) load (descriptive)$
(%i2) s1 : read_list (file_search ("pidigits.data"))$
(%i3) a1 : make_array (fixnum, length (s1))$
(%i4) fillarray (a1, s1);
(%o4)  {Lisp Array:
      #(3 1 4 1 5 9 2 6 5 3 5 8 9 7 9 3 2 3 8 4 6 2 6 4 3 3 8 3 2 7 9 \n        5 0 2 8 8 4 1 9 7 1 6 9 3 9 9 3 7 5 1 0 5 8 2 0 9 7 4 9 4 4 5 9
        2 3 0 7 8 1 6 4 0 6 2 8 6 2 0 8 9 8 6 2 8 0 3 4 8 2 5 3 4 2 \n        1 1 7 0 6 7})
(%i5) continuous_freq (a1);
      9  9  27  18  9  27  63  36  81
(%o5)  [[0, --, --, --, --, --, --, --, --, -9],
       10  5  10  5  2  5  10  5  10
       [8, 8, 12, 12, 10, 8, 9, 8, 12, 13]]

discrete_freq (data)  [Function]
Counts absolute frequencies in discrete samples, both numeric and categorical. Its
unique argument is a list, or 1-dimensional array (as created by make_array).

(%i1) load (descriptive)$
(%i2) s1 : read_list (file_search ("pidigits.data"))$
(%i3) discrete_freq (s1);
(%o3)  [[0, 1, 2, 3, 4, 5, 6, 7, 8, 9],
       [8, 8, 12, 12, 10, 8, 9, 8, 12, 13]]

The first list gives the sample values and the second their absolute frequencies. Commands \texttt{col} and \texttt{transpose} should help you to understand the last input.

The argument may be an array.

(%i1) load (descriptive)$
(%i2) s1 : read_list (file_search ("pidigits.data"))$
(%i3) a1 : make_array (fixnum, length (s1))$
(%i4) fillarray (a1, s1);
(%o4)  {Lisp Array:
      #(3 1 4 1 5 9 2 6 5 3 5 8 9 7 9 3 2 3 8 4 6 2 6 4 3 3 8 3 2 7 9 \n        5 0 2 8 8 4 1 9 7 1 6 9 3 9 9 3 7 5 1 0 5 8 2 0 9 7 4 9 4 4 5 9
        2 3 0 7 8 1 6 4 0 6 2 8 6 2 0 8 9 8 6 2 8 0 3 4 8 2 5 3 4 2 \n        1 1 7 0 6 7})
(%i5) discrete_freq (a1);
(%o5)  [[0, 1, 2, 3, 4, 5, 6, 7, 8, 9],
       [8, 8, 12, 12, 10, 8, 9, 8, 12, 13]]
standardize

`standardize(list)`
`standardize(matrix)`

Subtracts to each element of the list the sample mean and divides the result by the standard deviation. When the input is a matrix, `standardize` subtracts to each row the multivariate mean, and then divides each component by the corresponding standard deviation.

subsample

`subsample(data_matrix, predicate_function)`
`subsample(data_matrix, predicate_function, col_num1, col_num2, ...)`

This is a sort of variant of the Maxima `submatrix` function. The first argument is the data matrix, the second is a predicate function and optional additional arguments are the numbers of the columns to be taken. Its behaviour is better understood with examples.

These are multivariate records in which the wind speed in the first meteorological station were greater than 18. See that in the lambda expression the \( i \)-th component is referred to as \( v[i] \).

```
(%i1) load (descriptive)$
(%i2) s2 : read_matrix (file_search ("wind.data"))$
(%i3) subsample (s2, lambda([v], v[1] > 18));
(%o3) 
[ 19.38 15.37 15.12 23.09 25.25 ]
[ ]
[ 18.29 18.66 19.08 26.08 27.63 ]
```

In the following example, we request only the first, second and fifth components of those records with wind speeds greater or equal than 16 in station number 1 and less than 25 knots in station number 4. The sample contains only data from stations 1, 2 and 5. In this case, the predicate function is defined as an ordinary Maxima function.

```
(%i1) load (descriptive)$
(%i2) s2 : read_matrix (file_search ("wind.data"))$
(%i3) g(x):= x[1] >= 16 and x[4] < 25$
(%i4) subsample (s2, g, 1, 2, 5);
(%o4) 
[ 19.38 15.37 25.25 ]
[ ]
[ 17.33 14.67 19.58 ]
```

Here is an example with the categorical variables of `biomed.data`. We want the records corresponding to those patients in group B who are older than 38 years.

```
(%i1) load (descriptive)$
(%i2) s2 : read_matrix (file_search ("biomed.data"))$
(%i3) g(x):= x[1] == 1 and x[2] > 38$
(%i4) subsample (s2, g, 1, 2, 5);
(%o4) 
[ 19.38 15.37 25.25 ]
[ ]
[ 17.33 14.67 19.58 ]
```

Here is an example with the categorical variables of `biomed.data`. We want the records corresponding to those patients in group B who are older than 38 years.
Probably, the statistical analysis will involve only the blood measures,
Here, the first component is meaningless, since A and B are categorical, the second component is the mean age of individuals in rational form, and the fourth and last values exhibit some strange behaviour. This is because symbol NA is used here to indicate non available data, and the two means are nonsense. A possible solution would be to take out from the matrix those rows with NA symbols, although this deserves some loss of information.

\begin{verbatim}
(%i1) load (descriptive)$
(%i2) data: matrix([3,2,7], [3,7,2], [8,2,4], [5,2,4])$
(%i3) transform_sample(data, [a,b,c], [makelist(1,k,length(data)),a,b])
\end{verbatim}

\textbf{transform_sample (matrix, varlist, exprlist)}

[Function]

Transforms the sample \textit{matrix}, where each column is called according to \textit{varlist}, following expressions in \textit{exprlist}.

Examples:

The second argument assigns names to the three columns. With these names, a list of expressions define the transformation of the sample.

\begin{verbatim}
(%i1) load (descriptive)$
(%i2) data: matrix([3,2,7], [3,7,2], [8,2,4], [5,2,4])$
(%i3) transform_sample(data, [a,b,c], [c, a*b, log(a)]);
\end{verbatim}

Add a constant column and remove the third variable.

\begin{verbatim}
(%i1) load (descriptive)$
(%i2) data: matrix([3,2,7], [3,7,2], [8,2,4], [5,2,4])$
(%i3) transform_sample(data, [a,b,c], [makelist(1,k,length(data)),a,b]);
\end{verbatim}
49.3 Functions and Variables for descriptive statistics

mean

mean(list)
mean(matrix)

This is the sample mean, defined as

\[
\bar{x} = \frac{1}{n} \sum_{i=1}^{n} x_i
\]

Example:

(%i1) load (descriptive)$
(%i2) s1 : read_list (file_search ("pidigits.data"))$
(%i3) mean (s1);
(%,o3) 471
---
100

(%i4) %, numer;
(%,o4) 4.71

(%i5) s2 : read_matrix (file_search ("wind.data"))$
(%i6) mean (s2); [9.9485, 10.1607, 10.8685, 15.7166, 14.8441]

var

var(list)
var(matrix)

This is the sample variance, defined as

\[
\frac{1}{n} \sum_{i=1}^{n} (x_i - \bar{x})^2
\]

Example:

(%i1) load (descriptive)$
(%i2) s1 : read_list (file_search ("pidigits.data"))$
(%i3) var (s1), numer;
(%,o3) 8.42589999999999

See also function var1.
This is the sample variance, defined as

$$\frac{1}{n-1} \sum_{i=1}^{n} (x_i - \bar{x})^2$$

Example:

(%i1) load (descriptive)$
(%i2) s1 : read_list (file_search ("pidigits.data"))$
(%i3) var1 (s1), numer;
(%o3) 8.5110101010101
(%i4) s2 : read_matrix (file_search ("wind.data"))$
(%i5) var1 (s2);
(%o5) [17.39586540404041, 15.13912778787879, 15.63204924242424, 32.50152569696971, 24.66977392929294]

See also function var.

This is the square root of the function var, the variance with denominator $n$.

Example:

(%i1) load (descriptive)$
(%i2) s1 : read_list (file_search ("pidigits.data"))$
(%i3) std (s1), numer;
(%o3) 2.902740084816414
(%i4) s2 : read_matrix (file_search ("wind.data"))$
(%i5) std (s2);
(%o5) [4.149928523480858, 3.871399812729241, 3.933920277534866, 5.672434260526957, 4.941970881136392]

See also functions var and std1.

This is the square root of the function var1, the variance with denominator $n - 1$.

Example:

(%i1) load (descriptive)$
(%i2) s1 : read_list (file_search ("pidigits.data"))$
(%i3) std1 (s1), numer;
(%o3) 2.917363553109228
(%i4) s2 : read_matrix (file_search ("wind.data"))$
(%i5) std1 (s2);
(%o5) [4.170835096721089, 3.89090320978032, 3.953738641137555, 5.701010936401517, 4.966867617451963]

See also functions var1 and std.
The non central moment of order $k$, defined as
\[
\frac{1}{n} \sum_{i=1}^{n} x_i^k
\]

Example:

```lisp
(%i1) load (descriptive)$
(%i2) s1 : read_list (file_search ("pidigits.data"))$
(%i3) noncentral_moment (s1, 1), numer; /* the mean */
(%o3) 4.71
(%i5) s2 : read_matrix (file_search ("wind.data"))$
(%i6) noncentral_moment (s2, 5);
(%o6) [319793.8724761505, 320532.1923892463, 391249.5621381556, 2502278.205988911, 1691881.797742255]
```

See also function `central_moment`.

The central moment of order $k$, defined as
\[
\frac{1}{n} \sum_{i=1}^{n} (x_i - \bar{x})^k
\]

Example:

```lisp
(%i1) load (descriptive)$
(%i2) s1 : read_list (file_search ("pidigits.data"))$
(%i3) central_moment (s1, 2), numer; /* the variance */
(%o3) 8.425899999999999
(%i5) s2 : read_matrix (file_search ("wind.data"))$
(%i6) central_moment (s2, 3);
(%o6) [11.29584771375004, 16.97988248298583, 5.626661952750102, 37.5986572057918, 25.85981904394192]
```

See also functions `central_moment` and `mean`.

The variation coefficient is the quotient between the sample standard deviation ($\text{std}$) and the mean.

```
(%i1) load (descriptive)$
(%i2) s1 : read_list (file_search ("pidigits.data"))$
(%i3) cv (s1), numer;
(%o3) .6193977819764815
```
(%i4) s2 : read_matrix (file_search ("wind.data"))$
(%i5) cv (s2);
(%o5) [.4192426091090204, .3829365309260502, 0.363779605385983,
    .3627381836021478, .3346021393989506]

See also functions std and mean.

smin

smin (list)
smin (matrix)

This is the minimum value of the sample list. When the argument is a matrix, smin returns a list containing the minimum values of the columns, which are associated to statistical variables.

(%i1) load (descriptive)$
(%i2) s1 : read_list (file_search ("pidigits.data"))$
(%i3) smin (s1);
    0
(%i4) s2 : read_matrix (file_search ("wind.data"))$
(%i5) smin (s2);
    [0.58, 0.5, 2.67, 5.25, 5.17]

See also function smax.

smax

smax (list)
smax (matrix)

This is the maximum value of the sample list. When the argument is a matrix, smax returns a list containing the maximum values of the columns, which are associated to statistical variables.

(%i1) load (descriptive)$
(%i2) s1 : read_list (file_search ("pidigits.data"))$
(%i3) smax (s1);
    9
(%i4) s2 : read_matrix (file_search ("wind.data"))$
(%i5) smax (s2);
    [20.25, 21.46, 20.04, 29.63, 27.63]

See also function smin.

range

range (list)
rangle (matrix)

The range is the difference between the extreme values.

Example:

(%i1) load (descriptive)$
(%i2) s1 : read_list (file_search ("pidigits.data"))$
(%i3) range (s1);
    9
(%i4) s2 : read_matrix (file_search ("wind.data"))$
quantile
quantile(list,p)
quantile(matrix,p)
This is the p-quantile, with p a number in [0,1], of the sample list. Although there are several definitions for the sample quantile (Hyndman, R. J., Fan, Y. (1996) Sample quantiles in statistical packages. American Statistician, 50, 361-365), the one based on linear interpolation is implemented in package Chapter 49 [descriptive-pkg], page 675.

Example:
(%i1) load (descriptive)$
(%i2) s1 : read_list (file_search ("pidigits.data"))$
(%i3) /* 1st and 3rd quartiles */
   [quantile (s1, 1/4), quantile (s1, 3/4)], numer;
(%o3) [2.0, 7.25]
(%i4) s2 : read_matrix (file_search ("wind.data"))$
(%i5) quantile (s2, 1/4);
(%o5) [7.2575, 7.477500000000001, 7.82, 11.28, 11.48]

median
median(list)
median(matrix)
Once the sample is ordered, if the sample size is odd the median is the central value, otherwise it is the mean of the two central values.

Example:
(%i1) load (descriptive)$
(%i2) s1 : read_list (file_search ("pidigits.data"))$
(%i3) median (s1);
9
(%o3) -
2
(%i4) s2 : read_matrix (file_search ("wind.data"))$
(%i5) median (s2);
(%o5) [10.06, 9.855, 10.73, 15.48, 14.105]
The median is the 1/2-quantile.
See also function quantile.

qrange
qrange(list)
qrange(matrix)
The interquartilic range is the difference between the third and first quartiles, quantile(list,3/4) - quantile(list,1/4),

(%i1) load (descriptive)$
(%i2) s1 : read_list (file_search ("pidigits.data"))$
(%i3) qrange (s1);
   21
(%o3) --
   4
(%i4) s2 : read_matrix (file_search ("wind.data"))$
(%i5) qrange (s2);
(%o5) [5.385, 5.572499999999998, 6.022500000000001,
    8.729999999999999, 6.649999999999999]

See also function quantile.

mean_deviation

    mean_deviation (list)
    mean_deviation (matrix)

The mean deviation, defined as

\[ \frac{1}{n} \sum_{i=1}^{n} |x_i - \bar{x}| \]

Example:

(%i1) load (descriptive)$
(%i2) s1 : read_list (file_search ("pidigits.data"))$
(%i3) mean_deviation (s1);
   51
(%o3) --
   20
(%i4) s2 : read_matrix (file_search ("wind.data"))$
(%i5) mean_deviation (s2);
(%o5) [3.287959999999999, 3.075342, 3.23907, 4.715664000000001,
    4.028546000000002]

See also function mean.

median_deviation

    median_deviation (list)
    median_deviation (matrix)

The median deviation, defined as

\[ \frac{1}{n} \sum_{i=1}^{n} |x_i - \text{med}| \]

where \text{med} is the median of \text{list}.

Example:

(%i1) load (descriptive)$
(%i2) s1 : read_list (file_search ("pidigits.data"))$
(%i3) median_deviation (s1);
   5
(%o3) -
   2
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```lisp
(%i4) s2 : read_matrix (file_search ("wind.data"))$
(%i5) median_deviation (s2);
(%o5) [2.75, 2.755, 3.08, 4.315, 3.31]
```

See also function `mean`.

**harmonic_mean**

```
harmonic_mean (list)
harmonic_mean (matrix)
```

The harmonic mean, defined as

\[
\frac{n}{\sum_{i=1}^{n} \frac{1}{x_i}}
\]

Example:

```lisp
(%i1) load (descriptive)$
(%i2) y : [5, 7, 2, 5, 9, 5, 6, 4, 9, 2, 4, 2, 5]$
(%i3) harmonic_mean (y), numer;
(%o3) 3.901858027632205
(%i4) s2 : read_matrix (file_search ("wind.data"))$
(%i5) harmonic_mean (s2);
(%o5) [6.948015590052786, 7.391967752360356, 9.055658197151745, 13.44199028193692, 13.01439145898509]
```

See also functions `mean` and `geometric_mean`.

**geometric_mean**

```
geometric_mean (list)
geometric_mean (matrix)
```

The geometric mean, defined as

\[
\left( \prod_{i=1}^{n} x_i \right)^{\frac{1}{n}}
\]

Example:

```lisp
(%i1) load (descriptive)$
(%i2) y : [5, 7, 2, 5, 9, 5, 6, 4, 9, 2, 4, 2, 5]$
(%i3) geometric_mean (y), numer;
(%o3) 4.454845412337012
(%i4) s2 : read_matrix (file_search ("wind.data"))$
(%i5) geometric_mean (s2);
(%o5) [8.82476274347979, 9.22652604739361, 10.0442675714889, 14.61274126349021, 13.96184163444275]
```

See also functions `mean` and `harmonic_mean`.

**kurtosis**

```
kurtosis (list)
kurtosis (matrix)
```

The kurtosis coefficient, defined as

\[
\frac{1}{ns^4} \sum_{i=1}^{n} (x_i - \bar{x})^4 - 3
\]
Example:

```lisp
(%i1) load (descriptive)$
(%i2) s1 : read_list (file_search ("pidigits.data"))$
(%i3) kurtosis (s1), numer;
(%o3) -1.273247946514421
(%i4) s2 : read_matrix (file_search ("wind.data"))$
(%i5) kurtosis (s2);
(%o5) [- .2715445622195385, 0.119998784429451,
    - .427523490482861, - .6405361979019522, - .4952382132352935]
```

See also functions `mean`, `var` and `skewness`.

**skewness**

```
skewness (list)
skewness (matrix)
```

The skewness coefficient, defined as

\[
\frac{1}{ns^3} \sum_{i=1}^{n} (x_i - \bar{x})^3
\]

Example:

```lisp
(%i1) load (descriptive)$
(%i2) s1 : read_list (file_search ("pidigits.data"))$
(%i3) skewness (s1), numer;
(%o3) .009196180476450424
(%i4) s2 : read_matrix (file_search ("wind.data"))$
(%i5) skewness (s2);
(%o5) [.1580509020000978, .2926379232061854, .09242174416107717,
    .2059984348148687, .2142520248890831]
```

See also functions `mean`, `var` and `kurtosis`.

**pearson_skewness**

```
pearson_skewness (list)
pearson_skewness (matrix)
```

Pearson’s skewness coefficient, defined as

\[
\frac{3 (\bar{x} - \text{med})}{s}
\]

where `med` is the median of `list`.

Example:

```lisp
(%i1) load (descriptive)$
(%i2) s1 : read_list (file_search ("pidigits.data"))$
(%i3) pearson_skewness (s1), numer;
(%o3) .2159484029093895
(%i4) s2 : read_matrix (file_search ("wind.data"))$
(%i5) pearson_skewness (s2);
(%o5) [- .08019976629211892, .2357036272952649,
    .1050904062491204, .1245042340592368, .4464181795804519]
```

See also functions `mean`, `var` and `median`. 
quartile_skewness

quartile_skewness (list)
quartile_skewness (matrix)

The quartile skewness coefficient, defined as
\[
\frac{c_3 - 2c_2 + c_1}{c_2 - c_1}
\]
where \(c_p\) is the \(p\)-quantile of sample list.

Example:

```plaintext
(%i1) load (descriptive)$
(%i2) s1 : read_list (file_search ("pidigits.data"))$
(%i3) quartile_skewness (s1), numer;
(%o3) .04761904761904762
(%i4) s2 : read_matrix (file_search ("wind.data"))$
(%i5) quartile_skewness (s2);
(%o5) [- 0.0408542246982353, .1467025572005382,
   0.0336239103362392, .03780068728522298, .2105263157894735]
```

See also function `quantile`.

km

km (list, option ...)
km (matrix, option ...)

Kaplan Meier estimator of the survival, or reliability, function \(S(x) = 1 - F(x)\).

Data can be introduced as a list of pairs, or as a two column matrix. The first component is the observed time, and the second component a censoring index (1 = non censored, 0 = right censored).

The optional argument is the name of the variable in the returned expression, which is \(x\) by default.

Examples:

Sample as a list of pairs.

```plaintext
(%i1) load ("descriptive")$
(%i2) S: km([[2,1], [3,1], [5,0], [8,1]]);
   charfun((3 <= x) and (x < 8))
(%o2) charfun(x < 0) + -----------------------------
     2 3 charfun((2 <= x) and (x < 3))
   + -------------------------------
     4 + charfun((0 <= x) and (x < 2))
(%i3) load ("draw")$
(%i4) draw2d(
   line_width = 3, grid = true,
   explicit(S, x, -0.1, 10))$
```

Estimate survival probabilities.

```plaintext
(%i11) load ("descriptive")$
```
(\%i2) S(t):= 'mk([2,1], [3,1], [5,0], [8,1], t))$
(\%i3) S(6);
1
(\%o3) -
2

\textbf{cdf\_empirical}
\begin{itemize}
\item \texttt{cdf\_empirical(list, option ...)}
\item \texttt{cdf\_empirical(matrix, option ...)}
\end{itemize}
Empirical distribution function $F(x)$.

Data can be introduced as a list of numbers, or as a one column matrix.

The optional argument is the name of the variable in the returned expression, which is $x$ by default.

Example:

Empirical distribution function.

\begin{verbatim}
(%i1) load ("descriptive")$
(%i2) F(x):= '(cdf\_empirical([1,3,3,5,7,7,7,8,9]));
(%o2) F(x) := (charfun(x >= 9) + charfun(x >= 8)
+ 3 charfun(x >= 7) + charfun(x >= 5)
+ 2 charfun(x >= 3) + charfun(x >= 1))/9
(%i3) F(6);
4
(%o3) -
9
(%i4) load(draw)$
(%i5) draw2d(
   line_width = 3,
   grid = true,
   explicit(F(z), z, -2, 12)) $
\end{verbatim}

\textbf{cov(matrix)}
\begin{itemize}
\item \texttt{cov(matrix)}
\end{itemize}
The covariance matrix of the multivariate sample, defined as

$$
S = \frac{1}{n} \sum_{j=1}^{n} (X_j - \bar{X}) (X_j - \bar{X})'
$$

where $X_j$ is the $j$-th row of the sample matrix.

Example:

\begin{verbatim}
(%i1) load ("descriptive")$
(%i2) s2 : read_matrix (file_search ("wind.data"))$
(%i3) fpprintprec : 7$ /* change precision for pretty output */
\end{verbatim}
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(%i4) cov (s2);
[ ]
[ ]
[ ]
[ 19.39624 15.15834 17.32544 32.17651 20.44685 ]
[ ]

See also function \texttt{cov1}.

\textbf{cov1 (matrix)}

The covariance matrix of the multivariate sample, defined as

\[
\frac{1}{n-1} \sum_{j=1}^{n} (X_j - \bar{X}) (X_j - \bar{X})'
\]

where \(X_j\) is the \(j\)-th row of the sample matrix.

Example:

(%i11) load ("descriptive")$
(%i12) s2 : read_matrix (file_search ("wind.data"))$
(%i13) fpprintprec : 7$ /* change precision for pretty output */
(%i14) cov1 (s2);
[ ]
[ 13.75567 15.13913 13.43887 15.31145 15.12232 ]
[ ]
[ ]
[ 19.59216 15.31145 17.50044 32.50153 20.65338 ]
[ ]
[ 15.5774 15.12232 16.34516 20.65338 24.66977 ]

See also function \texttt{cov}.

\textbf{global_variances (matrix)}

Function \texttt{global_variances} returns a list of global variance measures:

- total variance: \(\text{trace}(S_1)\),
- mean variance: \(\text{trace}(S_1)/p\),
- generalized variance: \(\text{determinant}(S_1)\),
- generalized standard deviation: \(\sqrt{\text{determinant}(S_1)}\),
- effective standard deviation: \(\text{determinant}(S_1)^{(1/(2*p))}\).
where \( p \) is the dimension of the multivariate random variable and \( S_1 \) the covariance matrix returned by \texttt{cov1}.

Option:

- \texttt{'data}, default \texttt{'true}, indicates whether the input matrix contains the sample data, in which case the covariance matrix \texttt{cov1} must be calculated, or not, and then the covariance matrix (symmetric) must be given, instead of the data.

Example:

```
(%i1) load ("descriptive")$
(%i2) s2 : read_matrix (file_search ("wind.data"))$
(%i3) global_variances (s2);
(%o3) [105.338342060606, 21.06766841212119, 12874.34690469686,
                   113.4651792608501, 6.636590811800795, 2.576158149609762]
```

Calculate the \texttt{global_variances} from the covariance matrix.

```
(%i1) load ("descriptive")$
(%i2) s2 : read_matrix (file_search ("wind.data"))$
(%i3) s : cov1 (s2)$
(%i4) global_variances (s, data=false);
(%o4) [105.338342060606, 21.06766841212119, 12874.34690469686,
                   113.4651792608501, 6.636590811800795, 2.576158149609762]
```

See also \texttt{cov} and \texttt{cov1}.

\texttt{cor} [Function]

```
cor (matrix)
cor (matrix, logical_value)
```

The correlation matrix of the multivariate sample.

Option:

- \texttt{'data}, default \texttt{'true}, indicates whether the input matrix contains the sample data, in which case the covariance matrix \texttt{cov1} must be calculated, or not, and then the covariance matrix (symmetric) must be given, instead of the data.

Example:

```
(%i1) load ("descriptive")$
(%i2) fpprintprec : 7$
(%i3) s2 : read_matrix (file_search ("wind.data"))$
(%i4) cor (s2);
[ 1.0 .8476339 .8803515 .8239624 .7519506 ]
[ ]
[ .8476339 1.0 .8735834 .6902622 0.782502 ]
[ ]
[ .8803515 .8735834 1.0 .7764065 .8323358 ]
[ ]
[ .8239624 .6902622 .7764065 1.0 .7293848 ]
[ ]
[ .7519506 0.782502 .8323358 .7293848 1.0 ]
```
Calculate the correlation matrix from the covariance matrix.

(%i1) load ("descriptive")$
(%i2) fpprintprec : 7$
(%i3) s2 : read_matrix (file_search ("wind.data"))$
(%i4) s : cov1 (s2)$
(%i5) cor (s, data=false); /* this is faster */
[ 1.0 .8476339 .8803515 .8239624 .7519506 ]
[ ]
[ .8476339 1.0 .8735834 .6902622 0.782502 ]
[ ]
[ ]
(%o5) [ .8803515 .8735834 1.0 .7764065 .8323358 ]
[ ]
[ .8239624 .6902622 .7764065 1.0 .7293848 ]
[ ]
[ ]
[ .7519506 0.782502 .8323358 .7293848 1.0 ]

See also `cov` and `cov1`.

list_correlations

`list_correlations (matrix)`
`list_correlations (matrix, options ...)`

Function `list_correlations` returns a list of correlation measures:

- **precision matrix**: the inverse of the covariance matrix $S_1$,

\[ S_1^{-1} = (s_{ij})_{i,j=1,2,\ldots,p} \]

- **multiple correlation vector**: $(R_2^1, R_2^2, \ldots, R_2^p)$, with

\[ R_i^2 = 1 - \frac{1}{s_{ii} s_{ii}} \]

being an indicator of the goodness of fit of the linear multivariate regression model on $X_i$ when the rest of variables are used as regressors.

- **partial correlation matrix**: with element $(i,j)$ being

\[ r_{ij,\text{rest}} = -\frac{s_{ij}}{\sqrt{s_{ii} s_{jj}}} \]

**Option:**

- `'data`, default `'true`, indicates whether the input matrix contains the sample data, in which case the covariance matrix `cov1` must be calculated, or not, and then the covariance matrix (symmetric) must be given, instead of the data.

**Example:**

(%i1) load ("descriptive")$
(%i2) s2 : read_matrix (file_search ("wind.data"))$
(%i3) z : list_correlations (s2)$
(%i4) fpprintprec : 5$ /* for pretty output */
(\%i5) \(z[1]\); /* precision matrix */
[ [.38486 - .13856 - .15626 - .10239 .031179 ]
[ ]
[ [ - .13856 .34107 - .15233 .038447 - .052842 ]
[ ]
(\%o5) [ - .15626 - .15233 .47296 - .024816 - .10054 ]
[ ]
[ [ - .10239 .038447 - .024816 .10937 - .034033 ]
[ ]
[ .031179 - .052842 - .10054 - .034033 .14834 ]
(\%i6) \(z[2]\); /* multiple correlation vector */
(\%o6) [.85063, .80634, .86474, .71867, .72675]
(\%i7) \(z[3]\); /* partial correlation matrix */
[ [ - 1.0 .38244 .36627 .49908 - .13049 ]
[ ]
[ [ .38244 - 1.0 .37927 - .19907 .23492 ]
[ ]
(\%o7) [ .36627 .37927 - 1.0 .10911 .37956 ]
[ ]
[ [ .49908 - .19907 .10911 - 1.0 .26719 ]
[ ]
[ - .13049 .23492 .37956 .26719 - 1.0 ]

See also \(\texttt{cov}\) and \(\texttt{cov1}\).

\texttt{principal\_components}\hspace{1em}[\text{Function}]

\texttt{principal\_components}\hspace{1em}(\text{matrix})
\texttt{principal\_components}\hspace{1em}(\text{matrix, options} ...)

Calculates the principal components of a multivariate sample. Principal components are used in multivariate statistical analysis to reduce the dimensionality of the sample.

Option:

- \('\texttt{data}', default \('\texttt{true}'\), indicates whether the input matrix contains the sample data, in which case the covariance matrix \(\texttt{cov1}\) must be calculated, or not, and then the covariance matrix (symmetric) must be given, instead of the data.

The output of function \texttt{principal\_components} is a list with the following results:

- variances of the principal components,
- percentage of total variance explained by each principal component,
- rotation matrix.

Examples:

In this sample, the first component explains 83.13 per cent of total variance.

(\%i11) \texttt{load ("descriptive")}$
(\%i12) \texttt{s2 : read\_matrix (file\_search ("wind.data"))}$
(\%i13) \texttt{fpprintprec:4 }$
(\%i14) \texttt{res: principal\_components(s2);}$
0 errors, 0 warnings
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(%o4) \[
\begin{bmatrix}
87.57 & 8.753 & 5.515 & 1.889 & 1.613 \\
83.13 & 8.31 & 5.235 & 1.793 & 1.531 \\
0.369 & 0.3657 & 0.4298 & 0.7237 & 0.1469 \\
0.3959 & 0.2178 & 0.2181 & 0.2749 & 0.8201 \\
0.369 & 0.3657 & 0.4298 & 0.7237 & 0.1469 \\
0.3959 & 0.2178 & 0.2181 & 0.2749 & 0.8201 \\
0.5548 & 0.7744 & 0.1857 & 0.2319 & 0.06498 \\
0.4765 & 0.4669 & 0.712 & 0.09605 & 0.1969
\end{bmatrix}
\]

(%i5) /* accumulated percentages */
block([ap: copy(res[2])],
for k:2 thru length(ap) do ap[k]: ap[k]+ap[k-1],
ap);
(%o5) [83.13, 91.44, 96.68, 98.47, 100.0]

(%i6) /* sample dimension */
p: length(first(res));
(%o6) 5

(%i7) /* plot percentages to select number of principal components for further work */
draw2d(
    fill_density = 0.2,
    apply(bars, makelist([k, res[2][k], 1/2], k, p)),
    points_joined = true,
    point_type = filled_circle,
    point_size = 3,
    points(makelist([k, res[2][k]], k, p)),
    xlabel = "Variances",
    ylabel = "Percentages",
    xtics = setify(makelist([concat("PC",k),k], k, p)))$

In case the covariance matrix is known, it can be passed to the function, but option data=false must be used.

(%i1) load ("descriptive")$
(%i2) S: matrix([1,-2,0], [-2,5,0], [0,0,2]);
(%o2) [1 - 2 0 ]
[ - 2 5 0 ]
[ 0 0 2 ]

(%i3) fpprintprec:4$
(%i4) /* the argument is a covariance matrix */
principal_components(S, data=false);
0 errors, 0 warnings
(%o4) [ - .3827 0.0 .9239 ]
[ ]
(%o4) \[
\begin{bmatrix}
5.828, 2.0, .1716 \\
72.86, 25.0, 2.145 \\
.9239, 0.0, .3827
\end{bmatrix}
\]

(%i5) /* transformation to get the principal components from original records */
matrix([a1,b2,c3],[a2,b2,c2]).last(res);
\[
\begin{bmatrix}
.9239 b2 - .3827 a1 & 1.0 c3 & .3827 b2 + .9239 a1 \\
.9239 b2 - .3827 a2 & 1.0 c2 & .3827 b2 + .9239 a2
\end{bmatrix}
\]

49.4 Functions and Variables for statistical graphs

barsplot(data1, data2, ..., option_1, option_2, ...)

Plots bars diagrams for discrete statistical variables, both for one or multiple samples. data can be a list of outcomes representing one sample, or a matrix of m rows and n columns, representing n samples of size m each.

Available options are:

- **box_width** (default, 3/4): relative width of rectangles. This value must be in the range [0,1].
- **grouping** (default, clustered): indicates how multiple samples are shown. Valid values are: clustered and stacked.
- **groups_gap** (default, 1): a positive integer number representing the gap between two consecutive groups of bars.
- **bars_colors** (default, []): a list of colors for multiple samples. When there are more samples than specified colors, the extra necessary colors are chosen at random. See color to learn more about them.
- **frequency** (default, absolute): indicates the scale of the ordinates. Possible values are: absolute, relative, and percent.
- **ordering** (default, orderlessp): possible values are orderlessp or ordergreatp, indicating how statistical outcomes should be ordered on the x-axis.
- **sample_keys** (default, []): a list with the strings to be used in the legend. When the list length is other than 0 or the number of samples, an error message is returned.
- **start_at** (default, 0): indicates where the plot begins to be plotted on the x axis.
- All global draw options, except xtics, which is internally assigned by barsplot. If you want to set your own values for this option or want to build complex scenes, make use of barsplot_description. See example below.

The following local Chapter 52 [draw-pkg], page 747, options: key, color_draw, fill_color, fill_density and line_width. See also barsplot.

There is also a function wxbarsplot for creating embedded histograms in interfaces wxMaxima and iMaxima. barsplot in a multiplot context.

Examples:

Univariate sample in matrix form. Absolute frequencies.

(%i11) load ("descriptive")$
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(%i2) m : read_matrix (file_search ("biomed.data"))$
(%i3) barsplot(
   col(m,2),
   title = "Ages",
   xlabel = "years",
   box_width = 1/2,
   fill_density = 3/4)$

Two samples of different sizes, with relative frequencies and user declared colors.

(%i1) load ("descriptive")$
(%i2) l1:makelist(random(10),k,1,50)$
(%i3) l2:makelist(random(10),k,1,100)$
(%i4) barsplot(
   l1,l2,
   box_width = 1,
   fill_density = 1,
   bars_colors = [black, grey],
   frequency = relative,
   sample_keys = ["A", "B"])$

Four non numeric samples of equal size.

(%i1) load ("descriptive")$
(%i2) barsplot(
   makelist([Yes, No, Maybe][random(3)+1],k,1,50),
   makelist([Yes, No, Maybe][random(3)+1],k,1,50),
   makelist([Yes, No, Maybe][random(3)+1],k,1,50),
   makelist([Yes, No, Maybe][random(3)+1],k,1,50),
   title = "Asking for something to four groups",
   ylabel = "# of individuals",
   groups_gap = 3,
   fill_density = 0.5,
   ordering = ordergreatp)$

Stacked bars.

(%i1) load ("descriptive")$
(%i2) barsplot(
   makelist([Yes, No, Maybe][random(3)+1],k,1,50),
   makelist([Yes, No, Maybe][random(3)+1],k,1,50),
   makelist([Yes, No, Maybe][random(3)+1],k,1,50),
   makelist([Yes, No, Maybe][random(3)+1],k,1,50),
   title = "Asking for something to four groups",
   ylabel = "# of individuals",
   grouping = stacked,
   fill_density = 0.5,
   ordering = ordergreatp)$

For bars diagrams related options, see barsplot of package Chapter 52 [draw-pkg], page 747. See also functions histogram and piechart.
barsplot_description (...)

Function barsplot_description creates a graphic object suitable for creating complex scenes, together with other graphic objects.

Example: barsplot in a multiplot context.

(%i1) load ("descriptive")$
(%i2) l1:makelist(random(10),k,1,50)$
(%i3) l2:makelist(random(10),k,1,100)$
(%i4) bp1 : barsplot_description(
  l1,
  box_width = 1,
  fill_density = 0.5,
  bars_colors = [blue],
  frequency = relative)$
(%i5) bp2 : barsplot_description(
  l2,
  box_width = 1,
  fill_density = 0.5,
  bars_colors = [red],
  frequency = relative)$
(%i6) draw(gr2d(bp1), gr2d(bp2))$

boxplot (data)

boxplot (data, option_1, option_2, ...)

This function plots box-and-whisker diagrams. Argument data can be a list, which is not of great interest, since these diagrams are mainly used for comparing different samples, or a matrix, so it is possible to compare two or more components of a multivariate statistical variable. But it is also allowed data to be a list of samples with possible different sample sizes, in fact this is the only function in package descriptive that admits this type of data structure.

The box is plotted from the first quartile to the third, with an horizontal segment situated at the second quartile or median. By default, lower and upper whiskers are plotted at the minimum and maximum values, respectively. Option range can be used to indicate that values greater than \( \text{quantile}(x,3/4) + \text{range} \times (\text{quantile}(x,3/4) - \text{quantile}(x,1/4)) \) or less than \( \text{quantile}(x,1/4) - \text{range} \times (\text{quantile}(x,3/4) - \text{quantile}(x,1/4)) \) must be considered as outliers, in which case they are plotted as isolated points, and the whiskers are located at the extremes of the rest of the sample.

Available options are:

- box_width (default, 3/4): relative width of boxes. This value must be in the range \([0,1]\).
- box_orientation (default, vertical): possible values: vertical and horizontal.
- range (default, inf): positive coefficient of the interquartile range to set outliers boundaries.
- outliers_size (default, 1): circle size for isolated outliers.
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- All `draw` options, except `pointsJoined`, `pointSize`, `pointType`, `xtics`, `ytics`, `xrange`, and `yrange`, which are internally assigned by `boxplot`. If you want to set your own values for this options or want to build complex scenes, make use of `boxplot_description`.

- The following local `draw` options: `key`, `color`, and `lineWidth`.

There is also a function `wxboxplot` for creating embedded histograms in interfaces `wxMaxima` and `iMaxima`.

Examples:

Box-and-whisker diagram from a multivariate sample.

```maxima
(%i1) load ("descriptive")$
(%i2) s2 : read_matrix(file_search("wind.data"))$
(%i3) boxplot(s2,
   box_width = 0.2,
   title = "Windspeed in knots",
   xlabel = "Stations",
   color = red,
   line_width = 2)$
```

Box-and-whisker diagram from three samples of different sizes.

```maxima
(%i1) load ("descriptive")$
(%i2) A : [[6, 4, 6, 2, 4, 8, 6, 4, 6, 4, 3, 2],
   [8, 10, 7, 9, 12, 8, 10],
   [16, 13, 17, 12, 11, 18, 13, 18, 14, 12]]$
(%i3) boxplot (A, box_orientation = horizontal)$
```

Option `range` can be used to handle outliers.

```maxima
(%i1) load ("descriptive")$
(%i2) B : [[7, 15, 5, 8, 6, 5, 7, 3, 1],
   [10, 8, 12, 8, 11, 9, 20],
   [23, 17, 19, 7, 22, 19]]$
(%i3) boxplot (B, range=1)$
(%i4) boxplot (B, range=1.5, box_orientation = horizontal)$
(%i5) draw2d(
   boxplot_description (B,
   range = 1.5,
   line_width = 3,
   outliers_size = 2,
   color = red,
   background_color = light_gray),
   xtics = "["Low",1],["Medium",2],["High",3])$`
histogram (list)
histogram (list, option_1, option_2, ...)
histogram (one_column_matrix)
histogram (one_column_matrix, option_1, option_2, ...)
histogram (one_row_matrix)
histogram (one_row_matrix, option_1, option_2, ...)

This function plots an histogram from a continuous sample. Sample data must be stored in a list of numbers or a one dimensional matrix.

Available options are:

- **nclasses** (default, 10): number of classes of the histogram, or a list indicating the limits of the classes and the number of them, or only the limits. This option also accepts bounds for varying bin widths, or a symbol with the name of one of the three optimal algorithms available for the number of classes: 'fd (Freedman, D. and Diaconis, P. (1981) On the histogram as a density estimator: L_2 theory. Zeitschrift fuer Wahrscheinlichkeitstheorie und verwandte Gebiete 57, 453-476.), 'scott (Scott, D. W. (1979) On optimal and data-based histograms. Biometrika 66, 605-610.), and 'sturges (Sturges, H. A. (1926) The choice of a class interval. Journal of the American Statistical Association 21, 65-66).

- **frequency** (default, absolute): indicates the scale of the ordinates. Possible values are: absolute, relative, percent, and density. With density, the histogram area has a total area of one.

- **htics** (default, auto): format of the histogram tics. Possible values are: auto, endpoints, intervals, or a list of labels.

- All global **draw** options, except xrange, yrange, and xtics, which are internally assigned by histogram. If you want to set your own values for these options, make use of histogram_description. See examples below.

- The following local Chapter 52 [draw-pkg], page 747, options: key, color, fill_color, fill_density and line_width. See also barsplot.

There is also a function **wxhistogram** for creating embedded histograms in interfaces wxMaxima and iMaxima.

Examples:

A simple with eight classes:

```maxima
(%i1) load ("descriptive")$
(%i2) s1 : read_list (file_search ("pidigits.data"))$
(%i3) histogram (s1,
   nclasses = 8,
   title = "pi digits",
   xlabel = "digits",
   ylabel = "Absolute frequency",
   fill_color = grey,
   fill_density = 0.6)$
```

Setting the limits of the histogram to -2 and 12, with 3 classes. Also, we introduce predefined tics:
Chapter 49: descriptive

(%i1) load ("descriptive")$
(%i2) s1 : read_list (file_search ("pidigits.data"))$
(%i3) histogram (s1,
   nclasses = [-2,12,3],
   htics = ["A", "B", "C"],
   terminal = png,
   fill_color = "#23afa0",
   fill_density = 0.6)$

Bounds for varying bin widths.

(%i1) load ("descriptive")$
(%i2) s1 : read_list (file_search ("pidigits.data"))$
(%i3) histogram(s1, nclasses=fd) $

histogram_description (...)  [Function]
Function histogram_description creates a graphic object suitable for creating
complex scenes, together with other graphic objects. We make use of histogram_
description for setting the xrange and adding an explicit curve into the
scene:

(%i1) load ("descriptive")$
(%i2) ( load("distrib"),
  m: 14, s: 2,
  s2: random_normal(m, s, 1000) ) $
(%i3) draw2d(
  grid = true,
  xrange = [5, 25],
  histogram_description( 
    s2,
    nclasses = 9,
    frequency = density,
    fill_density = 0.5),
  explicit(pdf_normal(x,m,s), x, m - 3*s, m + 3* s))$

piechart  [Function]

piechart (list)
piechart (list, option_1, option_2, ...)
piechart (one_column_matrix)
piechart (one_column_matrix, option_1, option_2, ...)
piechart (one_row_matrix)
piechart (one_row_matrix, option_1, option_2, ...)

Similar to barsplot, but plots sectors instead of rectangles.
Available options are:
- `sector_colors` (default, `[]`): a list of colors for sectors. When there are more sectors than specified colors, the extra necessary colors are chosen at random. See `color` to learn more about them.

- `pie_center` (default, `[0,0]`): diagram’s center.

- `pie_radius` (default, 1): diagram’s radius.

- All global `draw` options, except `key`, which is internally assigned by `piechart`. If you want to set your own values for this option or want to build complex scenes, make use of `piechart_description`.

- The following local `draw` options: `key`, `color`, `fill_density` and `line_width`. See also `ellipse`.

There is also a function `wxpiechart` for creating embedded histograms in interfaces wxMaxima and iMaxima.

Example:

```maxima
(%i1) load ("descriptive")$
(%i2) s1 : read_list (file_search ("pidigits.data"))$
(%i3) piechart(
    s1,
    xrange = [-1.1, 1.3],
yrange = [-1.1, 1.1],
title = "Digit frequencies in pi")$
```

See also function `barsplot`.

`piechart_description (...)` [Function]

Function `piechart_description` creates a graphic object suitable for creating complex scenes, together with other graphic objects.

`scatterplot` [Function]

```
scatterplot (list)
scatterplot (list, option_1, option_2, ...)
scatterplot (matrix)
scatterplot (matrix, option_1, option_2, ...)
```

Plots scatter diagrams both for univariate (`list`) and multivariate (`matrix`) samples. Available options are the same admitted by `histogram`.

There is also a function `wxscatterplot` for creating embedded histograms in interfaces wxMaxima and iMaxima.

Examples:

Univariate scatter diagram from a simulated Gaussian sample.

```maxima
(%i1) load ("descriptive")$
(%i2) load ("distrib")$
(%i3) scatterplot(
    random_normal(0,1,200),
    xaxis = true,
    point_size = 2,
    dimensions = [600,150])$
```
Two dimensional scatter plot.

```
(%i1) load ("descriptive")$
(%i2) s2 : read_matrix (file_search ("wind.data"))$
(%i3) scatterplot(  
    submatrix(s2, 1,2,3),  
    title = "Data from stations #4 and #5",  
    point_type = diamant,  
    point_size = 2,  
    color = blue)$
```

Three dimensional scatter plot.

```
(%i1) load ("descriptive")$
(%i2) s2 : read_matrix (file_search ("wind.data"))$
(%i3) scatterplot(submatrix (s2, 1,2), nclasses=4)$
```

Five dimensional scatter plot, with five classes histograms.

```
(%i1) load ("descriptive")$
(%i2) s2 : read_matrix (file_search ("wind.data"))$
(%i3) scatterplot(  
    s2,  
    nclasses = 5,  
    frequency = relative,  
    fill_color = blue,  
    fill_density = 0.3,  
    xtics = 5)$
```

For plotting isolated or line-joined points in two and three dimensions, see `points`. See also `histogram`.

`scatterplot_description` (Function)

Function `scatterplot_description` creates a graphic object suitable for creating complex scenes, together with other graphic objects.

`starplot (data1, data2, ..., option_1, option_2, ...)` (Function)

Plots star diagrams for discrete statistical variables, both for one or multiple samples.

data can be a list of outcomes representing one sample, or a matrix of m rows and n columns, representing n samples of size m each.

Available options are:

- `stars_colors` (default, `[]`): a list of colors for multiple samples. When there are more samples than specified colors, the extra necessary colors are chosen at random. See `color` to learn more about them.
- `frequency` (default, `absolute`): indicates the scale of the radii. Possible values are: `absolute` and `relative`.
- `ordering` (default, `orderlessp`): possible values are `orderlessp` or `ordergreatp`, indicating how statistical outcomes should be ordered.
- `sample_keys` (default, `[]`): a list with the strings to be used in the legend. When the list length is other than 0 or the number of samples, an error message is returned.
\begin{itemize}
  \item \texttt{star_center} (default, [0, 0]): diagram's center.
  \item \texttt{star_radius} (default, 1): diagram's radius.
  \item All global \texttt{draw} options, except \texttt{points_joined}, \texttt{point_type}, and \texttt{key}, which are internally assigned by \texttt{starplot}. If you want to set your own values for this options or want to build complex scenes, make use of \texttt{starplot\_description}.
  \item The following local \texttt{draw} option: \texttt{line\_width}.
\end{itemize}

There is also a function \texttt{wxstarplot} for creating embedded histograms in interfaces wxMaxima and iMaxima.

Example:

Plot based on absolute frequencies. Location and radius defined by the user.

\begin{verbatim}
(%i1) load ("descriptive")$
(%i2) l1: makelist(random(10),k,1,50)$
(%i3) l2: makelist(random(10),k,1,200)$
(%i4) starplot(
    l1, l2,
    stars_colors = [blue,red],
    sample_keys = ["1st sample", "2nd sample"],
    star_center = [1,2],
    star_radius = 4,
    proportional_axes = xy,
    line_width = 2 )$
\end{verbatim}

\texttt{starplot\_description (\ldots)} \hfill \texttt{[Function]}

Function \texttt{starplot\_description} creates a graphic object suitable for creating complex scenes, together with other graphic objects.

\texttt{stemplot} \hfill \texttt{[Function]}

\begin{verbatim}
stemplot (data)
stemplot (data, option)
\end{verbatim}

Plots stem and leaf diagrams.

Unique available option is:

\begin{itemize}
  \item \texttt{leaf\_unit} (default, 1): indicates the unit of the leaves; must be a power of 10.
\end{itemize}

Example:

\begin{verbatim}
(%i1) load ("descriptive")$
(%i2) load(distrib)$
\end{verbatim}
(%i3) stemplot(
    random_normal(15, 6, 100),
    leaf_unit = 0.1);

-5|4
  0|37
  1|7
  3|6
  4|4
  5|4
  6|57
  7|0149
  8|3
  9|1334588
  10|07888
  11|01144467789
  12|12566889
  13|24778
  14|047
  15|223458
  16|4
  17|11557
  18|000247
  19|4467799
  20|00
  21|1
  22|2335
  23|01457
  24|12356
  25|455
  27|79

key: 6|3 = 6.3

(%o3) done
50 diag

50.1 Functions and Variables for diag

**diag (lm)**

Constructs a matrix that is the block sum of the elements of *lm*. The elements of *lm* are assumed to be matrices; if an element is scalar, it is treated as a 1 by 1 matrix. The resulting matrix will be square if each of the elements of *lm* is square.

Example:

```
(%i1) load("diag")$
(%i2) a1: matrix([1,2,3],[0,4,5],[0,0,6])$
(%i3) a2: matrix([1,1],[1,0])$
(%i4) diag([a1,x,a2]);

[ 1 2 3 0 0 0 ]
[ ]
[ 0 4 5 0 0 0 ]
[ ]
[ 0 0 6 0 0 0 ]
(%o4)
[ ]
[ 0 0 0 x 0 0 ]
[ ]
[ 0 0 0 1 1 ]
[ ]
[ 0 0 0 1 0 ]
(%i5) diag([matrix([1,2]), 3]);

[ 1 2 0 ]
(%o5)
[ ]
[ 0 0 3 ]
```

To use this function write first `load("diag")`.

**JF (lambda,n)**

Returns the Jordan cell of order *n* with eigenvalue *lambda*.

Example:

```
(%i1) load("diag")$
(%i2) JF(2,5);

[ 2 1 0 0 0 ]
[ ]
[ 0 2 1 0 0 ]
[ ]
(%o2)
[ 0 0 2 1 0 ]
```

To use this function write first `load("diag")`.
(\%i3) JF(3,2);

To use this function write first load("diag").

\textbf{\texttt{jordan(mat)}}

Returns the Jordan form of matrix \texttt{mat}, encoded as a list in a particular format. To get the corresponding matrix, call the function \texttt{dispJordan} using the output of \texttt{jordan} as the argument.

The elements of the returned list are themselves lists. The first element of each is an eigenvalue of \texttt{mat}. The remaining elements are positive integers which are the lengths of the Jordan blocks for this eigenvalue. These integers are listed in decreasing order. Eigenvalues are not repeated.

The functions \texttt{dispJordan}, \texttt{minimalPoly} and \texttt{ModeMatrix} expect the output of a call to \texttt{jordan} as an argument. If you construct this argument by hand, rather than by calling \texttt{jordan}, you must ensure that each eigenvalue only appears once and that the block sizes are listed in decreasing order, otherwise the functions might give incorrect answers.

Example:

(\%i1) load("diag")$
(\%i2) A: matrix([2,0,0,0,0,0,0,0],
[1,2,0,0,0,0,0,0],
[-4,1,2,0,0,0,0,0],
[2,0,0,2,0,0,0,0],
[-7,2,0,0,2,0,0,0],
[9,0,-2,0,1,2,0,0],
[-34,7,1,-2,-1,1,2,0],
[145,-17,-16,3,9,-2,0,3])$
(\%i3) jordan (A);
(\%o3) \[
[2, 3, 3, 1], [3, 1]
\]
(\%i4) dispJordan (%);

```plaintext
[ 2 1 0 0 0 0 0 0 ]
[ 0 2 1 0 0 0 0 0 ]
[ 0 0 2 0 0 0 0 0 ]
[ 0 0 0 2 1 0 0 0 ]
[ 0 0 0 0 2 1 0 0 ]
[ 0 0 0 0 0 2 0 0 ]
```

(\%o4)
To use this function write first `load("diag")`. See also `dispJordan` and `minimalPoly`.

**dispJordan (l)**

Returns a matrix in Jordan canonical form (JCF) corresponding to the list of eigenvalues and multiplicities given by `l`. This list should be in the format given by the `jordan` function. See `jordan` for details of this format.

Example:

```lisp
(%i1) load("diag")$

(%i2) b1:matrix([[0,0,1,1,1],
            [0,0,0,1,1],
            [0,0,0,0,1],
            [0,0,0,0,0],
            [0,0,0,0,0]])$

(%i3) jordan(b1); [[0, 3, 2]]

(%i4) dispJordan(%);
```

To use this function write first `load("diag")`. See also `jordan` and `minimalPoly`.

**minimalPoly (l)**

Returns the minimal polynomial of the matrix whose Jordan form is described by the list `l`. This list should be in the format given by the `jordan` function. See `jordan` for details of this format.

Example:

```lisp
(%i1) load("diag")$

(%i2) a:matrix([2,1,2,0],
            [-2,2,1,2],
            [-2,-1,-1,1],
            [3,1,2,-1])$

(%i3) jordan(a);
To use this function write first load("diag"). See also jordan and dispJordan.

**ModeMatrix (A, [jordan_info])** [Function]

Returns an invertible matrix $M$ such that $(Mm^{-1})A.M$ is the Jordan form of $A$.

To calculate this, Maxima must find the Jordan form of $A$, which might be quite computationally expensive. If that has already been calculated by a previous call to jordan, pass it as a second argument, jordan_info. See jordan for details of the required format.

Example:

```
(%i1) load("diag")$
(%i2) A: matrix([2,1,2,0], [-2,2,1,2], [-2,-1,-1,1], [3,1,2,-1])$
(%i3) M: ModeMatrix (A);
[ 1 - 1 1 1 ]
[ 1 ]
[ -- - 1 0 0 ]
[ 9 ]
[ ]
(%o3)
[ 13 ]
[ -- 1 - 1 0 ]
[ 9 ]
[ ]
[ 17 ]
[ -- 1 1 1 ]
[ 9 ]
(%i4) is ((M^-1) . A . M = dispJordan (jordan (A)));
(%o4) true
```

Note that, in this example, the Jordan form of $A$ is computed twice. To avoid this, we could have stored the output of jordan(A) in a variable and passed that to both ModeMatrix and dispJordan.

To use this function write first load("diag"). See also jordan and dispJordan.

**mat_function (f,A)** [Function]

Returns $f(A)$, where $f$ is an analytic function and $A$ a matrix. This computation is based on the Taylor expansion of $f$. It is not efficient for numerical evaluation, but can give symbolic answers for small matrices.

Example 1:

The exponential of a matrix. We only give the first row of the answer, since the output is rather large.

```
(%i1) load("diag")$
(%i2) A: matrix ([0,1,0], [0,0,1], [-1,-3,-3])$
```
Example 2:

Comparison with the Taylor series for the exponential and also comparing \( \exp(%i*A) \) with sine and cosine.

\[
\begin{align*}
\textcolor{blue}{\text{(i1)}} & \quad \text{load("diag")}\$ \\
\textcolor{blue}{\text{(i2)}} & \quad \text{A: matrix ([[0,1,1,1],}
\begin{bmatrix}
[0,0,0,1],
[0,0,0,1],
[0,0,0,0]\end{bmatrix}\$ \\
\textcolor{blue}{\text{(i3)}} & \quad \text{ratsimp (mat_function (exp, t*A)[1]);}
\begin{bmatrix}
2 & -t \\
(t + 2) & -t \\
\text{---} & \text{---} \\
2 & 2
\end{bmatrix} %e^{2 - t}
\begin{bmatrix}
t & -t \\
t & 2 - t \\
\text{---} & \text{---} \\
2 & 2
\end{bmatrix} %e^{2 - t}
\textcolor{blue}{\text{(o3)}} & \quad \text{------------------------, (t + t) %e , --------} \\
2 & 2
\textcolor{blue}{\text{(i4)}} & \quad \text{minimalPoly (jordan (A));}
\begin{bmatrix}
& & & 3 \\
& & & x
\end{bmatrix}
\textcolor{blue}{\text{(i5)}} & \quad \text{ratsimp (ident(4) + t*A + 1/2*(t^2)*A^^2);}
\begin{bmatrix}
[ & & & ] \\
[ & t & t & t + t ] \\
[ & & & ] \\
[0 & 1 & 0 & t ] \\
[ & & & ] \\
[0 & 0 & 1 & t ] \\
[ & & & ] \\
[0 & 0 & 0 & 1 ] \\
\textcolor{blue}{\text{(o5)}} & \quad \text{------------------------, (t + t) %e , --------} \\
2 & 2
\textcolor{blue}{\text{(i6)}} & \quad \text{ratsimp (mat_function (exp, %i*t*A));}
\begin{bmatrix}
[ & & & ] \\
[ & %i & t & %i & t & %i & t & - t ] \\
[ & & & ] \\
[0 & 1 & 0 & %i & t ] \\
[ & & & ] \\
[0 & 0 & 1 & %i & t ] \\
[ & & & ] \\
[0 & 0 & 0 & 1 ]
\textcolor{blue}{\text{(o6)}} & \quad \text{------------------------, (t + t) %e , --------} \\
2 & 2
\end{align*}
\]
(%i7) ratsimp (mat_function (cos, t*A) + %i*mat_function (sin, t*A));
[ 2 ]
[ 1  %i t  %i t  %i t - t ]
[ ]
(%o7) [ 0 1 0  %i t ]
[ ]
[ 0 0 1  %i t ]
[ ]
[ 0 0 0 1 ]

Example 3:
Power operations.

(%i11) load("diag")$
(%i12) A: matrix([[1,2,0], [0,1,0], [1,0,1]])$
(%i13) integer_pow(x) := block ([k], declare (k, integer), x^k)$
(%i14) mat_function (integer_pow, A);
(%o14)
[ 1 2 k 0 ]
[ ]
[ 0 1 0 ]
[ ]
[ k (k - 1) k 1 ]

(%i15) A^^20;
(%o15)
[ 1 40 0 ]
[ ]
[ 0 1 0 ]
[ ]
[ 20 380 1 ]

To use this function write first load("diag").
51 distrib

51.1 Introduction to distrib

Package distrib contains a set of functions for making probability computations on both discrete and continuous univariate models.

What follows is a short reminder of basic probabilistic related definitions.

Let \( f(x) \) be the density function of an absolute continuous random variable \( X \). The distribution function is defined as

\[
F(x) = \int_{-\infty}^{x} f(u) \, du
\]

which equals the probability \( Pr(X \leq x) \).

The mean value is a localization parameter and is defined as

\[
E[X] = \int_{-\infty}^{\infty} x f(x) \, dx
\]

The variance is a measure of variation,

\[
V[X] = \int_{-\infty}^{\infty} f(x) (x - E[X])^2 \, dx
\]

which is a positive real number. The square root of the variance is the standard deviation, \( D[X] = \sqrt{V[X]} \), and it is another measure of variation.

The skewness coefficient is a measure of non-symmetry,

\[
SK[X] = \frac{\int_{-\infty}^{\infty} f(x) (x - E[X])^3 \, dx}{D[X]^3}
\]

And the kurtosis coefficient measures the peakedness of the distribution,

\[
KU[X] = \frac{\int_{-\infty}^{\infty} f(x) (x - E[X])^4 \, dx}{D[X]^4} - 3
\]

If \( X \) is gaussian, \( KU[X] = 0 \). In fact, both skewness and kurtosis are shape parameters used to measure the non-gaussianity of a distribution.

If the random variable \( X \) is discrete, the density, or probability, function \( f(x) \) takes positive values within certain countable set of numbers \( x_i \), and zero elsewhere. In this case, the distribution function is

\[
F(x) = \sum_{x_i \leq x} f(x_i)
\]

The mean, variance, standard deviation, skewness coefficient and kurtosis coefficient take the form

\[
E[X] = \sum_{x_i} x_i f(x_i),
\]
\[ V[X] = \sum_{x_i} f(x_i) (x_i - E[X])^2, \]
\[ D[X] = \sqrt{V[X]}, \]
\[ SK[X] = \frac{\sum_{x_i} f(x) (x - E[X])^3}{D[X]^3} \]
and
\[ KU[X] = \frac{\sum_{x_i} f(x) (x - E[X])^4}{D[X]^4} - 3, \]
respectively.

There is a naming convention in package distrib. Every function name has two parts, the first one makes reference to the function or parameter we want to calculate,

**Functions:**
- Density function (pdf_*)
- Distribution function (cdf_*)
- Quantile (quantile_*)
- Mean (mean_*)
- Variance (var_*)
- Standard deviation (std_*)
- Skewness coefficient (skewness_*)
- Kurtosis coefficient (kurtosis_*)
- Random variate (random_*)

The second part is an explicit reference to the probabilistic model,

**Continuous distributions:**
- Normal (*normal)
- Student (*student_t)
- Chi^2 (*chi2)
- Noncentral Chi^2 (*noncentral_chi2)
- F (*f)
- Exponential (*exp)
- Lognormal (*lognormal)
- Gamma (*gamma)
- Beta (*beta)
- Continuous uniform (*continuous_uniform)
- Logistic (*logistic)
- Pareto (*pareto)
- Weibull (*weibull)
- Rayleigh (*rayleigh)
- Laplace (*laplace)
- Cauchy (*cauchy)
- Gumbel (*gumbel)
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Discrete distributions:
- Binomial (*binomial)
- Poisson (*poisson)
- Bernoulli (*bernoulli)
- Geometric (*geometric)
- Discrete uniform (*discrete_uniform)
- Hypergeometric (*hypergeometric)
- Negative binomial (*negative_binomial)
- Finite discrete (*general_finite_discrete)

For example, pdf_student_t(x, n) is the density function of the Student distribution with n degrees of freedom, std_pareto(a, b) is the standard deviation of the Pareto distribution with parameters a and b and kurtosis_poisson(m) is the kurtosis coefficient of the Poisson distribution with mean m.

In order to make use of package distrib you need first to load it by typing

(%i1) load(distrib)$

For comments, bugs or suggestions, please contact the author at 'riotorto AT yahoo DOT com'.

51.2 Functions and Variables for continuous distributions

pdf_normal (x, m, s) [Function]
Returns the value at x of the density function of a Normal(m, s) random variable, with s > 0. To make use of this function, write first load(distrib).

cdf_normal (x, m, s) [Function]
Returns the value at x of the distribution function of a Normal(m, s) random variable, with s > 0. This function is defined in terms of Maxima’s built-in error function erf.

(%i1) load (distrib)$
(%i2) cdf_normal(x,m,s);
\[ \frac{x - m}{\text{erf}\left(\frac{1}{\sqrt{2}} \frac{x - m}{s}\right)} + \frac{1}{2} \]

See also erf.

quantile_normal (q, m, s) [Function]
Returns the q-quantile of a Normal(m, s) random variable, with s > 0; in other words, this is the inverse of cdf_normal. Argument q must be an element of [0, 1].
To make use of this function, write first load(distrib).

(%i1) load (distrib)$
(%i2) quantile_normal(95/100,0,1);
\[ \frac{9}{\sqrt{2} \text{ inverse erf}(10)} \]

(%i3) float(%);
(%o3) 1.644853626951472
**mean_normal** \((m,s)\)  
Returns the mean of a \(Normal(m,s)\) random variable, with \(s > 0\), namely \(m\). To make use of this function, write first `load(distrib)`.

**var_normal** \((m,s)\)  
Returns the variance of a \(Normal(m,s)\) random variable, with \(s > 0\), namely \(s^2\). To make use of this function, write first `load(distrib)`.

**std_normal** \((m,s)\)  
Returns the standard deviation of a \(Normal(m,s)\) random variable, with \(s > 0\), namely \(s\). To make use of this function, write first `load(distrib)`.

**skewness_normal** \((m,s)\)  
Returns the skewness coefficient of a \(Normal(m,s)\) random variable, with \(s > 0\), which is always equal to 0. To make use of this function, write first `load(distrib)`.

**kurtosis_normal** \((m,s)\)  
Returns the kurtosis coefficient of a \(Normal(m,s)\) random variable, with \(s > 0\), which is always equal to 0. To make use of this function, write first `load(distrib)`.

**random_normal** \((m,s)\)
**random_normal** \((m,s,n)\)  
Returns a \(Normal(m,s)\) random variate, with \(s > 0\). Calling `random_normal` with a third argument \(n\), a random sample of size \(n\) will be simulated. This is an implementation of the Box-Mueller algorithm, as described in Knuth, D.E. (1981) *Seminumerical Algorithms. The Art of Computer Programming*. Addison-Wesley. To make use of this function, write first `load(distrib)`.

**pdf_student_t** \((x,n)\)  
Returns the value at \(x\) of the density function of a Student random variable \(t(n)\), with \(n > 0\) degrees of freedom. To make use of this function, write first `load(distrib)`.

**cdf_student_t** \((x,n)\)  
Returns the value at \(x\) of the distribution function of a Student random variable \(t(n)\), with \(n > 0\) degrees of freedom.

```lisp
(%i1) load (distrib)$
(%i2) cdf_student_t(1/2, 7/3);
            7  1  28
(7) 1 - -------------------------------------
        6  2  31
(%o2) 1 - -------------------------------
        2
(%i3) float(%);
(%o3) .6698450596140415
```

**quantile_student_t** \((q,n)\)  
Returns the \(q\)-quantile of a Student random variable \(t(n)\), with \(n > 0\); in other words, this is the inverse of `cdf_student_t`. Argument \(q\) must be an element of \([0,1]\). To make use of this function, write first `load(distrib)`.
mean_student_t (n) [Function]
Returns the mean of a Student random variable t(n), with n > 0, which is always equal to 0. To make use of this function, write first load(distrib).

var_student_t (n) [Function]
Returns the variance of a Student random variable t(n), with n > 2.

\[
\text{(%i1) load (distrib)\$}
\text{(%i2) var_student_t(n);}\\
\text{(%o2) } \frac{n}{n - 2}
\]

std_student_t (n) [Function]
Returns the standard deviation of a Student random variable t(n), with n > 2. To make use of this function, write first load(distrib).

skewness_student_t (n) [Function]
Returns the skewness coefficient of a Student random variable t(n), with n > 3, which is always equal to 0. To make use of this function, write first load(distrib).

kurtosis_student_t (n) [Function]
Returns the kurtosis coefficient of a Student random variable t(n), with n > 4. To make use of this function, write first load(distrib).

random_student_t (n) [Function]
random_student_t (n,m)
Returns a Student random variate t(n), with n > 0. Calling random_student_t with a second argument m, a random sample of size m will be simulated.

The implemented algorithm is based on the fact that if Z is a normal random variable \(N(0,1)\) and \(S^2\) is a chi square random variable with n degrees of freedom, \(\text{Chi}^2(n)\), then

\[
X = \frac{Z}{\sqrt{\frac{S^2}{n}}}
\]

is a Student random variable with n degrees of freedom, t(n).

To make use of this function, write first load(distrib).

pdf_noncentral_student_t (x,n,ncp) [Function]
Returns the value at x of the density function of a noncentral Student random variable nc_t(n,ncp), with n > 0 degrees of freedom and noncentrality parameter ncp. To make use of this function, write first load(distrib).

Sometimes an extra work is necessary to get the final result.

\[
\text{(%i1) load (distrib)\$}
\text{(%i2) expand(pdf_noncentral_student_t(3,5,0.1));}
\]
\[ \frac{7/2}{5/2} \frac{0.04296414417400905}{1.323650307289301 \times 10^{-6}} \frac{1.94793720435093 \times 10^{-4}}{\sqrt{\pi}} + \frac{\frac{7/2}{5/2} \sqrt{\pi}}{\frac{1.94793720435093 \times 10^{-4}}{\sqrt{\pi}}} \]

\[ \frac{7/2}{1.94793720435093 \times 10^{-4}} + \frac{1.323650307289301 \times 10^{-6}}{5} \]

\[ \text{cdf_noncentral_student_t}(x,n,ncp) \]
This function returns the value at \( x \) of the distribution function of a noncentral Student random variable \( \text{nc}_t(n,ncp) \), with \( n > 0 \) degrees of freedom and noncentrality parameter \( ncp \). This function has no closed form and it is numerically computed.

\[ \text{quantile_noncentral_student_t}(q,n,ncp) \]
Returns the \( q \)-quantile of a noncentral Student random variable \( \text{nc}_t(n,ncp) \), with \( n > 0 \) degrees of freedom and noncentrality parameter \( ncp \); in other words, this is the inverse of \( \text{cdf_noncentral_student_t} \). Argument \( q \) must be an element of \([0,1]\). To make use of this function, write first \( \text{load(distrib)} \).

\[ \text{mean_noncentral_student_t}(n,ncp) \]
This function returns the mean of a noncentral Student random variable \( \text{nc}_t(n,ncp) \), with \( n > 1 \) degrees of freedom and noncentrality parameter \( ncp \). To make use of this function, write first \( \text{load(distrib)} \).

\[ \text{var_noncentral_student_t}(n,ncp) \]
This function returns the variance of a noncentral Student random variable \( \text{nc}_t(n,ncp) \), with \( n > 2 \) degrees of freedom and noncentrality parameter \( ncp \). To make use of this function, write first \( \text{load(distrib)} \).

\[ \text{std_noncentral_student_t}(n,ncp) \]
This function returns the standard deviation of a noncentral Student random variable \( \text{nc}_t(n,ncp) \), with \( n > 2 \) degrees of freedom and noncentrality parameter \( ncp \). To make use of this function, write first \( \text{load(distrib)} \).
skewness_noncentral_student_t (n,ncp)

Returns the skewness coefficient of a noncentral Student random variable \( nc_t(n,ncp) \), with \( n > 3 \) degrees of freedom and noncentrality parameter \( ncp \). To make use of this function, write first \texttt{load(distrib)}. 

kurtosis_noncentral_student_t (n,ncp)

Returns the kurtosis coefficient of a noncentral Student random variable \( nc_t(n,ncp) \), with \( n > 4 \) degrees of freedom and noncentrality parameter \( ncp \). To make use of this function, write first \texttt{load(distrib)}. 

random_noncentral_student_t (n,ncp)

\[ \text{random_noncentral_student_t}(n,ncp,m) \]

Returns a noncentral Student random variate \( nc_t(n,ncp) \), with \( n > 0 \). Calling \texttt{random_noncentral_student_t} with a third argument \( m \), a random sample of size \( m \) will be simulated. 

The implemented algorithm is based on the fact that if \( X \) is a normal random variable \( N(ncp,1) \) and \( S^2 \) is a chi square random variable with \( n \) degrees of freedom, \( Chi^2(n) \), then

\[
U = \frac{X}{\sqrt{\frac{S^2}{n}}} 
\]

is a noncentral Student random variable with \( n \) degrees of freedom and noncentrality parameter \( ncp, nc_t(n,ncp) \).

To make use of this function, write first \texttt{load(distrib)}. 

pdf_chi2 (x,n)

Returns the value at \( x \) of the density function of a Chi-square random variable \( Chi^2(n) \), with \( n > 0 \). The \( Chi^2(n) \) random variable is equivalent to the \( Gamma(n/2,2) \).

\[
\text{(i1)} \quad \text{load (distrib)}$
\text{(i2)} \quad \text{pdf\_chi2(x,n);} 
\text{(o2)} \quad \frac{n/2 - 1 - x/2}{x} \frac{\text{e}}{n/2^n} \frac{\text{gamma(-)}}{2} 
\]

cdf_chi2 (x,n)

Returns the value at \( x \) of the distribution function of a Chi-square random variable \( Chi^2(n) \), with \( n > 0 \).

\[
\text{(i1)} \quad \text{load (distrib)}$
\text{(i2)} \quad \text{cdf\_chi2(3,4);} 
\text{(o2)} \quad 1 - \text{gamma\_incomplete\_regularized(2, -)} \frac{3}{2} 
\text{(o3)} \quad \text{float(%) = .4421745996289256} 
\]
quantile_chi2 (q, n) [Function]
Returns the q-quantile of a Chi-square random variable $Chi^2(n)$, with $n > 0$; in other words, this is the inverse of cdf_chi2. Argument q must be an element of $[0, 1]$.
This function has no closed form and it is numerically computed.

(%i1) load (distrib)$
(%i2) quantile_chi2(0.99,9);
(%o2) 21.66599433346194

mean_chi2 (n) [Function]
Returns the mean of a Chi-square random variable $Chi^2(n)$, with $n > 0$.
The $Chi^2(n)$ random variable is equivalent to the $Gamma(n/2, 2)$.

(%i1) load (distrib)$
(%i2) mean_chi2(n);
(%o2) n

var_chi2 (n) [Function]
Returns the variance of a Chi-square random variable $Chi^2(n)$, with $n > 0$.
The $Chi^2(n)$ random variable is equivalent to the $Gamma(n/2, 2)$.

(%i1) load (distrib)$
(%i2) var_chi2(n);
(%o2) 2 n

std_chi2 (n) [Function]
Returns the standard deviation of a Chi-square random variable $Chi^2(n)$, with $n > 0$.
The $Chi^2(n)$ random variable is equivalent to the $Gamma(n/2, 2)$.

(%i1) load (distrib)$
(%i2) std_chi2(n); sqrt(2) sqrt(n)
(%o2) sqrt(2) sqrt(n)

skewness_chi2 (n) [Function]
Returns the skewness coefficient of a Chi-square random variable $Chi^2(n)$, with $n > 0$.
The $Chi^2(n)$ random variable is equivalent to the $Gamma(n/2, 2)$.

(%i1) load (distrib)$
(%i2) skewness_chi2(n);

(%o2) 3/2 ------
      sqrt(n)

kurtosis_chi2 (n) [Function]
Returns the kurtosis coefficient of a Chi-square random variable $Chi^2(n)$, with $n > 0$.
The $Chi^2(n)$ random variable is equivalent to the $Gamma(n/2, 2)$.

(%i1) load (distrib)$
(%i2) kurtosis_chi2(n);

(%o2) 12 --
      n
random\_chi2 (n) \hspace{1cm} \text{[Function]}  
random\_chi2 (n,m)  
Returns a Chi-square random variate $\chi^2(n)$, with $n > 0$. Calling random\_chi2 with a second argument $m$, a random sample of size $m$ will be simulated.

The simulation is based on the Ahrens-Cheng algorithm. See random\_gamma for details.

To make use of this function, write first load(distrib).

pdf\_noncentral\_chi2 (x,n,ncp) \hspace{1cm} \text{[Function]}  
Returns the value at $x$ of the density function of a noncentral Chi-square random variable $nc\chi^2(n,ncp)$, with $n > 0$ and noncentrality parameter $ncp \geq 0$. To make use of this function, write first load(distrib).

cdf\_noncentral\_chi2 (x,n,ncp) \hspace{1cm} \text{[Function]}  
Returns the value at $x$ of the distribution function of a noncentral Chi-square random variable $nc\chi^2(n,ncp)$, with $n > 0$ and noncentrality parameter $ncp \geq 0$. To make use of this function, write first load(distrib).

quantile\_noncentral\_chi2 (q,n,ncp) \hspace{1cm} \text{[Function]}  
Returns the $q$-quantile of a noncentral Chi-square random variable $nc\chi^2(n,ncp)$, with $n > 0$ and noncentrality parameter $ncp \geq 0$; in other words, this is the inverse of cdf\_noncentral\_chi2. Argument $q$ must be an element of $[0,1]$.

This function has no closed form and it is numerically computed.

mean\_noncentral\_chi2 (n,ncp) \hspace{1cm} \text{[Function]}  
Returns the mean of a noncentral Chi-square random variable $nc\chi^2(n,ncp)$, with $n > 0$ and noncentrality parameter $ncp \geq 0$.

var\_noncentral\_chi2 (n,ncp) \hspace{1cm} \text{[Function]}  
Returns the variance of a noncentral Chi-square random variable $nc\chi^2(n,ncp)$, with $n > 0$ and noncentrality parameter $ncp \geq 0$.

std\_noncentral\_chi2 (n,ncp) \hspace{1cm} \text{[Function]}  
Returns the standard deviation of a noncentral Chi-square random variable $nc\chi^2(n,ncp)$, with $n > 0$ and noncentrality parameter $ncp \geq 0$.

skewness\_noncentral\_chi2 (n,ncp) \hspace{1cm} \text{[Function]}  
Returns the skewness coefficient of a noncentral Chi-square random variable $nc\chi^2(n,ncp)$, with $n > 0$ and noncentrality parameter $ncp \geq 0$.

kurtosis\_noncentral\_chi2 (n,ncp) \hspace{1cm} \text{[Function]}  
Returns the kurtosis coefficient of a noncentral Chi-square random variable $nc\chi^2(n,ncp)$, with $n > 0$ and noncentrality parameter $ncp \geq 0$.

random\_noncentral\_chi2 (n,ncp) \hspace{1cm} \text{[Function]}  
random\_noncentral\_chi2 (n,ncp,m)  
Returns a noncentral Chi-square random variate $nc\chi^2(n,ncp)$, with $n > 0$ and noncentrality parameter $ncp \geq 0$. Calling random\_noncentral\_chi2 with a third argument $m$, a random sample of size $m$ will be simulated.

To make use of this function, write first load(distrib).
pdf_f (x,m,n)  [Function]
Returns the value at x of the density function of a F random variable \( F(m, n) \), with \( m, n > 0 \). To make use of this function, write first \texttt{load(distrib)}.

cdf_f (x,m,n)  [Function]
Returns the value at x of the distribution function of a F random variable \( F(m, n) \), with \( m, n > 0 \).

\[
\%i1) \text{load (distrib)}$
\%i2) \text{cdf}_f(2,3,9/4); \\
\%o2) \quad 1 - \text{beta_incomplete_regularized}(-, -, --) \\
\quad 8 \quad 2 \quad 11
\%
\%o3) \quad \text{float}(%); \\
\%
\%o4) \quad 0.66756728179008
\]

quantile_f (q,m,n)  [Function]
Returns the \( q \)-quantile of a F random variable \( F(m, n) \), with \( m, n > 0 \); in other words, this is the inverse of \texttt{cdf_f}. Argument \( q \) must be an element of \([0,1]\).

\[
\%i1) \text{load (distrib)}$
\%i2) \text{quantile}_f(2/5,sqrt(3),5); \\
\%o2) \quad 0.518947838573693
\%
\]

mean_f (m,n)  [Function]
Returns the mean of a F random variable \( F(m, n) \), with \( m > 0, n > 2 \). To make use of this function, write first \texttt{load(distrib)}.

var_f (m,n)  [Function]
Returns the variance of a F random variable \( F(m, n) \), with \( m > 0, n > 4 \). To make use of this function, write first \texttt{load(distrib)}.

std_f (m,n)  [Function]
Returns the standard deviation of a F random variable \( F(m, n) \), with \( m > 0, n > 2 \). To make use of this function, write first \texttt{load(distrib)}.

skewness_f (m,n)  [Function]
Returns the skewness coefficient of a F random variable \( F(m, n) \), with \( m > 0, n > 4 \). To make use of this function, write first \texttt{load(distrib)}.

kurtosis_f (m,n)  [Function]
Returns the kurtosis coefficient of a F random variable \( F(m, n) \), with \( m > 0, n > 6 \). To make use of this function, write first \texttt{load(distrib)}.

random_f (m,n)  [Function]
random_f (m,n,k)
Returns a F random variate \( F(m, n) \), with \( m, n > 0 \). Calling \texttt{random_f} with a third argument \( k \), a random sample of size \( k \) will be simulated.
The simulation algorithm is based on the fact that if \( X \) is a \( \text{Chi}^2(m) \) random variable and \( Y \) is a \( \text{Chi}^2(n) \) random variable, then

\[
F = \frac{nX}{mY}
\]
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is a F random variable with \( m \) and \( n \) degrees of freedom, \( F(m, n) \).

To make use of this function, write first \texttt{load(distrib)}.

\textbf{pdf\_exp (x,m)} \hspace{1cm} \textbf{[Function]}

Returns the value at \( x \) of the density function of an \textit{Exponential} \((m)\) random variable, with \( m > 0 \).

The \textit{Exponential} \((m)\) random variable is equivalent to the \textit{Weibull} \((1, 1/m)\).

\begin{verbatim}
(%i1) load (distrib)$
(%i2) pdf_exp(x,m);  
\hspace{2cm} - m x
(%o2) m \%e
\end{verbatim}

\textbf{cdf\_exp (x,m)} \hspace{1cm} \textbf{[Function]}

Returns the value at \( x \) of the distribution function of an \textit{Exponential} \((m)\) random variable, with \( m > 0 \).

The \textit{Exponential} \((m)\) random variable is equivalent to the \textit{Weibull} \((1, 1/m)\).

\begin{verbatim}
(%i1) load (distrib)$
(%i2) cdf_exp(x,m);  
\hspace{2cm} - m x
(%o2) 1 - \%e
\end{verbatim}

\textbf{quantile\_exp (q,m)} \hspace{1cm} \textbf{[Function]}

Returns the \( q \)-quantile of an \textit{Exponential} \((m)\) random variable, with \( m > 0 \); in other words, this is the inverse of \texttt{cdf\_exp}. Argument \( q \) must be an element of \([0,1]\).

The \textit{Exponential} \((m)\) random variable is equivalent to the \textit{Weibull} \((1, 1/m)\).

\begin{verbatim}
(%i1) load (distrib)$
(%i2) quantile_exp(0.56,5);  
\hspace{2cm} .1641961104139661
(%o2) 0.8209805520698303
\end{verbatim}

\textbf{mean\_exp (m)} \hspace{1cm} \textbf{[Function]}

Returns the mean of an \textit{Exponential} \((m)\) random variable, with \( m > 0 \).

The \textit{Exponential} \((m)\) random variable is equivalent to the \textit{Weibull} \((1, 1/m)\).

\begin{verbatim}
(%i1) load (distrib)$
(%i2) mean_exp(m);  
\hspace{2cm} 1
(%o2) \frac{1}{m}
\end{verbatim}

\textbf{var\_exp (m)} \hspace{1cm} \textbf{[Function]}

Returns the variance of an \textit{Exponential} \((m)\) random variable, with \( m > 0 \).

The \textit{Exponential} \((m)\) random variable is equivalent to the \textit{Weibull} \((1, 1/m)\).

\begin{verbatim}
(%i1) load (distrib)$
\end{verbatim}
\[(%i2) \text{var}\_exp(m); \quad 1\]
\[(%o2) \quad \frac{1}{2} \quad \frac{m}{m}\]

\textbf{std\_exp (m)}

Returns the standard deviation of an \textit{Exponential}(m) random variable, with \(m > 0\).

The \textit{Exponential}(m) random variable is equivalent to the \textit{Weibull}(1,1/m).

\[(%i1) \text{load (distrib)};\]
\[(%i2) \text{std}\_exp(m); \quad 1\]
\[(%o2) \quad - \quad \frac{m}{m}\]

\textbf{skewness\_exp (m)}

Returns the skewness coefficient of an \textit{Exponential}(m) random variable, with \(m > 0\).

The \textit{Exponential}(m) random variable is equivalent to the \textit{Weibull}(1,1/m).

\[(%i1) \text{load (distrib)};\]
\[(%i2) \text{skewness}\_exp(m); \quad 2\]
\[(%o2) \quad 2 \quad \frac{m}{m}\]

\textbf{kurtosis\_exp (m)}

Returns the kurtosis coefficient of an \textit{Exponential}(m) random variable, with \(m > 0\).

The \textit{Exponential}(m) random variable is equivalent to the \textit{Weibull}(1,1/m).

\[(%i1) \text{load (distrib)};\]
\[(%i2) \text{kurtosis}\_exp(m); \quad 6\]
\[(%o3) \quad 6 \quad \frac{m}{m}\]

\textbf{random\_exp (m)}

\textbf{random\_exp (m,k)}

Returns an \textit{Exponential}(m) random variate, with \(m > 0\). Calling \texttt{random\_exp} with a second argument \(k\), a random sample of size \(k\) will be simulated.

The simulation algorithm is based on the general inverse method.

To make use of this function, write first \texttt{load(distrib)}.

\textbf{pdf\_lognormal (x,m,s)}

Returns the value at \(x\) of the density function of a \textit{Lognormal}(m,s) random variable, with \(s > 0\). To make use of this function, write first \texttt{load(distrib)}.

\textbf{cdf\_lognormal (x,m,s)}

Returns the value at \(x\) of the distribution function of a \textit{Lognormal}(m,s) random variable, with \(s > 0\). This function is defined in terms of Maxima’s built-in error function \texttt{erf}.

\[(%i1) \text{load (distrib)};\]
\[(%i2) \text{cdf\_lognormal(x,m,s)};\]
\[
\frac{\log(x) - m}{\text{erf}\left(\frac{-\sqrt{2} s}{1}\right)} + \frac{1}{2}
\]

(%o2)
\[
\frac{-\sqrt{2}}{2} + \frac{1}{2}
\]

See also \texttt{erf}.

\textbf{quantile\_lognormal (q,m,s)}

\texttt{quantile\_lognormal(q,m,s)}

Returns the \(q\)-quantile of a \texttt{Lognormal}(m, s) random variable, with \(s > 0\); in other words, this is the inverse of \texttt{cdf\_lognormal}. Argument \(q\) must be an element of [0,1]. To make use of this function, write first \texttt{load(distrib)}.

\begin{verbatim}
(%i1) load (distrib)$
(%i2) quantile_lognormal(95/100,0,1);
                                      \[\sqrt{2} \text{ inverse\_erf}(9/10)\]
(%o2) \%e
(%i3) float(%);
(%o3) 5.180251602233015
\end{verbatim}

\textbf{mean\_lognormal (m,s)}

\texttt{mean\_lognormal(m,s)}

Returns the mean of a \texttt{Lognormal}(m, s) random variable, with \(s > 0\). To make use of this function, write first \texttt{load(distrib)}.

\textbf{var\_lognormal (m,s)}

\texttt{var\_lognormal(m,s)}

Returns the variance of a \texttt{Lognormal}(m, s) random variable, with \(s > 0\). To make use of this function, write first \texttt{load(distrib)}.

\textbf{std\_lognormal (m,s)}

\texttt{std\_lognormal(m,s)}

Returns the standard deviation of a \texttt{Lognormal}(m, s) random variable, with \(s > 0\). To make use of this function, write first \texttt{load(distrib)}.

\textbf{skewness\_lognormal (m,s)}

\texttt{skewness\_lognormal(m,s)}

Returns the skewness coefficient of a \texttt{Lognormal}(m, s) random variable, with \(s > 0\). To make use of this function, write first \texttt{load(distrib)}.

\textbf{kurtosis\_lognormal (m,s)}

\texttt{kurtosis\_lognormal(m,s)}

Returns the kurtosis coefficient of a \texttt{Lognormal}(m, s) random variable, with \(s > 0\). To make use of this function, write first \texttt{load(distrib)}.

\textbf{random\_lognormal (m,s)}

\texttt{random\_lognormal(m,s,n)}

Returns a \texttt{Lognormal}(m, s) random variate, with \(s > 0\). Calling \texttt{random\_lognormal} with a third argument \(n\), a random sample of size \(n\) will be simulated.

Log-normal variates are simulated by means of random normal variates. See \texttt{random\_normal} for details.

To make use of this function, write first \texttt{load(distrib)}.

\textbf{pdf\_gamma (x,a,b)}

\texttt{pdf\_gamma(x,a,b)}

Returns the value at \(x\) of the density function of a \texttt{Gamma}(a, b) random variable, with \(a, b > 0\). To make use of this function, write first \texttt{load(distrib)}.
cdf_gamma \((x,a,b)\)  
Returns the value at \(x\) of the distribution function of a \(Gamma(a,b)\) random variable, with \(a,b > 0\).

\[
\text{(i1)} \quad \text{load (distrib)}$
\text{(i2)} \quad \text{cdf gamma}(3,5,21);
\]
\[
\text{(o2)} \quad 1 - \text{gamma_incomplete_regularized}(5, -) \quad 7
\]
\[
\text{(i3)} \quad \text{float(\%)}; \quad 4.402663157376807E-7
\]

quantile_gamma \((q,a,b)\)  
Returns the \(q\)-quantile of a \(Gamma(a,b)\) random variable, with \(a,b > 0\); in other words, this is the inverse of \(\text{cdf gamma}\). Argument \(q\) must be an element of \([0,1]\). To make use of this function, write first \text{load(distrib)}.

mean_gamma \((a,b)\)  
Returns the mean of a \(Gamma(a,b)\) random variable, with \(a,b > 0\). To make use of this function, write first \text{load(distrib)}.

var_gamma \((a,b)\)  
Returns the variance of a \(Gamma(a,b)\) random variable, with \(a,b > 0\). To make use of this function, write first \text{load(distrib)}.

std_gamma \((a,b)\)  
Returns the standard deviation of a \(Gamma(a,b)\) random variable, with \(a,b > 0\). To make use of this function, write first \text{load(distrib)}.

skewness_gamma \((a,b)\)  
Returns the skewness coefficient of a \(Gamma(a,b)\) random variable, with \(a,b > 0\). To make use of this function, write first \text{load(distrib)}.

kurtosis_gamma \((a,b)\)  
Returns the kurtosis coefficient of a \(Gamma(a,b)\) random variable, with \(a,b > 0\). To make use of this function, write first \text{load(distrib)}.

random_gamma \((a,b)\)  
\text{random_gamma} \((a,b,n)\)  
Returns a \(Gamma(a,b)\) random variate, with \(a,b > 0\). Calling \text{random gamma} with a third argument \(n\), a random sample of size \(n\) will be simulated.

The implemented algorithm is a combinantion of two procedures, depending on the value of parameter \(a\):


To make use of this function, write first \text{load(distrib)}.  

---
pdf_beta (x,a,b)  [Function]
Returns the value at x of the density function of a Beta(a,b) random variable, with a,b > 0. To make use of this function, write first load(distrib).

cdf_beta (x,a,b)  [Function]
Returns the value at x of the distribution function of a Beta(a,b) random variable, with a,b > 0.

(%i1) load (distrib)$
(%i2) cdf_beta(1/3,15,2);
    11
   ------
  14348907
(%o2) 7.666089131388195E-7

quantile_beta (q,a,b)  [Function]
Returns the q-quantile of a Beta(a,b) random variable, with a,b > 0; in other words, this is the inverse of cdf_beta. Argument q must be an element of [0,1]. To make use of this function, write first load(distrib).

mean_beta (a,b)  [Function]
Returns the mean of a Beta(a,b) random variable, with a,b > 0. To make use of this function, write first load(distrib).

var_beta (a,b)  [Function]
Returns the variance of a Beta(a,b) random variable, with a,b > 0. To make use of this function, write first load(distrib).

std_beta (a,b)  [Function]
Returns the standard deviation of a Beta(a,b) random variable, with a,b > 0. To make use of this function, write first load(distrib).

skewness_beta (a,b)  [Function]
Returns the skewness coefficient of a Beta(a,b) random variable, with a,b > 0. To make use of this function, write first load(distrib).

kurtosis_beta (a,b)  [Function]
Returns the kurtosis coefficient of a Beta(a,b) random variable, with a,b > 0. To make use of this function, write first load(distrib).

random_beta (a,b)
random_beta (a,b,n)
Returns a Beta(a,b) random variate, with a,b > 0. Calling random_beta with a third argument n, a random sample of size n will be simulated.

To make use of this function, write first load(distrib).
pdf_continuous_uniform \((x,a,b)\)  
Returns the value at \(x\) of the density function of a \(ContinuousUniform(a,b)\) random variable, with \(a < b\). To make use of this function, write first \texttt{load(distrib)}.

cdf_continuous_uniform \((x,a,b)\)  
Returns the value at \(x\) of the distribution function of a \(ContinuousUniform(a,b)\) random variable, with \(a < b\). To make use of this function, write first \texttt{load(distrib)}.

quantile_continuous_uniform \((q,a,b)\)  
Returns the \(q\)-quantile of a \(ContinuousUniform(a,b)\) random variable, with \(a < b\); in other words, this is the inverse of \texttt{cdf_continuous_uniform}. Argument \(q\) must be an element of \([0,1]\). To make use of this function, write first \texttt{load(distrib)}.

mean_continuous_uniform \((a,b)\)  
Returns the mean of a \(ContinuousUniform(a,b)\) random variable, with \(a < b\). To make use of this function, write first \texttt{load(distrib)}.

var_continuous_uniform \((a,b)\)  
Returns the variance of a \(ContinuousUniform(a,b)\) random variable, with \(a < b\). To make use of this function, write first \texttt{load(distrib)}.

std_continuous_uniform \((a,b)\)  
Returns the standard deviation of a \(ContinuousUniform(a,b)\) random variable, with \(a < b\). To make use of this function, write first \texttt{load(distrib)}.

skewness_continuous_uniform \((a,b)\)  
Returns the skewness coefficient of a \(ContinuousUniform(a,b)\) random variable, with \(a < b\). To make use of this function, write first \texttt{load(distrib)}.

kurtosis_continuous_uniform \((a,b)\)  
Returns the kurtosis coefficient of a \(ContinuousUniform(a,b)\) random variable, with \(a < b\). To make use of this function, write first \texttt{load(distrib)}.

random_continuous_uniform \((a,b)\)  
\texttt{random_continuous_uniform \((a,b,n)\)}  
Returns a \(ContinuousUniform(a,b)\) random variate, with \(a < b\). Calling \texttt{random_continuous_uniform} with a third argument \(n\), a random sample of size \(n\) will be simulated.

This is a direct application of the \texttt{random} built-in Maxima function.

See also \texttt{random}. To make use of this function, write first \texttt{load(distrib)}.

pdf_logistic \((x,a,b)\)  
Returns the value at \(x\) of the density function of a \(Logistic(a,b)\) random variable, with \(b > 0\). To make use of this function, write first \texttt{load(distrib)}.

cdf_logistic \((x,a,b)\)  
Returns the value at \(x\) of the distribution function of a \(Logistic(a,b)\) random variable, with \(b > 0\). To make use of this function, write first \texttt{load(distrib)}. 


quantile_logistic (q,a,b)
Returns the q-quantile of a Logistic(a, b) random variable, with b > 0; in other words, this is the inverse of cdf_logistic. Argument q must be an element of [0, 1]. To make use of this function, write first load(distrib).

mean_logistic (a,b)
Returns the mean of a Logistic(a, b) random variable, with b > 0. To make use of this function, write first load(distrib).

var_logistic (a,b)
Returns the variance of a Logistic(a, b) random variable, with b > 0. To make use of this function, write first load(distrib).

std_logistic (a,b)
Returns the standard deviation of a Logistic(a, b) random variable, with b > 0. To make use of this function, write first load(distrib).

skewness_logistic (a,b)
Returns the skewness coefficient of a Logistic(a, b) random variable, with b > 0. To make use of this function, write first load(distrib).

kurtosis_logistic (a,b)
Returns the kurtosis coefficient of a Logistic(a, b) random variable, with b > 0. To make use of this function, write first load(distrib).

random_logistic (a,b)
random_logistic (a,b,n)
Returns a Logistic(a, b) random variate, with b > 0. Calling random_logistic with a third argument n, a random sample of size n will be simulated.
The implemented algorithm is based on the general inverse method.
To make use of this function, write first load(distrib).

df_pareto (x,a,b)
Returns the value at x of the density function of a Pareto(a, b) random variable, with a, b > 0. To make use of this function, write first load(distrib).

cdf_pareto (x,a,b)
Returns the value at x of the distribution function of a Pareto(a, b) random variable, with a, b > 0. To make use of this function, write first load(distrib).

quantile_pareto (q,a,b)
Returns the q-quantile of a Pareto(a, b) random variable, with a, b > 0; in other words, this is the inverse of cdf_pareto. Argument q must be an element of [0, 1]. To make use of this function, write first load(distrib).

mean_pareto (a,b)
Returns the mean of a Pareto(a, b) random variable, with a > 1, b > 0. To make use of this function, write first load(distrib).
**var_pareto (a,b)**

Returns the variance of a Pareto(a, b) random variable, with $a > 2, b > 0$. To make use of this function, write first `load(distrib)`.

**std_pareto (a,b)**

Returns the standard deviation of a Pareto(a, b) random variable, with $a > 2, b > 0$. To make use of this function, write first `load(distrib)`.

**skewness_pareto (a,b)**

Returns the skewness coefficient of a Pareto(a, b) random variable, with $a > 3, b > 0$. To make use of this function, write first `load(distrib)`.

**kurtosis_pareto (a,b)**

Returns the kurtosis coefficient of a Pareto(a, b) random variable, with $a > 4, b > 0$. To make use of this function, write first `load(distrib)`.

**random_pareto (a,b)**

`random_pareto (a,b,n)`

Returns a Pareto(a, b) random variate, with $a > 0, b > 0$. Calling `random_pareto` with a third argument $n$, a random sample of size $n$ will be simulated. The implemented algorithm is based on the general inverse method. To make use of this function, write first `load(distrib)`.

**pdf_weibull (x,a,b)**

Returns the value at $x$ of the density function of a Weibull(a, b) random variable, with $a, b > 0$. To make use of this function, write first `load(distrib)`.

**cdf_weibull (x,a,b)**

Returns the value at $x$ of the distribution function of a Weibull(a, b) random variable, with $a, b > 0$. To make use of this function, write first `load(distrib)`.

**quantile_weibull (q,a,b)**

Returns the $q$-quantile of a Weibull(a, b) random variable, with $a, b > 0$; in other words, this is the inverse of `cdf_weibull`. Argument $q$ must be an element of $[0, 1]$. To make use of this function, write first `load(distrib)`.

**mean_weibull (a,b)**

Returns the mean of a Weibull(a, b) random variable, with $a, b > 0$. To make use of this function, write first `load(distrib)`.

**var_weibull (a,b)**

Returns the variance of a Weibull(a, b) random variable, with $a, b > 0$. To make use of this function, write first `load(distrib)`.

**std_weibull (a,b)**

Returns the standard deviation of a Weibull(a, b) random variable, with $a, b > 0$. To make use of this function, write first `load(distrib)`.

**skewness_weibull (a,b)**

Returns the skewness coefficient of a Weibull(a, b) random variable, with $a, b > 0$. To make use of this function, write first `load(distrib)`.
**kurtosis_weibull (a,b)**

Returns the kurtosis coefficient of a $Weibull(a, b)$ random variable, with $a, b > 0$. To make use of this function, write first `load(distrib)`.

**random_weibull (a,b)**

Returns a $Weibull(a, b)$ random variate, with $a, b > 0$. Calling `random_weibull` with a third argument $n$, a random sample of size $n$ will be simulated. The implemented algorithm is based on the general inverse method. To make use of this function, write first `load(distrib)`.

**pdf_rayleigh (x,b)**

Returns the value at $x$ of the density function of a $Rayleigh(b)$ random variable, with $b > 0$.

The $Rayleigh(b)$ random variable is equivalent to the $Weibull(2, 1/b)$.

```
(%i1) load (distrib)$
(%i2) pdf_rayleigh(x,b);
```

```
2 2
2 - b x
2 b x %e
```

**cdf_rayleigh (x,b)**

Returns the value at $x$ of the distribution function of a $Rayleigh(b)$ random variable, with $b > 0$.

The $Rayleigh(b)$ random variable is equivalent to the $Weibull(2, 1/b)$.

```
(%i1) load (distrib)$
(%i2) cdf_rayleigh(x,b);
```

```2 2
- b x
1 - %e
```

**quantile_rayleigh (q,b)**

Returns the $q$-quantile of a $Rayleigh(b)$ random variable, with $b > 0$; in other words, this is the inverse of `cdf_rayleigh`. Argument $q$ must be an element of $[0, 1]$.

The $Rayleigh(b)$ random variable is equivalent to the $Weibull(2, 1/b)$.

```
(%i1) load (distrib)$
(%i2) quantile_rayleigh(0.99,b);
```

```2.145966026289347
-------------
b```

**mean_rayleigh (b)**

Returns the mean of a $Rayleigh(b)$ random variable, with $b > 0$.

The $Rayleigh(b)$ random variable is equivalent to the $Weibull(2, 1/b)$.

```
(%i1) load (distrib)$
(%i2) mean_rayleigh(b);
```
\textbf{var\_rayleigh (b)}

Returns the variance of a \textit{Rayleigh(b)} random variable, with \( b > 0 \).

The \textit{Rayleigh(b)} random variable is equivalent to the \textit{Weibull}(2,1/b).

\begin{verbatim}
(%i1) load (distrib)$
(%i2) var\_rayleigh(b);
   %pi
1 - ---
4

(%o2) --------
    2
        b
\end{verbatim}

\textbf{std\_rayleigh (b)}

Returns the standard deviation of a \textit{Rayleigh(b)} random variable, with \( b > 0 \).

The \textit{Rayleigh(b)} random variable is equivalent to the \textit{Weibull}(2,1/b).

\begin{verbatim}
(%i1) load (distrib)$
(%i2) std\_rayleigh(b);
   %pi
sqrt(1 - ---)
4

(%o2) -----------------
    b
\end{verbatim}

\textbf{skewness\_rayleigh (b)}

Returns the skewness coefficient of a \textit{Rayleigh(b)} random variable, with \( b > 0 \).

The \textit{Rayleigh(b)} random variable is equivalent to the \textit{Weibull}(2,1/b).

\begin{verbatim}
(%i1) load (distrib)$
(%i2) skewness\_rayleigh(b);
   3/2
%pi 3 sqrt(%pi)
------ - ---------
    4        4

(%o2) -------------------
    %pi 3/2
(1 - ---) 4
\end{verbatim}

\textbf{kurtosis\_rayleigh (b)}

Returns the kurtosis coefficient of a \textit{Rayleigh(b)} random variable, with \( b > 0 \).

The \textit{Rayleigh(b)} random variable is equivalent to the \textit{Weibull}(2,1/b).

\begin{verbatim}
(%i1) load (distrib)$
(%i2) kurtosis\_rayleigh(b);
\end{verbatim}
\[ \frac{2}{3} \frac{\pi}{2} - \frac{2}{16} \] 

\[ \frac{\pi}{2} \frac{2}{16} - 3 \] 

\[ (1 - \frac{\pi}{4}) \]

\textbf{random\_rayleigh (b)}

\textbf{random\_rayleigh (b,n)}

Returns a \textit{Rayleigh}(b) random variate, with \(b > 0\). Calling \texttt{random\_rayleigh} with a second argument \(n\), a random sample of size \(n\) will be simulated.

The implemented algorithm is based on the general inverse method.

To make use of this function, write first \texttt{load(distrib)}.

\textbf{pdf\_laplace (x,a,b)}

Returns the value at \(x\) of the density function of a \textit{Laplace}(a,b) random variable, with \(b > 0\). To make use of this function, write first \texttt{load(distrib)}.

\textbf{cdf\_laplace (x,a,b)}

Returns the value at \(x\) of the distribution function of a \textit{Laplace}(a,b) random variable, with \(b > 0\). To make use of this function, write first \texttt{load(distrib)}.

\textbf{quantile\_laplace (q,a,b)}

Returns the \(q\)-quantile of a \textit{Laplace}(a,b) random variable, with \(b > 0\); in other words, this is the inverse of \texttt{cdf\_laplace}. Argument \(q\) must be an element of \([0,1]\). To make use of this function, write first \texttt{load(distrib)}.

\textbf{mean\_laplace (a,b)}

Returns the mean of a \textit{Laplace}(a,b) random variable, with \(b > 0\). To make use of this function, write first \texttt{load(distrib)}.

\textbf{var\_laplace (a,b)}

Returns the variance of a \textit{Laplace}(a,b) random variable, with \(b > 0\). To make use of this function, write first \texttt{load(distrib)}.

\textbf{std\_laplace (a,b)}

Returns the standard deviation of a \textit{Laplace}(a,b) random variable, with \(b > 0\). To make use of this function, write first \texttt{load(distrib)}.

\textbf{skewness\_laplace (a,b)}

Returns the skewness coefficient of a \textit{Laplace}(a,b) random variable, with \(b > 0\). To make use of this function, write first \texttt{load(distrib)}.

\textbf{kurtosis\_laplace (a,b)}

Returns the kurtosis coefficient of a \textit{Laplace}(a,b) random variable, with \(b > 0\). To make use of this function, write first \texttt{load(distrib)}.
random_laplace (a, b)  
random_laplace (a, b, n)  
Returns a Laplace\((a, b)\) random variate, with \(b > 0\). Calling random_laplace with a third argument \(n\), a random sample of size \(n\) will be simulated.

The implemented algorithm is based on the general inverse method.

To make use of this function, write first \texttt{load(distrib)}.

pdf_cauchy (x, a, b)  
Returns the value at \(x\) of the density function of a Cauchy\((a, b)\) random variable, with \(b > 0\). To make use of this function, write first \texttt{load(distrib)}.

cdf_cauchy (x, a, b)  
Returns the value at \(x\) of the distribution function of a Cauchy\((a, b)\) random variable, with \(b > 0\). To make use of this function, write first \texttt{load(distrib)}.

quantile_cauchy (q, a, b)  
Returns the \(q\)-quantile of a Cauchy\((a, b)\) random variable, with \(b > 0\); in other words, this is the inverse of cdf_cauchy. Argument \(q\) must be an element of \([0, 1]\). To make use of this function, write first \texttt{load(distrib)}.

random_cauchy (a, b)  
random_cauchy (a, b, n)  
Returns a Cauchy\((a, b)\) random variate, with \(b > 0\). Calling random_cauchy with a third argument \(n\), a random sample of size \(n\) will be simulated.

The implemented algorithm is based on the general inverse method.

To make use of this function, write first \texttt{load(distrib)}.

pdf_gumbel (x, a, b)  
Returns the value at \(x\) of the density function of a Gumbel\((a, b)\) random variable, with \(b > 0\). To make use of this function, write first \texttt{load(distrib)}.

cdf_gumbel (x, a, b)  
Returns the value at \(x\) of the distribution function of a Gumbel\((a, b)\) random variable, with \(b > 0\). To make use of this function, write first \texttt{load(distrib)}.

quantile_gumbel (q, a, b)  
Returns the \(q\)-quantile of a Gumbel\((a, b)\) random variable, with \(b > 0\); in other words, this is the inverse of cdf_gumbel. Argument \(q\) must be an element of \([0, 1]\). To make use of this function, write first \texttt{load(distrib)}.

mean_gumbel (a, b)  
Returns the mean of a Gumbel\((a, b)\) random variable, with \(b > 0\).

\begin{verbatim}
(%i1) load (distrib)$
(%i2) mean_gumbel(a,b);
(%o2) %gamma b + a
\end{verbatim}

where symbol \%gamma stands for the Euler-Mascheroni constant. See also \%gamma.

var_gumbel (a, b)  
Returns the variance of a Gumbel\((a, b)\) random variable, with \(b > 0\). To make use of this function, write first \texttt{load(distrib)}.
std_gumbel (a,b)  [Function]
Returns the standard deviation of a Gumbel(a,b) random variable, with \( b > 0 \). To make use of this function, write first load(distrib).

skewness_gumbel (a,b)  [Function]
Returns the skewness coefficient of a Gumbel(a,b) random variable, with \( b > 0 \).

\[
\begin{align*}
%i1) \text{load (distrib)}$ &
%i2) \text{skewness_gumbel(a,b)};
& \frac{3/2}{26 \ \text{zeta}(3)}
& \frac{3}{3 \ \%pi}
\end{align*}
\]

where \text{zeta} stands for the Riemann’s zeta function.

kurtosis_gumbel (a,b)  [Function]
Returns the kurtosis coefficient of a Gumbel(a,b) random variable, with \( b > 0 \). To make use of this function, write first load(distrib).

random_gumbel (a,b)  [Function]
random_gumbel (a,b,n)
Returns a Gumbel(a,b) random variate, with \( b > 0 \). Calling \text{random_gumbel} with a third argument \( n \), a random sample of size \( n \) will be simulated.
The implemented algorithm is based on the general inverse method.
To make use of this function, write first load(distrib).

51.3 Functions and Variables for discrete distributions
pdf_general_finite_discrete (x,v)  [Function]
Returns the value at \( x \) of the probability function of a general finite discrete random variable, with vector probabilities \( v \), such that \( \Pr(X=i) = v_i \). Vector \( v \) can be a list of nonnegative expressions, whose components will be normalized to get a vector of probabilities. To make use of this function, write first load(distrib).

\[
\begin{align*}
%i1) \text{load (distrib)}$ &
%i2) \text{pdf_general_finite_discrete(2, [1/7, 4/7, 2/7])};
& 4
& -
& 7
\end{align*}
\]

\[
\begin{align*}
%i3) \text{pdf_general_finite_discrete(2, [1, 4, 2])};
& 4
& -
& 7
\end{align*}
\]

cdf_general_finite_discrete (x,v)  [Function]
Returns the value at \( x \) of the distribution function of a general finite discrete random variable, with vector probabilities \( v \).
See `pdf_general_finite_discrete` for more details.

\begin{verbatim}
(%i1) load (distrib)$
(%i2) cdf_general_finite_discrete(2, [1/7, 4/7, 2/7]);
5
(%o2) -
7
(%i3) cdf_general_finite_discrete(2, [1, 4, 2]);
5
(%o3) -
7
(%i4) cdf_general_finite_discrete(2+1/2, [1, 4, 2]);
5
(%o4) -
7
\end{verbatim}

\texttt{quantile\_general\_finite\_discrete} \texttt{(q,v)} \hspace{1cm} [Function]

Returns the \textit{q}-quantile of a general finite discrete random variable, with vector probabilities \textit{v}.

See `pdf_general_finite_discrete` for more details.

\texttt{mean\_general\_finite\_discrete} \texttt{(v)} \hspace{1cm} [Function]

Returns the mean of a general finite discrete random variable, with vector probabilities \textit{v}.

See `pdf_general_finite_discrete` for more details.

\texttt{var\_general\_finite\_discrete} \texttt{(v)} \hspace{1cm} [Function]

Returns the variance of a general finite discrete random variable, with vector probabilities \textit{v}.

See `pdf_general_finite_discrete` for more details.

\texttt{std\_general\_finite\_discrete} \texttt{(v)} \hspace{1cm} [Function]

Returns the standard deviation of a general finite discrete random variable, with vector probabilities \textit{v}.

See `pdf_general_finite_discrete` for more details.

\texttt{skewness\_general\_finite\_discrete} \texttt{(v)} \hspace{1cm} [Function]

Returns the skewness coefficient of a general finite discrete random variable, with vector probabilities \textit{v}.

See `pdf_general_finite_discrete` for more details.

\texttt{kurtosis\_general\_finite\_discrete} \texttt{(v)} \hspace{1cm} [Function]

Returns the kurtosis coefficient of a general finite discrete random variable, with vector probabilities \textit{v}.

See `pdf_general_finite_discrete` for more details.
random_general_finite_discrete (v)  
random_general_finite_discrete (v,m)

Returns a general finite discrete random variate, with vector probabilities v. Calling random_general_finite_discrete with a second argument m, a random sample of size m will be simulated.

See pdf_general_finite_discrete for more details.

(%i1) load (distrib)$
(%i2) random_general_finite_discrete([1,3,1,5]);
(%o2) 4
(%i3) random_general_finite_discrete([1,3,1,5], 10);
(%o3) [4, 2, 2, 3, 2, 4, 4, 1, 2, 2]

pdf_binomial (x,n,p)  

Returns the value at x of the probability function of a Binomial(n,p) random variable, with 0 ≤ p ≤ 1 and n a positive integer. To make use of this function, write first load(distrib).

(%i1) load (distrib)$
(%i2) pdf_binomial(5,7,1/6);
(%o2) 7775
(%o2)
(%i3) float(%);
(%o3) .9998713991769548

cdf_binomial (x,n,p)  

Returns the value at x of the distribution function of a Binomial(n,p) random variable, with 0 ≤ p ≤ 1 and n a positive integer.

(%i1) load (distrib)$
(%i2) cdf_binomial(5,7,1/6);
(%o2) 7775
(%o2)
(%i3) float(%);
(%o3) .9998713991769548

quantile_binomial (q,n,p)  

Returns the q-quantile of a Binomial(n,p) random variable, with 0 ≤ p ≤ 1 and n a positive integer; in other words, this is the inverse of cdf_binomial. Argument q must be an element of [0, 1]. To make use of this function, write first load(distrib).

mean_binomial (n,p)  

Returns the mean of a Binomial(n,p) random variable, with 0 ≤ p ≤ 1 and n a positive integer. To make use of this function, write first load(distrib).

var_binomial (n,p)  

Returns the variance of a Binomial(n,p) random variable, with 0 ≤ p ≤ 1 and n a positive integer. To make use of this function, write first load(distrib).

std_binomial (n,p)  

Returns the standard deviation of a Binomial(n,p) random variable, with 0 ≤ p ≤ 1 and n a positive integer. To make use of this function, write first load(distrib).

skewness_binomial (n,p)  

Returns the skewness coefficient of a Binomial(n,p) random variable, with 0 ≤ p ≤ 1 and n a positive integer. To make use of this function, write first load(distrib).
kurtosis_binomial (n,p)
Returns the kurtosis coefficient of a Binomial(n, p) random variable, with \( 0 \leq p \leq 1 \) and \( n \) a positive integer. To make use of this function, write first \texttt{load(distrib)}.

random_binomial (n,p)
random_binomial (n,p,m)
Returns a Binomial(n, p) random variate, with \( 0 \leq p \leq 1 \) and \( n \) a positive integer. Calling \texttt{random_binomial} with a third argument \( m \), a random sample of size \( m \) will be simulated.


To make use of this function, write first \texttt{load(distrib)}.

pdf_poisson (x,m)
Returns the value at \( x \) of the probability function of a Poisson\((m)\) random variable, with \( m > 0 \). To make use of this function, write first \texttt{load(distrib)}.

cdf_poisson (x,m)
Returns the value at \( x \) of the distribution function of a Poisson\((m)\) random variable, with \( m > 0 \).

\begin{verbatim}
(%i1) load (distrib)$
(%i2) cdf_poisson(3,5);
(%o2) gamma_incomplete_regularized(4, 5)
(%i3) float(%);
(%o3) .2650259152973623
\end{verbatim}

quantile_poisson (q,m)
Returns the \( q \)-quantile of a Poisson\((m)\) random variable, with \( m > 0 \); in other words, this is the inverse of \texttt{cdf_poisson}. Argument \( q \) must be an element of \([0, 1]\). To make use of this function, write first \texttt{load(distrib)}.

mean_poisson (m)
Returns the mean of a Poisson\((m)\) random variable, with \( m > 0 \). To make use of this function, write first \texttt{load(distrib)}.

var_poisson (m)
Returns the variance of a Poisson\((m)\) random variable, with \( m > 0 \). To make use of this function, write first \texttt{load(distrib)}.

std_poisson (m)
Returns the standard deviation of a Poisson\((m)\) random variable, with \( m > 0 \). To make use of this function, write first \texttt{load(distrib)}.

skewness_poisson (m)
Returns the skewness coefficient of a Poisson\((m)\) random variable, with \( m > 0 \). To make use of this function, write first \texttt{load(distrib)}.
kurtosis_poisson \((m)\) \[\text{Function}\]
Returns the kurtosis coefficient of a Poisson random variable \(Poi(m)\), with \(m > 0\).
To make use of this function, write first \texttt{load(distrib)}.

random_poisson \((m)\) \[\text{Function}\]
random_poisson \((m, n)\)
Returns a \(Poi(m)\) random variate, with \(m > 0\). Calling \texttt{random_poisson} with a second argument \(n\), a random sample of size \(n\) will be simulated.
To make use of this function, write first \texttt{load(distrib)}.

pdf_bernoulli \((x, p)\) \[\text{Function}\]
Returns the value at \(x\) of the probability function of a \(Bernoulli(p)\) random variable, with \(0 \leq p \leq 1\).
The \(Bernoulli(p)\) random variable is equivalent to the \(Binomial(1, p)\).

\[
\begin{align*}
\texttt{(i1) load (distrib)}$ \\
\texttt{(i2) pdf_bernoulli(1,p)}; \\
\texttt{(o2)} p
\end{align*}
\]

cdf_bernoulli \((x, p)\) \[\text{Function}\]
Returns the value at \(x\) of the distribution function of a \(Bernoulli(p)\) random variable, with \(0 \leq p \leq 1\). To make use of this function, write first \texttt{load(distrib)}.

quantile_bernoulli \((q, p)\) \[\text{Function}\]
Returns the \(q\)-quantile of a \(Bernoulli(p)\) random variable, with \(0 \leq p \leq 1\); in other words, this is the inverse of \texttt{cdf_bernoulli}. Argument \(q\) must be an element of \([0, 1]\).
To make use of this function, write first \texttt{load(distrib)}.

mean_bernoulli \((p)\) \[\text{Function}\]
Returns the mean of a \(Bernoulli(p)\) random variable, with \(0 \leq p \leq 1\).
The \(Bernoulli(p)\) random variable is equivalent to the \(Binomial(1, p)\).

\[
\begin{align*}
\texttt{(i1) load (distrib)}$ \\
\texttt{(i2) mean_bernoulli(p)}; \\
\texttt{(o2)} p
\end{align*}
\]

var_bernoulli \((p)\) \[\text{Function}\]
Returns the variance of a \(Bernoulli(p)\) random variable, with \(0 \leq p \leq 1\).
The \(Bernoulli(p)\) random variable is equivalent to the \(Binomial(1, p)\).

\[
\begin{align*}
\texttt{(i1) load (distrib)}$ \\
\texttt{(i2) var_bernoulli(p)}; \\
\texttt{(o2)} (1 - p) p
\end{align*}
\]

std_bernoulli \((p)\) \[\text{Function}\]
Returns the standard deviation of a \(Bernoulli(p)\) random variable, with \(0 \leq p \leq 1\).
The *Bernoulli*(p) random variable is equivalent to the *Binomial*(1,p).

(%i1) load (distrib)$
(%i2) std_bernoulli(p);
(%o2) \sqrt{(1 - p) p}

skewness_bernoulli (p)
 Returns the skewness coefficient of a *Bernoulli*(p) random variable, with 0 ≤ p ≤ 1.

The *Bernoulli*(p) random variable is equivalent to the *Binomial*(1,p).

(%i1) load (distrib)$
(%i2) skewness_bernoulli(p);
(%o2) \frac{1 - 2 p}{\sqrt{(1 - p) p}}

kurtosis_bernoulli (p)
 Returns the kurtosis coefficient of a *Bernoulli*(p) random variable, with 0 ≤ p ≤ 1.

The *Bernoulli*(p) random variable is equivalent to the *Binomial*(1,p).

(%i1) load (distrib)$
(%i2) kurtosis_bernoulli(p);
(%o2) \frac{1 - 6 (1 - p) p}{(1 - p) p}

random_bernoulli (p)
random_bernoulli (p,n)
 Returns a *Bernoulli*(p) random variate, with 0 ≤ p ≤ 1. Calling *random_bernoulli* with a second argument n, a random sample of size n will be simulated.

This is a direct application of the *random* built-in Maxima function.

See also *random*. To make use of this function, write first load(distrib).

pdf_geometric (x,p)
 Returns the value at x of the probability function of a *Geometric*(p) random variable, with 0 < p ≤ 1. To make use of this function, write first load(distrib).

cdf_geometric (x,p)
 Returns the value at x of the distribution function of a *Geometric*(p) random variable, with 0 < p ≤ 1. To make use of this function, write first load(distrib).

quantile_geometric (q,p)
 Returns the q-quantile of a *Geometric*(p) random variable, with 0 < p ≤ 1; in other words, this is the inverse of *cdf_geometric*. Argument q must be an element of [0,1]. To make use of this function, write first load(distrib).

mean_geometric (p)
 Returns the mean of a *Geometric*(p) random variable, with 0 < p ≤ 1. To make use of this function, write first load(distrib).
var_geometric (p) [Function]
Returns the variance of a Geometric(p) random variable, with 0 < p \leq 1. To make use of this function, write first load(distrib).

std_geometric (p) [Function]
Returns the standard deviation of a Geometric(p) random variable, with 0 < p \leq 1. To make use of this function, write first load(distrib).

skewness_geometric (p) [Function]
Returns the skewness coefficient of a Geometric(p) random variable, with 0 < p \leq 1. To make use of this function, write first load(distrib).

kurtosis_geometric (p) [Function]
Returns the kurtosis coefficient of a geometric random variable Geo(p), with 0 < p \leq 1. To make use of this function, write first load(distrib).

random_geometric (p) [Function]
random_geometric (p,n)
Returns a Geometric(p) random variate, with 0 < p \leq 1. Calling random_geometric with a second argument n, a random sample of size n will be simulated.
The algorithm is based on simulation of Bernoulli trials.
To make use of this function, write first load(distrib).

pdf_discrete_uniform (x,n) [Function]
Returns the value at x of the probability function of a DiscreteUniform(n) random variable, with n a strictly positive integer. To make use of this function, write first load(distrib).

cdf_discrete_uniform (x,n) [Function]
Returns the value at x of the distribution function of a DiscreteUniform(n) random variable, with n a strictly positive integer. To make use of this function, write first load(distrib).

quantile_discrete_uniform (q,n) [Function]
Returns the q-quantile of a DiscreteUniform(n) random variable, with n a strictly positive integer; in other words, this is the inverse of cdf_discrete_uniform. Argument q must be an element of [0,1]. To make use of this function, write first load(distrib).

mean_discrete_uniform (n) [Function]
Returns the mean of a DiscreteUniform(n) random variable, with n a strictly positive integer. To make use of this function, write first load(distrib).

var_discrete_uniform (n) [Function]
Returns the variance of a DiscreteUniform(n) random variable, with n a strictly positive integer. To make use of this function, write first load(distrib).

std_discrete_uniform (n) [Function]
Returns the standard deviation of a DiscreteUniform(n) random variable, with n a strictly positive integer. To make use of this function, write first load(distrib).
skewness_discrete_uniform (n)
Returns the skewness coefficient of a DiscreteUniform(n) random variable, with n a strictly positive integer. To make use of this function, write first load(distrib).

kurtosis_discrete_uniform (n)
Returns the kurtosis coefficient of a DiscreteUniform(n) random variable, with n a strictly positive integer. To make use of this function, write first load(distrib).

random_discrete_uniform (n)
random_discrete_uniform (n, m)
Returns a DiscreteUniform(n) random variate, with n a strictly positive integer. Calling random_discrete_uniform with a second argument m, a random sample of size m will be simulated.
This is a direct application of the random built-in Maxima function.
See also random. To make use of this function, write first load(distrib).

pdf_hypergeometric (x,n1,n2,n)
Returns the value at x of the probability function of a Hypergeometric(n1, n2, n) random variable, with n1, n2 and n non negative integers and n <= n1 + n2. Being n1 the number of objects of class A, n2 the number of objects of class B, and n the size of the sample without replacement, this function returns the probability of event "exactly x objects are of class A".
To make use of this function, write first load(distrib).

cdf_hypergeometric (x,n1,n2,n)
Returns the value at x of the distribution function of a Hypergeometric(n1, n2, n) random variable, with n1, n2 and n non negative integers and n <= n1 + n2. See pdf_hypergeometric for a more complete description.
To make use of this function, write first load(distrib).

quantile_hypergeometric (q,n1,n2,n)
Returns the q-quantile of a Hypergeometric(n1, n2, n) random variable, with n1, n2 and n non negative integers and n <= n1 + n2; in other words, this is the inverse of cdf_hypergeometric. Argument q must be an element of [0, 1]. To make use of this function, write first load(distrib).

mean_hypergeometric (n1,n2,n)
Returns the mean of a discrete uniform random variable Hyp(n1, n2, n), with n1, n2 and n non negative integers and n <= n1 + n2. To make use of this function, write first load(distrib).

var_hypergeometric (n1,n2,n)
Returns the variance of a hypergeometric random variable Hyp(n1, n2, n), with n1, n2 and n non negative integers and n <= n1 + n2. To make use of this function, write first load(distrib).

std_hypergeometric (n1,n2,n)
Returns the standard deviation of a Hypergeometric(n1, n2, n) random variable, with n1, n2 and n non negative integers and n <= n1 + n2. To make use of this function, write first load(distrib).
skewness_hypergeometric (n1,n2,n)

Returns the skewness coefficient of a Hypergeometric(n1,n2,n) random variable, with n1, n2 and n non negative integers and n <= n1 + n2. To make use of this function, write first load(distrib).

kurtosis_hypergeometric (n1,n2,n)

Returns the kurtosis coefficient of a Hypergeometric(n1,n2,n) random variable, with n1, n2 and n non negative integers and n <= n1 + n2. To make use of this function, write first load(distrib).

random_hypergeometric (n1,n2,n)

random_hypergeometric (n1,n2,n,m)

Returns a Hypergeometric(n1,n2,n) random variate, with n1, n2 and n non negative integers and n <= n1 + n2. Calling random_hypergeometric with a fourth argument m, a random sample of size m will be simulated.


To make use of this function, write first load(distrib).

pdf_negative_binomial (x,n,p)

Returns the value at x of the probability function of a NegativeBinomial(n,p) random variable, with 0 < p ≤ 1 and n a positive number. To make use of this function, write first load(distrib).

cdf_negative_binomial (x,n,p)

Returns the value at x of the distribution function of a NegativeBinomial(n,p) random variable, with 0 < p ≤ 1 and n a positive number.

(%i1) load (distrib)$

(%i2) cdf_negative_binomial(3,4,1/8);

3271

524288

quantile_negative_binomial (q,n,p)

Returns the q-quantile of a NegativeBinomial(n,p) random variable, with 0 < p ≤ 1 and n a positive number; in other words, this is the inverse of cdf_negative_binomial. Argument q must be an element of [0,1]. To make use of this function, write first load(distrib).

mean_negative_binomial (n,p)

Returns the mean of a NegativeBinomial(n,p) random variable, with 0 < p ≤ 1 and n a positive number. To make use of this function, write first load(distrib).

var_negative_binomial (n,p)

Returns the variance of a NegativeBinomial(n,p) random variable, with 0 < p ≤ 1 and n a positive number. To make use of this function, write first load(distrib).
std_negative_binomial (n,p)  
Returns the standard deviation of a \texttt{NegativeBinomial}(n,p) random variable, with \(0 < p \leq 1\) and \(n\) a positive number. To make use of this function, write first \texttt{load(distrib)}.

skewness_negative_binomial (n,p)  
Returns the skewness coefficient of a \texttt{NegativeBinomial}(n,p) random variable, with \(0 < p \leq 1\) and \(n\) a positive number. To make use of this function, write first \texttt{load(distrib)}.

kurtosis_negative_binomial (n,p)  
Returns the kurtosis coefficient of a \texttt{NegativeBinomial}(n,p) random variable, with \(0 < p \leq 1\) and \(n\) a positive number. To make use of this function, write first \texttt{load(distrib)}.

random_negative_binomial (n,p)  
random_negative_binomial (n,p,m)  
Returns a \texttt{NegativeBinomial}(n,p) random variate, with \(0 < p \leq 1\) and \(n\) a positive number. Calling \texttt{random_negative_binomial} with a third argument \(m\), a random sample of size \(m\) will be simulated.


To make use of this function, write first \texttt{load(distrib)}.
52 draw

52.1 Introduction to draw

draw is a Maxima-Gnuplot and a Maxima-vtk interface.

There are three main functions to be used at Maxima level: draw2d, draw3d and draw.

Follow these links for more elaborated examples of this package:

http://tecnostats.net/Maxima/gnuplot
http://tecnostats.net/Maxima/vtk

You need Gnuplot 4.2 or newer to run this program.

52.2 Functions and Variables for draw

52.2.1 Scenes

gr2d (graphic option, ..., graphic object, ...)  [Scene constructor]

Function gr2d builds an object describing a 2D scene. Arguments are graphic options, graphic objects, or lists containing both graphic options and objects. This scene is interpreted sequentially: graphic options affect those graphic objects placed on its right. Some graphic options affect the global appearance of the scene.

This is the list of graphic objects available for scenes in two dimensions: bars, ellipse, explicit, image, implicit, label, parametric, points, polar, polygon, quadrilateral, rectangle, triangle, vector and geomap (this one defined in package worldmap).

See also draw and draw2d.

(%i1) load("draw");
(%o1) /maxima/share/draw/draw.lisp
(%i2) draw(
   gr2d(
      key="sin (x)", grid=[2,2],
      explicit(
         sin(x),
         x,0,2*%pi
      )
   ),
   gr2d(
      key="cos (x)", grid=[2,2],
      explicit(
         cos(x),
         x,0,2*%pi
      )
   )
);
(%o2) [gr2d(explicit), gr2d(explicit)]
gr3d (graphic option, ..., graphic object, ...)  [Scene constructor]
Function gr3d builds an object describing a 3d scene. Arguments are graphic options, graphic objects, or lists containing both graphic options and objects. This scene is interpreted sequentially: graphic options affect those graphic objects placed on its right. Some graphic options affect the global appearance of the scene.

This is the list of graphic objects available for scenes in three dimensions: cylindrical, elevation_grid, explicit, implicit, label, mesh, parametric, parametric_surface, points, quadrilateral, spherical, triangle, tube, vector, and geomap (this one defined in package worldmap).
See also draw and draw3d.

52.2.2 Functions
draw (gr2d, ..., gr3d, ..., options, ...)  [Function]
Plots a series of scenes; its arguments are gr2d and/or gr3d objects, together with some options, or lists of scenes and options. By default, the scenes are put together in one column.

Function draw accepts the following global options: terminal, columns, dimensions, file_name and delay.

Functions draw2d and draw3d are short cuts to be used when only one scene is required, in two or three dimensions, respectively.
See also gr2d and gr3d.

Example:
(%i1) scene1: gr2d(title="Ellipse",
               nticks=30,
               parametric(2*cos(t),5*sin(t),t,0,2*%pi))$
(%i2) scene2: gr2d(title="Triangle",
               polygon([[4,5,7],[6,4,2]]))$
(%i3) draw(scene1, scene2, columns = 2)$
The two draw sentences are equivalent:
(%i1) draw(gr3d(explicit(x^2+y^2,x,-1,1,y,-1,1)));
Chapter 52: draw

(%o1)          [gr3d(explicit)]
(%i2)  draw3d(explicit(x^2+y^2,x,-1,1,y,-1,1));
(%o2)    [gr3d(explicit)]

An animated gif file:

(%i1)  draw(
    delay    = 100,
    file_name = "zzz",
    terminal  = 'animated_gif,
    gr2d(explicit(x^2,x,-1,1)),
    gr2d(explicit(x^3,x,-1,1)),
    gr2d(explicit(x^4,x,-1,1)));

End of animation sequence

(%o1) [gr2d(explicit), gr2d(explicit), gr2d(explicit)]

See also gr2d, gr3d, draw2d and draw3d.

draw2d (option, graphic_object, ...)
       [Function]
This function is a short cut for draw(gr2d(options, ..., graphic_object, ...)).
It can be used to plot a unique scene in 2d.
See also draw and gr2d.

draw3d (option, graphic_object, ...)
       [Function]
This function is a short cut for draw(gr3d(options, ..., graphic_object, ...)).
It can be used to plot a unique scene in 3d.
See also draw and gr3d.

draw_file (graphic option, ..., graphic object, ...)
       [Function]
Saves the current plot into a file. Accepted graphics options are: terminal, dimensions and file_name.
Example:

(%i1) /* screen plot */
\texttt{draw(gr3d(explicit(x^2+y^2,x,-1,1,y,-1,1)))}$

\texttt{(\%i2) /* same plot in eps format */}
\texttt{draw_file(terminal = eps,}
\texttt{dimensions = \[5,5\]) $}

\textbf{multiplot_mode (term)} \hfill [Function]

This function enables Maxima to work in one-window multiplot mode with terminal \textit{term}; accepted arguments for this function are \texttt{screen}, \texttt{wxt}, \texttt{aquaterm} and \texttt{none}.

When multiplot mode is enabled, each call to \texttt{draw} sends a new plot to the same window, without erasing the previous ones. To disable the multiplot mode, write \texttt{multiplot_mode(none)}.

When multiplot mode is enabled, global option \texttt{terminal} is blocked and you have to disable this working mode before changing to another terminal.

This feature does not work in Windows platforms.

Example:

\texttt{(\%i1) set_draw_defaults(}
\texttt{   xrange = \[-1,1\],}
\texttt{   yrange = \[-1,1\],}
\texttt{   grid = true,}
\texttt{   title = "Step by step plot" )}$
\texttt{(\%i2) multiplot_mode(screen)$}
\texttt{(\%i3) draw2d(color=blue, explicit(x^2,x,-1,1))}$
\texttt{(\%i4) draw2d(color=red, explicit(x^3,x,-1,1))}$
\texttt{(\%i5) draw2d(color=brown, explicit(x^4,x,-1,1))}$
\texttt{(\%i6) multiplot_mode(none)$}
set_draw_defaults (graphic option, ..., graphic object, ...)  
[Function] 
Sets user graphics options. This function is useful for plotting a sequence of graphics with common graphics options. Calling this function without arguments removes user defaults.

Example:

(\%i1) set_draw_defaults(
    xrange = [-10,10],
    yrange = [-2, 2],
    color = blue,
    grid = true)$
(%i2) /* plot with user defaults */
   draw2d(explicit(((1+x)**2/(1+x*x))-1,x,-10,10))$
(%i3) set_draw_defaults()$
(%i4) /* plot with standard defaults */
   draw2d(explicit(((1+x)**2/(1+x*x))-1,x,-10,10))$

52.2.3 Graphics options

adapt_depth  
[Graphic option]  
Default value: 10

adapt_depth is the maximum number of splittings used by the adaptive plotting routine.

This option is relevant only for 2d explicit functions.

See also ntics

allocation  
[Graphic option]  
Default value: false

With option allocation it is possible to place a scene in the output window at will; this is of interest in multiplots. When false, the scene is placed automatically, depending on the value assigned to option columns. In any other case, allocation must be set to a list of two pairs of numbers; the first corresponds to the position of the lower left corner of the scene, and the second pair gives the width and height of the plot. All quantities must be given in relative coordinates, between 0 and 1.

Examples:
In site graphics.

(\%i1) draw(
   gr2d(
       explicit(x^2,x,-1,1)),
   gr2d(
       allocation = [[1/4, 1/4],[1/2, 1/2]],
       explicit(x^3,x,-1,1),
       grid = true) )$
Multiplot with selected dimensions.

(%i1) draw(
    terminal = wxt,
    gr2d(
        grid=[5,5],
        allocation = [[0, 0],[1, 1/4]],
        explicit(x^2,x,-1,1)),
    gr3d(
        allocation = [[0, 1/4],[1, 3/4]],
        explicit(x^2+y^2,x,-1,1,y,-1,1) )))$

See also option columns.

axis_3d

[Graphic option]

Default value: true

If axis_3d is true, the x, y and z axis are shown in 3d scenes.

Since this is a global graphics option, its position in the scene description does not matter.
Example:

```latex
(%i1) draw3d(axis_3d = false,
    explicit(sin(x^2+y^2),x,-2,2,y,-2,2) )$
```

See also `axis_bottom`, `axis_left`, `axis_top`, and `axis_right` for axis in 2d.

**axis_bottom**

[Graphic option]

Default value: true

If `axis_bottom` is true, the bottom axis is shown in 2d scenes.

Since this is a global graphics option, its position in the scene description does not matter.

Example:

```latex
(%i1) draw2d(axis_bottom = false,
    explicit(x^3,x,-1,1))$
```

See also `axis_left`, `axis_top`, `axis_right` and `axis_3d`. 

axis_left

Default value: true

If axis_left is true, the left axis is shown in 2d scenes.

Since this is a global graphics option, its position in the scene description does not matter.

Example:

(%i1) draw2d(axis_left = false, explicit(x^3,x,-1,1))$

See also axis_bottom, axis_top, axis_right and axis_3d.

axis_right

Default value: true

If axis_right is true, the right axis is shown in 2d scenes.

Since this is a global graphics option, its position in the scene description does not matter.

Example:

(%i1) draw2d(axis_right = false, explicit(x^3,x,-1,1))$

See also axis_bottom, axis_left, axis_top and axis_3d.

axis_top

Default value: true

If axis_top is true, the top axis is shown in 2d scenes.

Since this is a global graphics option, its position in the scene description does not matter.

Example:

(%i1) draw2d(axis_top = false, explicit(x^3,x,-1,1))$

See also axis_bottom, axis_left, axis_right, and axis_3d.

background_color

Default value: white

Sets the background color for terminals. Default background color is white.

Since this is a global graphics option, its position in the scene description does not matter.

This option does not work with terminals epslatex and epslatex_standalone.

See also color

border

Default value: true

If border is true, borders of polygons are painted according to line_type and line_width.

This option affects the following graphic objects:

• gr2d: polygon, rectangle and ellipse.
Example:

```
(%i1) draw2d(color = brown,
       line_width = 8,
       polygon([[3,2],[7,2],[5,5]]),
       border = false,
       fill_color = blue,
       polygon([[5,2],[9,2],[7,5]]) )$
```

![Diagram of a triangle with different color and line width](attachment:image.png)

**capping**

[Graphic option]

Default value: [false, false]

A list with two possible elements, true and false, indicating whether the extremes of a graphic object tube remain closed or open. By default, both extremes are left open.

Setting `capping = false` is equivalent to `capping = [false, false]`, and `capping = true` is equivalent to `capping = [true, true]`.

Example:

```
(%i1) draw3d(
    capping = [false, true],
    tube(0, 0, a, 1,
         a, 0, 8))$
```
cbrange

[Graphic option]

Default value: auto

If cbrange is auto, the range for the values which are colored when enhanced3d is not false is computed automatically. Values outside of the color range use color of the nearest extreme.

When enhanced3d or colorbox is false, option cbrange has no effect.

If the user wants a specific interval for the colored values, it must be given as a Maxima list, as in cbrange=[-2, 3].

Since this is a global graphics option, its position in the scene description does not matter.

Example:

(%i1) draw3d (  
  enhanced3d = true,  
  color = green,  
  cbrange = [-3,10],  
  explicit(x^2+y^2, x,-2,2,y,-2,2))$


See also enhanced3d, colorbox and cbtics.

**cbtics**

[Graphic option]

Default value: auto

This graphic option controls the way tic marks are drawn on the colorbox when option enhanced3d is not false.

When enhanced3d or colorbox is false, option cbtics has no effect.

See xtics for a complete description.

Example:

```lisp
(%i1) draw3d (enhanced3d = true, color = green, cbtics = {"High",10}, colorbox, cbrange = [0, 10], explicit(x^2+y^2, x,-2,2,y,-2,2)) $
```

See also enhanced3d, colorbox and cbrange.

**color**

[Graphic option]

Default value: blue

**color** specifies the color for plotting lines, points, borders of polygons and labels.

Colors can be given as names or in hexadecimal rgb code. If a gnuplot version >= 5.0 is used and the terminal that is in use supports this rgba colors with transparency information are also supported.

Available color names are:

<table>
<thead>
<tr>
<th>White</th>
<th>Black</th>
<th>Gray0</th>
<th>Grey0</th>
</tr>
</thead>
<tbody>
<tr>
<td>gray10</td>
<td>grey10</td>
<td>gray20</td>
<td>grey20</td>
</tr>
<tr>
<td>gray30</td>
<td>grey30</td>
<td>gray40</td>
<td>grey40</td>
</tr>
<tr>
<td>gray50</td>
<td>grey50</td>
<td>gray60</td>
<td>grey60</td>
</tr>
<tr>
<td>gray70</td>
<td>grey70</td>
<td>gray80</td>
<td>grey80</td>
</tr>
<tr>
<td>gray90</td>
<td>grey90</td>
<td>gray100</td>
<td>grey100</td>
</tr>
</tbody>
</table>
Cromatic components in hexadecimal code are introduced in the form "#rrggbb".

Example:

(%i1) draw2d(explicit(x^2,x,-1,1), /* default is black */
    color = red,
    explicit(0.5 + x^2,x,-1,1),
    color = blue,
    explicit(1 + x^2,x,-1,1),
    color = light_blue,
    explicit(1.5 + x^2,x,-1,1),
    color = "#23ab0f",
    label(["This is a label",0,1.2]) )$

(%i1) draw2d(
    line_width=50,
    color="#FF0000",
    explicit(sin(x),x,0,10),
See also \texttt{fill\_color}.

\textbf{colorbox} \hspace{1cm} \textbf{[Graphic option]}

\textit{Default value: true}

If \texttt{colorbox} is \texttt{true}, a color scale without label is drawn together with \texttt{image} 2D objects, or coloured 3d objects. If \texttt{colorbox} is \texttt{false}, no color scale is shown. If \texttt{colorbox} is a string, a color scale with label is drawn.

Since this is a global graphics option, its position in the scene description does not matter.

Example:

Color scale and images.

\begin{verbatim}
(%i1) im: apply('matrix,
   makelist(makelist(random(200),i,1,30),i,1,30))$
(%i2) draw(
  gr2d(image(im,0,0,30,30)),
  gr2d(colorbox = false, image(im,0,0,30,30))
)$
\end{verbatim}
Color scale and 3D coloured object.

(%i1) draw3d(
    colorbox = "Magnitude",
    enhanced3d = true,
    explicit(x^2+y^2,x,-1,1,y,-1,1))$

See also \texttt{palette\_draw}.

\texttt{columns} \hspace{1cm} \textbf{[Graphic option]}

Default value: 1

\texttt{columns} is the number of columns in multiple plots.

Since this is a global graphics option, its position in the scene description does not matter. It can be also used as an argument of function \texttt{draw}.

Example:

(%i1) scene1: gr2d(title="Ellipse",
    nticks=30,
    parametric(2*cos(t),5*sin(t),t,0,2*%pi))$
\texttt{(\%i2) scene2: gr2d(title="Triangle",}
\texttt{ polygon([4,5,7],[6,4,2]))$}
\texttt{(\%i3) draw(scene1, scene2, columns = 2)$}

\textbf{contour} \quad [\text{Graphic option}]

Default value: \texttt{none}

Option \texttt{contour} enables the user to select where to plot contour lines. Possible values are:

- \texttt{none}: no contour lines are plotted.
- \texttt{base}: contour lines are projected on the xy plane.
- \texttt{surface}: contour lines are plotted on the surface.
- \texttt{both}: two contour lines are plotted: on the xy plane and on the surface.
- \texttt{map}: contour lines are projected on the xy plane, and the view point is set just in the vertical.

Since this is a global graphics option, its position in the scene description does not matter.

Example:

\texttt{(\%i1) draw3d(explicit(20*exp(-x^2-y^2)-10,x,0,2,y,-3,3),}
\texttt{ contour_levels = 15,}
\texttt{ contour = both,}
\texttt{ surface_hide = true) $}
(%i1) draw3d(explicit(20*exp(-x^2-y^2)-10,x,0,2,y,-3,3),
    contour_levels = 15,
    contour = map
)

contour_levels

[Graphic option]

Default value: 5

This graphic option controls the way contours are drawn. contour_levels can be set to a positive integer number, a list of three numbers or an arbitrary set of numbers:

- When option contour_levels is bounded to positive integer \( n \), \( n \) contour lines will be drawn at equal intervals. By default, five equally spaced contours are plotted.

- When option contour_levels is bounded to a list of length three of the form \([\text{lowest}, s, \text{highest}]\), contour lines are plotted from \( \text{lowest} \) to \( \text{highest} \) in steps of \( s \).

- When option contour_levels is bounded to a set of numbers of the form \( \{\text{n1, n2, \ldots}\} \), contour lines are plotted at values \( \text{n1, n2, \ldots} \).
Since this is a global graphics option, its position in the scene description does not matter.
Examples:
Ten equally spaced contour lines. The actual number of levels can be adjusted to give simple labels.

```lisp
(%i1) draw3d(color = green,
     explicit(20*exp(-x^2-y^2)-10,x,0,2,y,-3,3),
     contour_levels = 10,
     contour = both,
     surface_hide = true) $
```

From -8 to 8 in steps of 4.

```lisp
(%i1) draw3d(color = green,
     explicit(20*exp(-x^2-y^2)-10,x,0,2,y,-3,3),
     contour_levels = [-8,4,8],
     contour = both,
     surface_hide = true) $
```

Isolines at levels -7, -6, 0.8 and 5.

```lisp
(%i1) draw3d(color = green,
     explicit(20*exp(-x^2-y^2)-10,x,0,2,y,-3,3),
     contour_levels = {-7, -6, 0.8, 5},
     contour = both,
     surface_hide = true) $
```

See also `contour`.

data_file_name [Graphic option]
Default value: "data.gnuplot"
This is the name of the file with the numeric data needed by Gnuplot to build the requested plot.
Since this is a global graphics option, its position in the scene description does not matter. It can be also used as an argument of function `draw`.
See example in `gnuplot_file_name`.

delay [Graphic option]
Default value: 5
This is the delay in 1/100 seconds of frames in animated gif files.
Since this is a global graphics option, its position in the scene description does not matter. It can be also used as an argument of function `draw`.
Example:
```lisp
(%i1) draw(
    delay = 100,
    file_name = "zzz",
    terminal = 'animated_gif,
    gr2d(explicit(x^2,x,-1,1)),
    gr2d(explicit(x^3,x,-1,1)),
```
\texttt{gr2d(explicit(x^4,x,-1,1))};

End of animation sequence

(\%o2) \[\text{gr2d(explicit), gr2d(explicit), gr2d(explicit)}\]

Option \texttt{delay} is only active in animated gif’s; it is ignored in any other case.

See also \texttt{terminal}, and \texttt{dimensions}.

\textbf{dimensions} \hspace{1cm} \textit{[Graphic option]}

Default value: \[600,500\]

Dimensions of the output terminal. Its value is a list formed by the width and the height. The meaning of the two numbers depends on the terminal you are working with.

With terminals \texttt{gif}, \texttt{animated_gif}, \texttt{png}, \texttt{jpg}, \texttt{svg}, \texttt{screen}, \texttt{wxt}, and \texttt{aquaterm}, the integers represent the number of points in each direction. If they are not integers, they are rounded.

With terminals \texttt{eps}, \texttt{eps_color}, \texttt{pdf}, and \texttt{pdfcairo}, both numbers represent hundredths of cm, which means that, by default, pictures in these formats are 6 cm in width and 5 cm in height.

Since this is a global graphics option, its position in the scene description does not matter. It can be also used as an argument of function \texttt{draw}.

Examples:

Option \texttt{dimensions} applied to file output and to \texttt{wxt} canvas.

(\%i1) \texttt{draw2d(}
    dimensions = \[300,300\],
    terminal = 'png,
    explicit(x^4,x,-1,1)) $

(\%i2) \texttt{draw2d(}
    dimensions = \[300,300\],
    terminal = 'wxt,
    explicit(x^4,x,-1,1)) $

Option \texttt{dimensions} applied to eps output. We want an eps file with A4 portrait dimensions.

(\%i1) \texttt{A4portrait: 100*[21, 29.7]}$
(\%i2) \texttt{draw3d(}
    dimensions = \texttt{A4portrait},
    terminal = 'eps,
    explicit(x^2-y^2,x,-2,2,y,-2,2)) $

\textbf{draw_realpart} \hspace{1cm} \textit{[Graphic option]}

Default value: \texttt{true}

When \texttt{true}, functions to be drawn are considered as complex functions whose real part value should be plotted; when \texttt{false}, nothing will be plotted when the function does not give a real value.

This option affects objects \texttt{explicit} and \texttt{parametric} in 2D and 3D, and \texttt{parametric_surface}.
Example:

Option `draw_realpart` affects objects `explicit` and `parametric`.

```
(%i1) draw2d(
   draw_realpart = false,
   explicit(sqrt(x^2 - 4*x) - x, x, -1, 5),
   color = red,
   draw_realpart = true,
   parametric(x,sqrt(x^2 - 4*x) - x + 1, x, -1, 5) );
```

**enhanced3d**  
Default value: none

If `enhanced3d` is none, surfaces are not colored in 3D plots. In order to get a colored surface, a list must be assigned to option `enhanced3d`, where the first element is an expression and the rest are the names of the variables or parameters used in that expression. A list such `[f(x,y,z), x, y, z]` means that point `[x,y,z]` of the surface is assigned number `f(x,y,z)`, which will be colored according to the actual palette. For those 3D graphic objects defined in terms of parameters, it is possible to define the color number in terms of the parameters, as in `[f(u), u]`, as in objects `parametric` and `tube`, or `[f(u,v), u, v]`, as in object `parametric_surface`. While all 3D objects admit the model based on absolute coordinates, `[f(x,y,z), x, y, z]`, only two of them, namely `explicit` and `elevation_grid`, accept also models defined on the `[x,y]` coordinates, `[f(x,y), x, y]`. 3D graphic object `implicit` accepts only the `[f(x,y,z), x, y, z]` model. Object `points` accepts also the `[f(x,y,z), x, y, z]` model, but when points have a chronological nature, model `[f(k), k]` is also valid, being `k` an ordering parameter.

When `enhanced3d` is assigned something different to none, options `color` and `surface_hide` are ignored.

The names of the variables defined in the lists may be different to those used in the definitions of the graphic objects.

In order to maintain back compatibility, `enhanced3d = false` is equivalent to `enhanced3d = none`, and `enhanced3d = true` is equivalent to `enhanced3d = [z, x, y, z]`. If an expression is given to `enhanced3d`, its variables must be the same used in the surface definition. This is not necessary when using lists.

See option `palette` to learn how palettes are specified.

Examples:

`explicit` object with coloring defined by the `[f(x,y,z), x, y, z]` model.

```
(%i1) draw3d(
   enhanced3d = [x-z/10,x,y,z],
   palette = gray,
   explicit(20*exp(-x^2-y^2)-10,x,-3,3,y,-3,3))$
```
explicit object with coloring defined by the \([f(x,y), x, y]\) model. The names of the variables defined in the lists may be different to those used in the definitions of the graphic objects; in this case, \(r\) corresponds to \(x\), and \(s\) to \(y\).

\[
(%i1) \text{draw3d(}
  \hspace{1em} \text{enhanced3d = \{sin(r*s), r,s\},}
  \hspace{1em} \text{explicit(20*exp(-x^2-y^2)-10,x,-3,3,y,-3,3))}
\]

parametric object with coloring defined by the \([f(x,y,z), x, y, z]\) model.

\[
(%i1) \text{draw3d(}
  \hspace{1em} \text{nticks = 100},
  \hspace{1em} \text{line_width = 2},
  \hspace{1em} \text{enhanced3d = \{if y>= 0 then 1 else 0, x, y, z\},}
  \hspace{1em} \text{parametric(sin(u)^2, cos(u), u, u, 0, 4*%pi))}
\]
parametric object with coloring defined by the \([f(u), u]\) model. In this case, 
\((u-1)^2\) is a shortcut for \([(u-1)^2, u]\).

(%i1) draw3d(
  nticks = 60,
  line_width = 3,
  enhanced3d = (u-1)^2,
  parametric(cos(5*u)^2, sin(7*u), u-2, u, 0, 2))$

elevation_grid object with coloring defined by the \([f(x, y), x, y]\) model.

(%i1) m: apply(
  matrix,
  makelist(makelist(cos(i^2/80-k/30), k, 1, 30), i, 1, 20))$

(%i2) draw3d(
  enhanced3d = [cos(x*y*10), x, y],
  elevation_grid(m, -1, -1, 2, 2),
  xlabel = "x",
  ylabel = "y");
tube object with coloring defined by the \([f(x,y,z), x, y, z]\) model.

(%i1) draw3d(
   enhanced3d = [cos(x-y),x,y,z],
   palette = gray,
   xu_grid = 50,
   tube(cos(a), a, 0, 1, a, 0, 4*%pi) )$

\[\]

\[\]

tube object with coloring defined by the \([f(u), u]\) model. Here, \(\text{enhanced3d} = -a\) would be the shortcut for \(\text{enhanced3d} = [-\text{foo, foo}].\)

(%i1) draw3d(
   capping = [true, false],
   palette = [26,15,-2],
   enhanced3d = [-foo, foo],
   tube(a, a, a^2, 1, a, -2, 2) )$
implicit and points objects with coloring defined by the \([f(x,y,z), x, y, z]\) model.

(%i1) draw3d(
   enhanced3d = [x-y,x,y,z],
   implicit((x^2+y^2+z^2-1)*(x^2+(y-1.5)^2+z^2-0.5)=0.015,
           x,-1,1,y,-1.2,2.3,z,-1,1)) $

(%i2) m: makelist([random(1.0),random(1.0),random(1.0)],k,1,2000)$

(%i3) draw3d(
   point_type = filled_circle,
   point_size = 2,
   enhanced3d = [u+v-w,u,v,w],
   points(m) ) $
When points have a chronological nature, model \([f(k), k]\) is also valid, being \(k\) an ordering parameter.

\[
\text{(\%i1) } m:\text{makelist([random(1.0), random(1.0), random(1.0)],k,1,5)}$
\]

\[
\text{(\%i2) draw3d(}
\text{enhanced3d = [sin(j), j],}
\text{point_size = 3,}
\text{point_type = filled_circle,}
\text{points_joined = true,}
\text{points(m))}$
\]

**error_type**

[Graphic option]

Default value: \(y\)

Depending on its value, which can be \(x\), \(y\), or \(xy\), graphic object *errors* will draw points with horizontal, vertical, or both, error bars. When \(\text{error_type=boxes}\), boxes will be drawn instead of crosses.

See also *errors*.
file_name

Default value: "maxima_out"

This is the name of the file where terminals \texttt{png}, \texttt{jpg}, \texttt{gif}, \texttt{eps}, \texttt{eps\_color}, \texttt{pdf}, \texttt{pdfcairo} and \texttt{svg} will save the graphic.

Since this is a global graphics option, its position in the scene description does not matter. It can be also used as an argument of function \texttt{draw}.

Example:

```latex
(%i1) draw2d(file_name = "myfile",
    explicit(x^2,x,-1,1),
    terminal = 'png)$
```

See also \texttt{terminal}, \texttt{dimensions\_draw}.

fill_color

Default value: "red"

\texttt{fill\_color} specifies the color for filling polygons and 2d \texttt{explicit} functions.

See \texttt{color} to learn how colors are specified.

fill_density

Default value: 0

\texttt{fill\_density} is a number between 0 and 1 that specifies the intensity of the \texttt{fill\_color} in \texttt{bars} objects.

See \texttt{bars} for examples.

filled_func

Default value: false

Option \texttt{filled_func} controls how regions limited by functions should be filled. When \texttt{filled_func} is \texttt{true}, the region bounded by the function defined with object \texttt{explicit} and the bottom of the graphic window is filled with \texttt{fill\_color}. When \texttt{filled_func} contains a function expression, then the region bounded by this function and the function defined with object \texttt{explicit} will be filled. By default, explicit functions are not filled.

A useful special case is \texttt{filled_func=0}, which generates the region bond by the horizontal axis and the explicit function.

This option affects only the 2d graphic object \texttt{explicit}.

Example:

Region bounded by an \texttt{explicit} object and the bottom of the graphic window.

```latex
(%i1) draw2d(fill_color = red,
    filled_func = true,
    explicit(sin(x),x,0,10))$
```
Region bounded by an `explicit` object and the function defined by option `filled_func`. Note that the variable in `filled_func` must be the same as that used in `explicit`.

```
(%i1) draw2d(fill_color = grey,
    filled_func = sin(x),
    explicit(-sin(x),x,0,%pi));
```

See also `fill_color` and `explicit`.

**font**  
Default value: "" (empty string)  
This option can be used to set the font face to be used by the terminal. Only one font face and size can be used throughout the plot.  
Since this is a global graphics option, its position in the scene description does not matter.  
See also `font_size`.  

[Graphic option]
Gnuplot doesn’t handle fonts by itself, it leaves this task to the support libraries of the different terminals, each one with its own philosophy about it. A brief summary follows:

- **x11**: Uses the normal x11 font server mechanism.

  Example:

  ```
  (%i1) draw2d(font = "Arial",
              font_size = 20,
              label(["Arial font, size 20",1,1]));
  ```

- **windows**: The windows terminal doesn’t support changing of fonts from inside the plot. Once the plot has been generated, the font can be changed right-clicking on the menu of the graph window.

- **png, jpeg, gif**: The `libgd` library uses the font path stored in the environment variable `GDFONTPATH`; in this case, it is only necessary to set option `font` to the font’s name. It is also possible to give the complete path to the font file.

  Examples:

  Option `font` can be given the complete path to the font file:

  ```
  (%i1) path: "/usr/share/fonts/truetype/freefont/"$
  (%i2) file: "FreeSerifBoldItalic.ttf"$
  (%i3) draw2d(
              font = concat(path, file),
              font_size = 20,
              color = red,
              label(["FreeSerifBoldItalic font, size 20",1,1]),
              terminal = png)$
  ```

  If environment variable `GDFONTPATH` is set to the path where font files are allocated, it is possible to set graphic option `font` to the name of the font.

  ```
  (%i1) draw2d(
              font = "FreeSerifBoldItalic",
              font_size = 20,
              color = red,
              label(["FreeSerifBoldItalic font, size 20",1,1]),
              terminal = png)$
  ```

- **Postscript**: Standard Postscript fonts are:

  "Helvetica", "Helvetica-Oblique", "Helvetica-Bold",
  and "Courier-BoldOblique".

  Example:

  ```
  (%i1) draw2d(
              font = "Courier-Oblique",
              font_size = 15,
              label(["Courier-Oblique font, size 15",1,1]),
              terminal = eps)$
  ```

- **pdf**: Uses same fonts as Postscript.
• *pdfcairo*: Uses same fonts as *wxt*.
• *wxt*: The *pango* library finds fonts via the *fontconfig* utility.
• *aqua*: Default is "Times-Roman".

The gnuplot documentation is an important source of information about terminals and fonts.

**font_size**

[Graphic option]

Default value: 10

This option can be used to set the font size to be used by the terminal. Only one font face and size can be used throughout the plot. *font_size* is active only when option *font* is not equal to the empty string.

Since this is a global graphics option, its position in the scene description does not matter.

See also *font*.

**gnuplot_file_name**

[Graphic option]

Default value: "maxout_xxx.gnuplot" with "xxx" being a number that is unique to each concurrently-running maxima process.

This is the name of the file with the necessary commands to be processed by Gnuplot.

Since this is a global graphics option, its position in the scene description does not matter. It can be also used as an argument of function *draw*.

Example:

```lisp
(%i1) draw2d(
    file_name = "my_file",
    gnuplot_file_name = "my_commands_for_gnuplot",
    data_file_name = "my_data_for_gnuplot",
    terminal = png,
    explicit(x^2,x,-1,1)) $
```

See also *data_file_name*.

**grid**

[Graphic option]

Default value: *false*

If *grid* is *not* *false*, a grid will be drawn on the *xy* plane. If *grid* is assigned *true*, one grid line per tick of each axis is drawn. If *grid* is assigned a list *nx*,*ny* with *[nx,ny]* > *[0,0]* instead *nx* lines per tick of the *x* axis and *ny* lines per tick of the *y* axis are drawn.

Since this is a global graphics option, its position in the scene description does not matter.

Example:

```lisp
(%i1) draw2d(grid = true,
    explicit(exp(u),u,-2,2))$
```
Chapter 52: draw

(%i1) draw2d(grid = [2,2],
   explicit(sin(x),x,0,2*%pi))$
head_angle = 40,
vector([4,1],[0,6]),
head_angle = 60,
vector([5,1],[0,6]),
head_angle = 90,
vector([6,1],[0,6]),
head_angle = 120,
vector([7,1],[0,6]),
head_angle = 160,
vector([8,1],[0,6]),
head_angle = 180,
vector([9,1],[0,6]) )$

See also head_both, head_length, and head_type.

head_both

[Graphic option]

Default value: false

If head_both is true, vectors are plotted with two arrow heads. If false, only one arrow is plotted.

This option is relevant only for vector objects.

Example:

(%i1) draw2d(xrange = [0,8],
        yrange = [0,8],
        head_length = 0.7,
        vector([1,1],[6,0]),
        head_both = true,
        vector([1,7],[6,0]) )$

See also head_both, head_length, and head_type.
See also `head_length`, `head_angle`, and `head_type`.

**head_length**

[Graphic option]

Default value: 2

`head_length` indicates, in x-axis units, the length of arrow heads.

This option is relevant only for `vector` objects.

Example:

```
(%i1) draw2d(xrange = [0,12],
    yrange = [0,8],
    vector([0,1],[5,5]),
    head_length = 1,
    vector([2,1],[5,5]),
    head_length = 0.5,
    vector([4,1],[5,5]),
    head_length = 0.25,
    vector([6,1],[5,5]) )$
```

See also `head_both`, `head_angle`, and `head_type`. 
head_type

Default value: filled

head_type is used to specify how arrow heads are plotted. Possible values are: filled (closed and filled arrow heads), empty (closed but not filled arrow heads), and nofilled (open arrow heads).

This option is relevant only for vector objects.

Example:

(%i1) draw2d(xrange = [0,12],
               yrange = [0,10],
               head_length = 1,
               vector([0,1],[5,5]), /* default type */
               head_type = 'empty,
               vector([3,1],[5,5]),
               head_type = 'nofilled,
               vector([6,1],[5,5]))$}

See also head_both, head_angle, and head_length.

interpolate_color

Default value: false

This option is relevant only when enhanced3d is not false.

When interpolate_color is false, surfaces are colored with homogeneous quadrangles. When true, color transitions are smoothed by interpolation.

interpolate_color also accepts a list of two numbers, [m,n]. For positive m and n, each quadrangle or triangle is interpolated m times and n times in the respective direction. For negative m and n, the interpolation frequency is chosen so that there will be at least |m| and |n| points drawn; you can consider this as a special gridding function. Zeros, i.e. interpolate_color=[0,0], will automatically choose an optimal number of interpolated surface points.

Also, interpolate_color=true is equivalent to interpolate_color=[0,0].
Examples:

Color interpolation with explicit functions.

(%i1) draw3d(
     enhanced3d = sin(x*y),
     explicit(20*exp(-x^2-y^2)-10, x, -3, 3, y, -3, 3))$

(%i2) draw3d(
     interpolate_color = true,
     enhanced3d = sin(x*y),
     explicit(20*exp(-x^2-y^2)-10, x, -3, 3, y, -3, 3))$

(%i3) draw3d(
     interpolate_color = [-10,0],
     enhanced3d = sin(x*y),
     explicit(20*exp(-x^2-y^2)-10, x, -3, 3, y, -3, 3))$
Interpolating colors in parametric surfaces can give unexpected results.

(%i1) draw3d(
    enhanced3d = true,
    mesh([[1,1,3], [7,3,1], [12,-2,4], [15,0,5]],
          [[2,7,8], [4,3,1], [10,5,8], [12,7,1]],
          [[-2,11,10], [6,9,5], [6,15,1], [20,15,2]])) $
(\%i3) draw3d(
  enhanced3d = true,
  interpolate_color = true,
  view=map,
  mesh([[1,1,3], [7,3,1],[12,-2,4],[15,0,5]],
       [[2,7,8], [4,3,1],[10,5,8], [12,7,1]],
       [[-2,11,10],[6,9,5],[6,15,1], [20,15,2]])$
This option is relevant only for implicit objects.

### key

Default value: "" (empty string)

key is the name of a function in the legend. If key is an empty string, no key is assigned to the function.

This option affects the following graphic objects:

- **gr2d**: points, polygon, rectangle, ellipse, vector, explicit, implicit, parametric and polar.
- **gr3d**: points, explicit, parametric and parametric_surface.

Example:

```maxima
(%i1) draw2d(key = "Sinus",
    explicit(sin(x),x,0,10),
    key = "Cosinus",
    color = red,
    explicit(cos(x),x,0,10) )$
```

### key_pos

Default value: "" (empty string)

key_pos defines at which position the legend will be drawn. If key is an empty string, "top_right" is used. Available position specifiers are: top_left, top_center, top_right, center_left, center, center_right, bottom_left, bottom_center, and bottom_right.

Since this is a global graphics option, its position in the scene description does not matter.

Example:

```maxima
(%i1) draw2d(
    key_pos = top_left,
    key = "x",
```
explicit(x, x,0,10),
color= red,
key = "x squared",
explicit(x^2,x,0,10))$

(%i3) draw3d(
    key_pos = center,
    key = "x",
    explicit(x+y,x,0,10,y,0,10),
    color= red,
    key = "x squared",
    explicit(x^2+y^2,x,0,10,y,0,10))$

**label_alignment**

[Graphic option]

Default value: center

`label_alignment` is used to specify where to write labels with respect to the given coordinates. Possible values are: center, left, and right.

This option is relevant only for `label` objects.

Example:

(%i11) draw2d(xrange = [0,10],
            yrange = [0,10],
            points_joined = true,
            points([[5,0],[5,10]]),
            color = blue,
            label(["Centered alignment (default)",5,2]),
            label_alignment = 'left,
            label(["Left alignment",5,5]),
            label_alignment = 'right,
            label(["Right alignment",5,8]))$
See also `label_orientation`, and `color`

`label_orientation` [Graphic option]

Default value: `horizontal`

`label_orientation` is used to specify orientation of labels. Possible values are: `horizontal`, and `vertical`.

This option is relevant only for `label` objects.

Example:

In this example, a dummy point is added to get an image. Package `draw` needs always data to draw an scene.

```
(%i1) draw2d(xrange = [0,10],
            yrange = [0,10],
            point_size = 0,
            points([[5,5]]),
            color = navy,
            label(["Horizontal orientation (default)",5,2]),
            label_orientation = 'vertical,
            color = "#654321",
            label(["Vertical orientation",1,5]))$
```
See also \texttt{label\_alignment} and \texttt{color}

**line\_type**

[Graphic option]

Default value: \texttt{solid}

\texttt{line\_type} indicates how lines are displayed; possible values are \texttt{solid} and \texttt{dots}, both available in all terminals, and \texttt{dashes}, \texttt{short\_dashes}, \texttt{short\_long\_dashes}, \texttt{short\_short\_long\_dashes}, and \texttt{dot\_dash}, which are not available in \texttt{png}, \texttt{jpg}, and \texttt{gif} terminals.

This option affects the following graphic objects:

- \texttt{gr2d}: \texttt{points}, \texttt{polygon}, \texttt{rectangle}, \texttt{ellipse}, \texttt{vector}, \texttt{explicit}, \texttt{implicit}, \texttt{parametric} and \texttt{polar}.

- \texttt{gr3d}: \texttt{points}, \texttt{explicit}, \texttt{parametric} and \texttt{parametric\_surface}.

Example:

```latex
(\%i1) draw2d(line\_type = dots,
    explicit(1 + x^2,x,-1,1),
    line\_type = solid, /* default */
    explicit(2 + x^2,x,-1,1))$
```
See also line_width.

**line_width**

[Graphic option]

Default value: 1

line_width is the width of plotted lines. Its value must be a positive number.

This option affects the following graphic objects:

- **gr2d**: points, polygon, rectangle, ellipse, vector, explicit, implicit, parametric and polar.
- **gr3d**: points and parametric.

Example:

```maxima
(%i1) draw2d(explicit(x^2,x,-1,1), /* default width */
   line_width = 5.5,
   explicit(1 + x^2,x,-1,1),
   line_width = 10,
   explicit(2 + x^2,x,-1,1))$
```

See also line_type.
logcb

Default value: false

If logcb is true, the tics in the colorbox will be drawn in the logarithmic scale.

When enhanced3d or colorbox is false, option logcb has no effect.

Since this is a global graphics option, its position in the scene description does not matter.

Example:

```plaintext
(%i1) draw3d (enhanced3d = true, color = green, logcb = true, logz = true, palette = [-15,24,-9], explicit(exp(x^2-y^2), x,-2,2,y,-2,2))$
```

See also enhanced3d, colorbox and cbrange.

logx

Default value: false

If logx is true, the x axis will be drawn in the logarithmic scale.

Since this is a global graphics option, its position in the scene description does not matter, with the exception that it should be written before any 2D explicit object, so that draw can produce a better plot.

Example:

```plaintext
(%i1) draw2d(logx = true, explicit(log(x),x,0.01,5))$
```

See also logy, logx_secondary, logy_secondary, and logz.

logx_secondary

Default value: false
If \texttt{logx_secondary} is \texttt{true}, the secondary \( x \) axis will be drawn in the logarithmic scale.

This option is relevant only for 2d scenes.

Since this is a global graphics option, its position in the scene description does not matter.

Example:

\begin{verbatim}
(%i1) draw2d(
    grid = true,
    key="x^2, linear scale",
    color=red,
    explicit(x^2,x,1,100),
    xaxis_secondary = true,
    xtics_secondary = true,
    logx_secondary = true,
    key = "x^2, logarithmic x scale",
    color = blue,
    explicit(x^2,x,1,100) )$
\end{verbatim}

See also \texttt{logx_draw}, \texttt{logy_draw}, \texttt{logy_secondary}, and \texttt{logz}.

\textbf{logy} \hspace{1cm} [Graphic option]

Default value: \texttt{false}

If \texttt{logy} is \texttt{true}, the \( y \) axis will be drawn in the logarithmic scale.

Since this is a global graphics option, its position in the scene description does not matter.

Example:

\begin{verbatim}
(%i1) draw2d(logy = true,
    explicit(exp(x),x,0,5))$
\end{verbatim}

See also \texttt{logx_draw}, \texttt{logx_secondary}, \texttt{logy_secondary}, and \texttt{logz}. 
logy_secondary

Default value: false

If logy_secondary is true, the secondary y axis will be drawn in the logarithmic scale.

This option is relevant only for 2d scenes.

Since this is a global graphics option, its position in the scene description does not matter.

Example:

(%i1) draw2d(
    grid = true,
    key="x^2, linear scale",
    color=red,
    explicit(x^2,x,1,100),
    yaxis_secondary = true,
    ytics_secondary = true,
    logy_secondary = true,
    key = "x^2, logarithmic y scale",
    color = blue,
    explicit(x^2,x,1,100) )$

See also logx_draw, logy_draw, logx_secondary, and logz.

logz

Default value: false

If logz is true, the z axis will be drawn in the logarithmic scale.

Since this is a global graphics option, its position in the scene description does not matter.

Example:

(%i1) draw3d(logz = true,
    explicit(exp(u^2+v^2),u,-2,2,v,-2,2))$

See also logx_draw and logy_draw.

nticks

Default value: 29

In 2d, nticks gives the initial number of points used by the adaptive plotting routine for explicit objects. It is also the number of points that will be shown in parametric and polar curves.

This option affects the following graphic objects:

- gr2d: ellipse, explicit, parametric and polar.
- gr3d: parametric.

See also adapt_depth

Example:

(%i1) draw2d(transparent = true,
    ellipse(0,0,4,2,0,180),
nticks = 5,
ellipse(0,0,4,2,180,180) $

\text{palette} \quad [\text{Graphic option}]

\text{palette}$ indicates how to map gray levels onto color components. It works together
with option \text{enhanced3d} in 3D graphics, who associates every point of a surfaces to a
real number or gray level. It also works with gray images. With \text{palette}, levels are
transformed into colors.

There are two ways for defining these transformations.

First, \text{palette} can be a vector of length three with components ranging from -36 to
+36; each value is an index for a formula mapping the levels onto red, green and blue
colors, respectively:

\begin{align*}
0: & \quad 0 \\
3: & \quad x \\
6: & \quad x^4 \\
9: & \quad \sin(90x) \\
12: & \quad (2x-1)^2 \\
15: & \quad \sin(360x) \\
18: & \quad |\cos(360x)| \\
21: & \quad 3x \\
24: & \quad |3x-1| \\
27: & \quad (3x-2)/2 \\
30: & \quad x/0.32-0.78125 \\
33: & \quad |2x - 0.5| \\
36: & \quad 2x - 1
\end{align*}

negative numbers mean negative colour component. \text{palette = gray} and \text{palette =}
color are short cuts for \text{palette = [3,3,3]} and \text{palette = [7,5,15]}, respectively.

Second, \text{palette} can be a user defined lookup table. In this case, the format for
building a lookup table of length \text{n} is \text{palette=[color \_1, color \_2, \ldots, color \_}
n], where \texttt{color\_i} is a well formed color (see option \texttt{color}) such that \texttt{color\_1} is assigned to the lowest gray level and \texttt{color\_n} to the highest. The rest of colors are interpolated.

Since this is a global graphics option, its position in the scene description does not matter.

Examples:

It works together with option \texttt{enhanced3d} in 3D graphics.

\begin{verbatim}
(%i1) draw3d(
enhanced3d = [z-x+2*y,x,y,z],
palette = [32, -8, 17],
explicit(20*exp(-x^2-y^2)-10,x,-3,3,y,-3,3))$
\end{verbatim}

\begin{figure}
\centering
\includegraphics[width=\textwidth]{image}
\caption{Example of enhanced3d in 3D graphics.}
\end{figure}

It also works with gray images.

\begin{verbatim}
(%i1) im: apply(
    'matrix,
    makelist(makelist(random(200),i,1,30),i,1,30))$
(%i2) /* palette = color, default */
draw2d(image(im,0,0,30,30))$
(%i3) draw2d(palette = gray, image(im,0,0,30,30))$
(%i4) draw2d(palette = [15,20,-4],
    colorbox=false,
    image(im,0,0,30,30))$
\end{verbatim}
palette can be a user defined lookup table. In this example, low values of $x$ are colored in red, and higher values in yellow.

(%i1) draw3d(
    palette = [red, blue, yellow],
    enhanced3d = x,
    explicit(x^2+y^2,x,-1,1,y,-1,1))$

See also colorbox and enhanced3d.

point_size

[Graphic option]

Default value: 1

point_size sets the size for plotted points. It must be a non negative number.

This option has no effect when graphic option point_type is set to dot.

This option affects the following graphic objects:

- gr2d: points.
- gr3d: points.
Example:

```lisp
(%i1) draw2d(points(makelist([[random(20),random(50)],k,1,10]),
    point_size = 5,
    points(makelist(k,k,1,20),makelist(random(30),k,1,20))))$
```

![Graph Example]

**point_type**  
[Graphic option]

Default value: 1

point_type indicates how isolated points are displayed; the value of this option can be any integer index greater or equal than -1, or the name of a point style: 
$none$ (-1), $dot$ (0), $plus$ (1), $multiply$ (2), $asterisk$ (3), $square$ (4), $filled_square$ (5), $circle$ (6), $filled_circle$ (7), $up_triangle$ (8), $filled_up_triangle$ (9), $down_triangle$ (10), $filled_down_triangle$ (11), $diamant$ (12) and $filled_diamant$ (13).

This option affects the following graphic objects:
- **gr2d**: points.
- **gr3d**: points.

Example:

```lisp
(%i1) draw2d(xrange = [0,10],
    yrange = [0,10],
    point_size = 3,
    point_type = diamant,
    points([[1,1],[5,1],[9,1]]),
    point_type = filled_down_triangle,
    points([[1,2],[5,2],[9,2]]),
    point_type = asterisk,
    points([[1,3],[5,3],[9,3]]),
    point_type = filled_diamant,
    points([[1,4],[5,4],[9,4]]),
    point_type = 5,
    points([[1,5],[5,5],[9,5]]),
    point_type = 6,
```
points_joined  [Graphic option]

Default value: false

When points_joined is true, points are joined by lines; when false, isolated points are drawn. A third possible value for this graphic option is impulses; in such case, vertical segments are drawn from points to the x-axis (2D) or to the xy-plane (3D).

This option affects the following graphic objects:

- gr2d: points.
- gr3d: points.

Example:

(%i1) draw2d(xrange = [0,10],
             yrange = [0,4],
             point_size = 3,
             point_type = up_triangle,
             color = blue,
             points([[1,1],[5,1],[9,1]]),
             points_joined = true,
             point_type = square,
             line_type = dots,
             points([[1,2],[5,2],[9,2]]),
             point_type = circle,
             color = red,
             line_width = 7,
proportional_axes

Default value: none

When `proportional_axes` is equal to `xy` or `xyz`, the aspect ratio of the axis units will be set to 1:1 resulting in a 2D or 3D scene that will be drawn with axes proportional to their relative lengths.

Since this is a global graphics option, its position in the scene description does not matter.

This option works with GnuPlot version 4.2.6 or greater.

Examples:

Single 2D plot.

(\%i1) draw2d(
    ellipse(0,0,1,1,0,360),
    transparent=true,
    color = blue,
    line_width = 4,
    ellipse(0,0,2,1/2,0,360),
    proportional_axes = 'xy) $
Multiplot.

(%i1) draw(
   terminal = wxt,
   gr2d(proportional_axes = 'xy,
       explicit(x^2,x,0,1)),
   gr2d(explicit(x^2,x,0,1),
       xrange = [0,1],
       yrange = [0,2],
       proportional_axes='xy),
   gr2d(explicit(x^2,x,0,1)))$

surface_hide

Default value: false

If surface_hide is true, hidden parts are not plotted in 3d surfaces.
Since this is a global graphics option, its position in the scene description does not matter.
Example:
terminal

Default value: screen

Selects the terminal to be used by Gnuplot; possible values are: screen (default), png, pngcairo, jpg, gif, eps, eps_color, epslatex, epslatex_standalone, svg, canvas, dumb, dumb_file, pdf, pdfcairo, wxt, animated_gif, multipage_pdfcairo, multipage_pdf, multipage_eps, multipage_eps_color, and aquaterm.

Terminals screen, wxt and aquaterm can be also defined as a list with two elements: the name of the terminal itself and a non negative integer number. In this form, multiple windows can be opened at the same time, each with its corresponding number. This feature does not work in Windows platforms.

Since this is a global graphics option, its position in the scene description does not matter. It can be also used as an argument of function draw.

N.B. pdfcairo requires Gnuplot 4.3 or newer. pdf requires Gnuplot to be compiled with the option --enable-pdf and libpdf must be installed. The pdf library is available from: http://www.pdflib.com/en/download/pdflib-family/pdflib-lite/

Examples:

(\%i1) /* screen terminal (default) */
   draw2d(explicit(x^2,x,-1,1))$

(\%i2) /* png file */
   draw2d(terminal = 'png,
           explicit(x^2,x,-1,1))$

(\%i3) /* jpg file */
   draw2d(terminal = 'jpg,
$$\text{dimensions} = [300,300],$$

$$\text{explicit}(x^2, x, -1, 1))$$

(\%i4) /* eps file */

draw2d(file_name = "myfile",
       explicit(x^2, x, -1, 1),
       terminal = 'eps)$

(\%i5) /* pdf file */

draw2d(file_name = "mypdf",
       dimensions = 100*[12.0,8.0],
       explicit(x^2, x, -1, 1),
       terminal = 'pdf)$

(\%i6) /* wxwidgets window */

draw2d(explicit(x^2, x, -1, 1),
       terminal = 'wxt)$

Multiple windows.

(\%i1) draw2d(explicit(x^5, x, -2, 2), terminal=[screen, 3])$

(\%i2) draw2d(explicit(x^2, x, -2, 2), terminal=[screen, 0])$

An animated gif file.

(\%i1) draw(
    delay = 100,
    file_name = "zzz",
    terminal = 'animated_gif,
    gr2d(explicit(x^2, x, -1, 1)),
    gr2d(explicit(x^3, x, -1, 1)),
    gr2d(explicit(x^4, x, -1, 1)));

End of animation sequence

(\%o1) [gr2d(explicit), gr2d(explicit), gr2d(explicit)]

Option delay is only active in animated gif's; it is ignored in any other case.

Multipage output in eps format.

(\%i1) draw(
    file_name = "parabol",
    terminal = multipage_eps,
    dimensions = 100*[10,10],
    gr2d(explicit(x^2, x, -1, 1)),
    gr3d(explicit(x^2+y^2, x, -1,1, y, -1,1)))$

See also file_name, dimensions_draw and delay.

title [Graphic option]

Default value: "" (empty string)

Option title, a string, is the main title for the scene. By default, no title is written.

Since this is a global graphics option, its position in the scene description does not matter.

Example:

(\%i1) draw2d(explicit(exp(u),u,-2,2),
transform

Default value: none

If `transform` is none, the space is not transformed and graphic objects are drawn as defined. When a space transformation is desired, a list must be assigned to option `transform`. In case of a 2D scene, the list takes the form \([f_1(x,y), f_2(x,y), x, y]\). In case of a 3D scene, the list is of the form \([f_1(x,y,z), f_2(x,y,z), f_3(x,y,z), x, y, z]\).

The names of the variables defined in the lists may be different to those used in the definitions of the graphic objects.

Examples:

Rotation in 2D.

\[(%i1) \; \text{th} : \pi / 4\]
\[(%i2) \; \text{draw2d}(\]
  color = "#e245f0",
  proportional_axes = 'xy,
  line_width = 8,
  triangle([3,2],[7,2],[5,5]),
  border = false,
  fill_color = yellow,
  transform = [cos(th)*x - sin(th)*y,
               sin(th)*x + cos(th)*y, x, y],
  triangle([3,2],[7,2],[5,5]) \)\]
Translation in 3D.

\[
\text{(\%i1) draw3d(}
\quad \text{color} = "\#a02c00",
\quad \text{explicit}(20*\exp(-x^2-y^2)-10,x,-3,3,y,-3,3),
\quad \text{transform} = \{x+10,y+10,z+10,x,y,z\},
\quad \text{color} = \text{blue},
\quad \text{explicit}(20*\exp(-x^2-y^2)-10,x,-3,3,y,-3,3) )$
\]

\begin{verbatim}
transparent [Graphic option]
Default value: \text{false}

If \texttt{transparent} is \texttt{false}, interior regions of polygons are filled according to \texttt{fill_color}.

This option affects the following graphic objects:

- \texttt{gr2d}: \texttt{polygon}, \texttt{rectangle} and \texttt{ellipse}.

Example:

\[
\text{(\%i1) draw2d(polygon([[3,2],[7,2],[5,5]]),}
\quad \text{transparent} = \text{true},
\quad \text{color} = \text{blue},
\quad \text{polygon}([[5,2],[9,2],[7,5]] )}$
\end{verbatim}
unit_vectors

[Graphic option]
Default value: false
If unit_vectors is true, vectors are plotted with module 1. This is useful for plotting vector fields. If unit_vectors is false, vectors are plotted with its original length.
This option is relevant only for vector objects.
Example:
(\%i1) draw2d(xrange = [-1,6],
yrange = [-1,6],
head_length = 0.1,
vector([0,0],[5,2]),
unit_vectors = true,
color = red,
vector([0,3],[5,2]))$

user_preamble

[Graphic option]
Default value: "" (empty string)
Expert Gnuplot users can make use of this option to fine tune Gnuplot’s behaviour by writing settings to be sent before the `plot` or `splot` command.

The value of this option must be a string or a list of strings (one per line).

Since this is a global graphics option, its position in the scene description does not matter.

Example:

Tell Gnuplot to draw axes and grid on top of graphics objects,

```
(%i1) draw2d(
   xaxis = true, xaxis_type = solid,
   yaxis = true, yaxis_type = solid,
   user_preamble = "set grid front",
   region(x^2+y^2<1 ,x,-1.5,1.5,y,-1.5,1.5))$
```

![Graph with axes and grid](image)

Tell gnuplot to draw all contour lines in black

```
(%i1) wxdraw3d(
   contour=both,
   surface_hide=true,enhanced3d=true,wired_surface=true,
   contour_levels=10,
   user_preamble="set for [i=1:8] linetype i dashtype i linecolor 0",
   explicit(sin(x)*cos(y),x,1,10,y,1,10)
);
```
view

Default value: [60, 30]

A pair of angles, measured in degrees, indicating the view direction in a 3D scene. The first angle is the vertical rotation around the x axis, in the range [0, 360]. The second one is the horizontal rotation around the z axis, in the range [0, 360].

If option view is given the value map, the view direction is set to be perpendicular to the xy-plane.

Since this is a global graphics option, its position in the scene description does not matter.

Example:

(%i1) draw3d(view = [170, 50],
       enhanced3d = true,
       explicit(sin(x^2+y^2),x,-2,2,y,-2,2) )$

(%i2) draw3d(view = map,
       enhanced3d = true,
explicit(sin(x^2+y^2),x,-2,2,y,-2,2)$

```
(%i1) draw3d(
    enhanced3d  = [sin(x),x,y],
    wired_surface = true,
    explicit(x^2+y^2,x,-1,1,y,-1,1)) $
```

### wired_surface

[Graphic option]

Default value: `false`

Indicates whether 3D surfaces in `enhanced3d` mode show the grid joining the points or not.

Since this is a global graphics option, its position in the scene description does not matter.

Example:

```
(%i1) draw3d(
    enhanced3d  = [sin(x),x,y],
    wired_surface = true,
    explicit(x^2+y^2,x,-1,1,y,-1,1)) $
```

### x_voxel

[Graphic option]

Default value: 10
x_voxel is the number of voxels in the x direction to be used by the marching cubes algorithm implemented by the 3d implicit object. It is also used by graphic object region.

xaxis

[Graphic option]
Default value: false
If xaxis is true, the x axis is drawn.
Since this is a global graphics option, its position in the scene description does not matter.
Example:

```
(%i1) draw2d(explicit(x^3,x,-1,1),
    xaxis = true,
    xaxis_color = blue)$
```

See also xaxis_width, xaxis_type and xaxis_color.

xaxis_color

[Graphic option]
Default value: "black"
xaxis_color specifies the color for the x axis. See color to know how colors are defined.
Since this is a global graphics option, its position in the scene description does not matter.
Example:

```
(%i1) draw2d(explicit(x^3,x,-1,1),
    xaxis = true,
    xaxis_color = red)$
```

See also xaxis, xaxis_width and xaxis_type.

xaxis_secondary

[Graphic option]
Default value: false
If `xaxis_secondary` is `true`, function values can be plotted with respect to the second x axis, which will be drawn on top of the scene.

Note that this is a local graphics option which only affects to 2d plots.

Example:

```maxima
(%i1) draw2d(
    key = "Bottom x-axis",
    explicit(x+1,x,1,2),
    color = red,
    key = "Above x-axis",
    xtics_secondary = true,
    xaxis_secondary = true,
    explicit(x^2,x,-1,1)) $
```

See also `xrange_secondary`, `xtics_secondary`, `xtics_rotate_secondary`, `xtics_axis_secondary` and `xaxis_secondary`.

**xaxis_type**

[Graphic option]

Default value: `dots`

`xaxis_type` indicates how the x axis is displayed; possible values are `solid` and `dots`

Since this is a global graphics option, its position in the scene description does not matter.

Example:

```maxima
(%i1) draw2d(explicit(x^3,x,-1,1),
    xaxis = true,
    xaxis_type = solid) $
```

See also `xaxis`, `xaxis_width` and `xaxis_color`.

**xaxis_width**

[Graphic option]

Default value: 1

`xaxis_width` is the width of the x axis. Its value must be a positive number.
Since this is a global graphics option, its position in the scene description does not matter.

Example:

```
(%i1) draw2d(explicit(x^3,x,-1,1),
    xaxis = true,
    xaxis_width = 3)$
```

See also `xaxis`, `axis_type` and `axis_color`.

**xlabel**

[Graphic option]

Default value: ""

Option `xlabel`, a string, is the label for the x axis. By default, the axis is labeled with string "x".

Since this is a global graphics option, its position in the scene description does not matter.

Example:

```
(%i1) draw2d(xlabel = "Time",
    explicit(exp(u),u,-2,2),
    ylabel = "Population")$
```

See also `xlabel_secondary`, `ylabel`, `ylabel_secondary` and `zlabel_draw`.

**xlabel_secondary**

[Graphic option]

Default value: "" (empty string)

Option `xlabel_secondary`, a string, is the label for the secondary x axis. By default, no label is written.

Since this is a global graphics option, its position in the scene description does not matter.

Example:

```
(%i1) draw2d(
    xaxis_secondary=true,yaxis_secondary=true,
    xtics_secondary=true,ytic_secondary=true,
    xlabel_secondary="t[\text{[s]}",
    ylabel_secondary="U[V]",
    explicit(sin(t),t,0,10) )$
```
See also xlabel_draw, ylabel_draw, ylabel_secondary and zlabel_draw.

**xrange**

[Graphic option]

Default value: auto

If `xrange` is auto, the range for the x coordinate is computed automatically.

If the user wants a specific interval for x, it must be given as a Maxima list, as in `xrange=[-2, 3]`.

Since this is a global graphics option, its position in the scene description does not matter.

Example:

```
(%i1) draw2d(xrange = [-3,5],
    explicit(x^2,x,-1,1))$
```

See also `yrange` and `zrange`.

**xrange_secondary**

[Graphic option]

Default value: auto

If `xrange_secondary` is auto, the range for the second x axis is computed automatically.

If the user wants a specific interval for the second x axis, it must be given as a Maxima list, as in `xrange_secondary=[-2, 3]`.

Since this is a global graphics option, its position in the scene description does not matter.

See also `xrange`, `yrange`, `zrange` and `yrange_secondary`.

**xtics**

[Graphic option]

Default value: true

This graphic option controls the way tic marks are drawn on the x axis.

- When option `xtics` is bounded to symbol true, tic marks are drawn automatically.
- When option `xtics` is bounded to symbol false, tic marks are not drawn.
• When option \texttt{xtics} is bounded to a positive number, this is the distance between two consecutive tic marks.
• When option \texttt{xtics} is bounded to a list of length three of the form \texttt{[start,incr,end]}, tic marks are plotted from \texttt{start} to \texttt{end} at intervals of length \texttt{incr}.
• When option \texttt{xtics} is bounded to a set of numbers of the form \{n1, n2, ...\}, tic marks are plotted at values n1, n2, ...
• When option \texttt{xtics} is bounded to a set of pairs of the form \{"label1", n1\}, \{"label2", n2\}, ..., tic marks corresponding to values n1, n2, ... are labeled with "label1", "label2", ..., respectively.

Since this is a global graphics option, its position in the scene description does not matter.

Examples:

Disable tics.

(\%i1) \texttt{draw2d(xtics = 'false,}
\texttt{    explicit(x^3,x,-1,1) )}$

Tics every 1/4 units.

(\%i1) \texttt{draw2d(xtics = 1/4,}
\texttt{    explicit(x^3,x,-1,1) )}$

Tics from -3/4 to 3/4 in steps of 1/8.

(\%i1) \texttt{draw2d(xtics = [-3/4,1/8,3/4],}
\texttt{    explicit(x^3,x,-1,1) )}$

Tics at points -1/2, -1/4 and 3/4.

(\%i1) \texttt{draw2d(xtics = {-1/2,-1/4,3/4},}
\texttt{    explicit(x^3,x,-1,1) )}$

Labeled tics.

(\%i1) \texttt{draw2d(xtics = {"High",0.75,"Medium",0,"Low",-0.75},}
\texttt{    explicit(x^3,x,-1,1) )}$

See also \texttt{ytics}, and \texttt{ztics}.

\texttt{xtics_axis} [Graphic option]

Default value: false

If \texttt{xtics_axis} is true, tic marks and their labels are plotted just along the x axis, if it is false tics are plotted on the border.

Since this is a global graphics option, its position in the scene description does not matter.

\texttt{xtics_rotate} [Graphic option]

Default value: false

If \texttt{xtics_rotate} is true, tic marks on the x axis are rotated 90 degrees.

Since this is a global graphics option, its position in the scene description does not matter.
**xtics_rotate_secondary**

[Graphic option]

Default value: false

If `xtics_rotate_secondary` is true, tic marks on the secondary x axis are rotated 90 degrees.

Since this is a global graphics option, its position in the scene description does not matter.

**xtics_secondary**

[Graphic option]

Default value: auto

This graphic option controls the way tic marks are drawn on the second x axis.

See `xtics` for a complete description.

**xtics_secondary_axis**

[Graphic option]

Default value: false

If `xtics_secondary_axis` is true, tic marks and their labels are plotted just along the secondary x axis, if it is false tics are plotted on the border.

Since this is a global graphics option, its position in the scene description does not matter.

**xu_grid**

[Graphic option]

Default value: 30

`xu_grid` is the number of coordinates of the first variable (x in explicit and u in parametric 3d surfaces) to build the grid of sample points.

This option affects the following graphic objects:

- `gr3d`: explicit and `parametric_surface`.

Example:

```lisp
(%i1) draw3d(xu_grid = 10,
            yv_grid = 50,
            explicit(x^2+y^2,x,-3,3,y,-3,3))$
```

See also `yv_grid`.

**xy_file**

[Graphic option]

Default value: "" (empty string)

`xy_file` is the name of the file where the coordinates will be saved after clicking with the mouse button and hitting the 'x' key. By default, no coordinates are saved.

Since this is a global graphics option, its position in the scene description does not matter.

**xyplane**

[Graphic option]

Default value: false

Allocates the xy-plane in 3D scenes. When `xyplane` is false, the xy-plane is placed automatically; when it is a real number, the xy-plane intersects the z-axis at this level. This option has no effect in 2D scenes.

Since this is a global graphics option, its position in the scene description does not matter.
Example:

```lisp
(%i1) draw3d(xyplane = %e-2,
   explicit(x^2+y^2,x,-1,1,y,-1,1))$
```

**y_voxel**  
Default value: 10

y_voxel is the number of voxels in the y direction to be used by the *marching cubes algorithm* implemented by the 3d implicit object. It is also used by graphic object region.

**yaxis**  
Default value: false

If yaxis is true, the y axis is drawn.

Since this is a global graphics option, its position in the scene description does not matter.

Example:

```lisp
(%i1) draw2d(explicit(x^3,x,-1,1),
   yaxis = true,
   yaxis_color = blue)$
```

See also yaxis_width, yaxis_type and yaxis_color.

**yaxis_color**  
Default value: "black"

yaxis_color specifies the color for the y axis. See color to know how colors are defined.

Since this is a global graphics option, its position in the scene description does not matter.

Example:

```lisp
(%i1) draw2d(explicit(x^3,x,-1,1),
   yaxis = true,
   yaxis_color = red)$
```

See also yaxis, yaxis_width and yaxis_type.

**yaxis_secondary**  
Default value: false

If yaxis_secondary is true, function values can be plotted with respect to the second y axis, which will be drawn on the right side of the scene.

Note that this is a local graphics option which only affects to 2d plots.

Example:

```lisp
(%i1) draw2d(
   explicit(sin(x),x,0,10),
   yaxis_secondary = true,
   ytics_secondary = true,
   color = blue,
   explicit(100*sin(x+0.1)+2,x,0,10));
```
See also `yrange_secondary`, `ytics_secondary`, `ytics_rotate_secondary` and 
ytics_axis_secondary

**yaxis_type**  
[Graphic option]  
Default value: `dots`  
`yaxis_type` indicates how the y axis is displayed; possible values are `solid` and `dots`. Since this is a global graphics option, its position in the scene description does not matter.  
Example:  
```lisp
(%i1) draw2d(explicit(x^3,x,-1,1),
    yaxis = true,
    yaxis_type = solid)$
```

See also `yaxis`, `yaxis_width` and `yaxis_color`.

**yaxis_width**  
[Graphic option]  
Default value: 1  
`yaxis_width` is the width of the y axis. Its value must be a positive number. Since this is a global graphics option, its position in the scene description does not matter.  
Example:  
```lisp
(%i1) draw2d(explicit(x^3,x,-1,1),
    yaxis = true,
    yaxis_width = 3)$
```

See also `yaxis`, `yaxis_type` and `yaxis_color`.

**ylabel**  
[Graphic option]  
Default value: ""  
Option `ylabel`, a string, is the label for the y axis. By default, the axis is labeled with string "y". Since this is a global graphics option, its position in the scene description does not matter.  
Example:  
```lisp
(%i1) draw2d(xlabel = "Time",
    ylabel = "Population",
    explicit(exp(u),u,-2,2) )$
```

See also `xlabel_draw`, `xlabel_secondary`, `ylabel_secondary`, and `zlabel_draw`.

**ylabel_secondary**  
[Graphic option]  
Default value: "" (empty string)  
Option `ylabel_secondary`, a string, is the label for the secondary y axis. By default, no label is written. Since this is a global graphics option, its position in the scene description does not matter.
Example:

```
(%i1) draw2d(
    key_pos=bottom_right,
    key="current",
    xlabel="t[s]",
    ylabel="I[A]",ylabel_secondary="P[W]",
    explicit(sin(t),t,0,10),
    yaxis_secondary=true,
    ytics_secondary=true,
    color=red,key="Power",
    explicit((sin(t))^2,t,0,10)
)
```

See also `xlabel_draw`, `xlabel_secondary`, `ylabel_draw` and `zlabel_draw`.

**yrange**

[Graphic option]

Default value: `auto`

If `yrange` is `auto`, the range for the y coordinate is computed automatically.

If the user wants a specific interval for y, it must be given as a Maxima list, as in `yrange=[-2, 3]`.

Since this is a global graphics option, its position in the scene description does not matter.

Example:

```
(%i1) draw2d(yrange = [-2,3],
    explicit(x^2,x,-1,1),
    xrange = [-3,3])$
```

See also `xrange`, `yrange_secondary` and `zrange`.

**yrange_secondary**

[Graphic option]

Default value: `auto`

If `yrange_secondary` is `auto`, the range for the second y axis is computed automatically.

If the user wants a specific interval for the second y axis, it must be given as a Maxima list, as in `yrange_secondary=[-2, 3]`.

Since this is a global graphics option, its position in the scene description does not matter.

Example:

```
(%i1) draw2d(
    explicit(sin(x),x,0,10),
    yaxis_secondary = true,
    ytics_secondary = true,
    yrange = [-3, 3],
    yrange_secondary = [-20, 20],
    color = blue,
    explicit(100*sin(x+0.1)+2,x,0,10)) $
```

See also `xrange`, `yrange` and `zrange`.

ytics

[Graphic option]

Default value: true

This graphic option controls the way tic marks are drawn on the y axis.

See xtics for a complete description.

ytics_axis

[Graphic option]

Default value: false

If ytics_axis is true, tic marks and their labels are plotted just along the y axis, if it is false tics are plotted on the border.

Since this is a global graphics option, its position in the scene description does not matter.

ytics_rotate

[Graphic option]

Default value: false

If ytics_rotate is true, tic marks on the y axis are rotated 90 degrees.

Since this is a global graphics option, its position in the scene description does not matter.

ytics_rotate_secondary

[Graphic option]

Default value: false

If ytics_rotate_secondary is true, tic marks on the secondary y axis are rotated 90 degrees.

Since this is a global graphics option, its position in the scene description does not matter.

ytics_secondary

[Graphic option]

Default value: auto

This graphic option controls the way tic marks are drawn on the second y axis.

See xtics for a complete description.

ytics_secondary_axis

[Graphic option]

Default value: false

If ytics_secondary_axis is true, tic marks and their labels are plotted just along the secondary y axis, if it is false tics are plotted on the border.

Since this is a global graphics option, its position in the scene description does not matter.

yv_grid

[Graphic option]

Default value: 30

yv_grid is the number of coordinates of the second variable (y in explicit and v in parametric 3d surfaces) to build the grid of sample points.

This option affects the following graphic objects:

- gr3d: explicit and parametric_surface.
Example:

```lisp
(%i1) draw3d(xu_grid = 10,
    yv_grid = 50,
    explicit(x^2+y^2,x,-3,3,y,-3,3) )$
```

See also `xu_grid`.

**z_voxel**

[Graphic option]

Default value: 10

`z_voxel` is the number of voxels in the z direction to be used by the *marching cubes algorithm* implemented by the 3d implicit object.

**zaxis**

[Graphic option]

Default value: `false`

If `zaxis` is `true`, the z axis is drawn in 3D plots. This option has no effect in 2D scenes.

Since this is a global graphics option, its position in the scene description does not matter.

Example:

```lisp
(%i1) draw3d(explicit(x^2+y^2,x,-1,1,y,-1,1),
    zaxis = true,
    zaxis_type = solid,
    zaxis_color = blue)$
```

See also `zaxis_width`, `zaxis_type` and `zaxis_color`.

**zaxis_color**

[Graphic option]

Default value: "black"

`zaxis_color` specifies the color for the z axis. See `color` to know how colors are defined. This option has no effect in 2D scenes.

Since this is a global graphics option, its position in the scene description does not matter.
Example:

(%i1) draw3d(explicit(x^2+y^2,x,-1,1,y,-1,1),
    zaxis     = true,
    zaxis_type = solid,
    zaxis_color = red)$

See also `zaxis`, `zaxis_width` and `zaxis_type`.

**zaxis_type**

[Graphic option]

Default value: `dots`

`zaxis_type` indicates how the z axis is displayed; possible values are `solid` and `dots`. This option has no effect in 2D scenes.

Since this is a global graphics option, its position in the scene description does not matter.

Example:

(%i1) draw3d(explicit(x^2+y^2,x,-1,1,y,-1,1),
    zaxis     = true,
    zaxis_type = solid)$

See also `zaxis`, `zaxis_width` and `zaxis_color`.

**zaxis_width**

[Graphic option]

Default value: 1

`zaxis_width` is the width of the z axis. Its value must be a positive number. This option has no effect in 2D scenes.

Since this is a global graphics option, its position in the scene description does not matter.

Example:

(%i1) draw3d(explicit(x^2+y^2,x,-1,1,y,-1,1),
    zaxis     = true,
    zaxis_type = solid,
    zaxis_width = 3)$

See also `zaxis`, `zaxis_type` and `zaxis_color`.

**zlabel**

[Graphic option]

Default value: `""`

Option `zlabel`, a string, is the label for the z axis. By default, the axis is labeled with string "z".

Since this is a global graphics option, its position in the scene description does not matter.

Example:

(%i1) draw3d(zlabel = "Z variable",
    ylabel = "Y variable",
    explicit(sin(x^2+y^2),x,-2,2,y,-2,2),
    xlabel = "X variable")$

See also `xlabel_draw`, `ylabel_draw`, and `zlabel_rotate`. 
zlabel_rotate  [Graphic option]
Default value: "auto"
This graphics option allows to choose if the z axis label of 3d plots is drawn horizontal (false), vertical (true) or if maxima automatically chooses an orientation based on the length of the label (auto).
Since this is a global graphics option, its position in the scene description does not matter.
Example:
(%i1) draw3d(
   explicit(sin(x)*sin(y),x,0,10,y,0,10),
   zlabel_rotate=false
)
See also zlabel_draw.
zrange  [Graphic option]
Default value: auto
If zrange is auto, the range for the z coordinate is computed automatically.
If the user wants a specific interval for z, it must be given as a Maxima list, as in zrange=[-2, 3].
Since this is a global graphics option, its position in the scene description does not matter.
Example:
(%i1) draw3d(yrange = [-3,3],
   zrange = [-2,5],
   explicit(x^2+y^2,x,-1,1,y,-1,1),
   xrange = [-3,3])$
See also xrange and yrange.
ztics  [Graphic option]
Default value: true
This graphic option controls the way tic marks are drawn on the z axis.
See xtics for a complete description.
ztics_axis  [Graphic option]
Default value: false
If ztics_axis is true, tic marks and their labels are plotted just along the z axis, if it is false tics are plotted on the border.
Since this is a global graphics option, its position in the scene description does not matter.
ztics_rotate  [Graphic option]
Default value: false
If ztics_rotate is true, tic marks on the z axis are rotated 90 degrees.
Since this is a global graphics option, its position in the scene description does not matter.
52.2.4 Graphics objects

bars ([x1,h1,w1], [x2,h2,w2, ...])  
Draws vertical bars in 2D.

2D
bars ([x1,h1,w1], [x2,h2,w2, ...]) draws bars centered at values x1, x2, ... with heights h1, h2, ... and widths w1, w2, ...

This object is affected by the following graphic options: key, fill_color, fill_density and line_width.

Example:

(%i1) draw2d(
   key = "Group A",
   fill_color = blue,
   fill_density = 0.2,
   bars([0.8,5,0.4],[1.8,7,0.4],[2.8,-4,0.4]),
   key = "Group B",
   fill_color = red,
   fill_density = 0.6,
   line_width = 4,
   bars([1.2,4,0.4],[2.2,-2,0.4],[3.2,5,0.4]),
   xaxis = true);

---

cylindrical (radius, z, minz, maxz, azi, minazi, maxazi)  
Draws 3D functions defined in cylindrical coordinates.

3D
cylindrical(radius, z, minz, maxz, azi, minazi, maxazi) plots the function radius(z, azi) defined in cylindrical coordinates, with variable z taking values from minz to maxz and azimuth azi taking values from minazi to maxazi.

This object is affected by the following graphic options: xu_grid, yv_grid, line_type, key, wired_surface, enhanced3d and color
Example:

(%i1) draw3d(cylindrical(1,z,-2,2,az,0,2*%pi))$

The `draw3d` function is used to draw a 3D cylindrical object defined by `cylindrical(1,z,-2,2,az,0,2*%pi)`.

**elevation_grid (mat,x0,y0,width,height)**

[Graphic object]

Draws matrix `mat` in 3D space. `z` values are taken from `mat`, the abscissas range from `x0` to `x0 + width` and ordinates from `y0` to `y0 + height`. Element `a(1,1)` is projected on point `(x0,y0 + height)`, `a(1,n)` on `(x0 + width,y0 + height)`, `a(m,1)` on `(x0,y0)`, and `a(m,n)` on `(x0 + width,y0)`.

This object is affected by the following graphic options: `line_type`, `line_width`, `key`, `wired_surface`, `enhanced3d` and `color`.

In older versions of Maxima, `elevation_grid` was called `mesh`. See also `mesh`.

Example:

(%i1) m: apply(
    matrix,
    makelist(makelist(random(10.0),k,1,30),i,1,20))$

(%i2) draw3d(
    color = blue,
    elevation_grid(m,0,0,3,2),
    xlabel = "x",
    ylabel = "y",
    surface_hide = true);
ellipse \((xc, yc, a, b, \text{ang1}, \text{ang2})\)  

Draws ellipses and circles in 2D.

**2D**

The `ellipse` function plots an ellipse centered at \((xc, yc)\) with horizontal and vertical semi axis \(a\) and \(b\), respectively, starting at angle \(\text{ang1}\) with an amplitude equal to angle \(\text{ang2}\).

This object is affected by the following graphic options: `nticks`, `transparent`, `fill_color`, `border`, `line_width`, `line_type`, `key`, and `color`.

Example:

```maxima
(%i1) draw2d(transparent = false,
fill_color = red,
color = gray30,
transparent = false,
line_width = 5,
ellipse(0,6,3,2,270,-270),
/* center (x,y), a, b, start & end in degrees */
transparent = true,
color = blue,
line_width = 3,
ellipse(2.5,6,2,3,30,-90),
xrange = [-3,6],
yrange = [2,9] )$
```
The `errors` function draws points with error bars, horizontally, vertically or both, depending on the value of the `error_type` option.

**2D**

If `error_type = x`, arguments to `errors` must be of the form `[x, y, xdelta]` or `[x, y, xlow, xhigh]`. If `error_type = y`, arguments must be of the form `[x, y, ydelta]` or `[x, y, ylow, yhigh]`. If `error_type = xy` or `error_type = boxes`, arguments to `errors` must be of the form `[x, y, xdelta, ydelta]` or `[x, y, xlow, xhigh, ylow, yhigh]`.

See also `error_type`.

This object is affected by the following graphic options: `error_type`, `pointsJoined`, `line_width`, `key`, `line_type`, `color`, `fill_density`, `xaxis_secondary`, and `yaxis_secondary`.

Option `fill_density` is only relevant when `error_type=boxes`.

Examples:

Horizontal error bars.

```lisp
(%i1) draw2d(
    error_type = y,
    errors([[1,2,1], [3,5,3], [10,3,1], [17,6,2]]))
```


Vertical and horizontal error bars.

(%i1) draw2d(
    error_type = 'xy,
    points_joined = true,
    color = blue,
    errors([[1,2,1,2], [3,5,2,1], [10,3,1,1], [17,6,1/2,2]]);

explicit

\texttt{explicit(fcn, var, minval, maxval)}
\texttt{explicit(fcn, var1, minval1, maxval1, var2, minval2, maxval2)}

Draws explicit functions in 2D and 3D.

2D

\texttt{explicit(fcn, var, minval, maxval)} plots explicit function \textit{fcn}, with variable \textit{var} taking values from \textit{minval} to \textit{maxval}.

This object is affected by the following graphic options: \texttt{nticks, adapt_depth, draw_realpart, line_width, line_type, key, filled_func, fill_color} and color
Example:

(%i1) draw2d(line_width = 3,
    color    = blue,
    explicit(x^2, x, -3, 3) )$

(%i2) draw2d(fill_color = brown,
    filled_func = true,
    explicit(x^2, x, -3, 3) )$

3D

explicit(fcn, var1, minval1, maxval1, var2, minval2, maxval2) plots the explicit function fcn, with variable var1 taking values from minval1 to maxval1 and variable var2 taking values from minval2 to maxval2.

This object is affected by the following graphic options: draw_realpart, xu_grid, yv_grid, line_type, line_width, key, wired_surface, enhanced3d and color.

Example:
(%i1) draw3d(key = "Gauss",
        color = "#a02c00",
        explicit(20*exp(-x^2-y^2)-10,x,-3,3,y,-3,3),
        yv_grid = 10,
        color = blue,
        key = "Plane",
        explicit(x+y,x,-5,5,y,-5,5),
        surface_hide = true)$

See also `filled_func` for filled functions.

**image (im,x0,y0,width,height)**

Renders images in 2D.

2D

`image (im,x0,y0,width,height)` plots image `im` in the rectangular region from vertex `(x0,y0)` to `(x0+width,y0+height)` on the real plane. Argument `im` must be a matrix of real numbers, a matrix of vectors of length three or a `picture` object.

If `im` is a matrix of real numbers or a `levels picture` object, pixel values are interpreted according to graphic option `palette`, which is a vector of length three with components ranging from -36 to +36; each value is an index for a formula mapping the levels onto red, green and blue colors, respectively:

<table>
<thead>
<tr>
<th></th>
<th>Formula</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>0.5</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>x</td>
</tr>
<tr>
<td>4</td>
<td>x^2</td>
</tr>
<tr>
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<td>x^3</td>
</tr>
<tr>
<td>6</td>
<td>x^4</td>
</tr>
<tr>
<td>7</td>
<td>sqrt(x)</td>
</tr>
<tr>
<td>8</td>
<td>sqrt(sqrt(x))</td>
</tr>
<tr>
<td>9</td>
<td>sin(90x)</td>
</tr>
<tr>
<td>10</td>
<td>cos(90x)</td>
</tr>
<tr>
<td>11</td>
<td></td>
</tr>
<tr>
<td>12</td>
<td>(2x-1)^2</td>
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<td>13</td>
<td>sin(180x)</td>
</tr>
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<td></td>
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<td>sin(360x)</td>
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<tr>
<td>16</td>
<td>cos(360x)</td>
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<td></td>
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<tr>
<td>19</td>
<td>sin(720x)</td>
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<tr>
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<td></td>
</tr>
<tr>
<td>21</td>
<td>3x</td>
</tr>
<tr>
<td>22</td>
<td>3x-1</td>
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<td>23</td>
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<td>(3x-1)/2</td>
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<td>27</td>
<td>(3x-2)/2</td>
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<tr>
<td>28</td>
<td></td>
</tr>
<tr>
<td>29</td>
<td></td>
</tr>
</tbody>
</table>
30: $x/0.32-0.78125$
31: $2x-0.84$
32: $4x;1;-2x+1.84;x/0.08-11.5$
33: $|2x - 0.5|$
34: $2x$
35: $2x - 0.5$
36: $2x - 1$

Negative numbers mean negative color component.

palette = gray and palette = color are short cuts for palette = [3,3,3] and palette = [7,5,15], respectively.

If \texttt{im} is a matrix of vectors of length three or an rgb picture object, they are interpreted as red, green and blue color components.

Examples:

If \texttt{im} is a matrix of real numbers, pixel values are interpreted according to graphic option palette.

\begin{verbatim}
(%i1) im: apply(
    'matrix,
    makelist(makelist(random(200),i,1,30),i,1,30))$
(%i2) /* palette = color, default */
    draw2d(image(im,0,0,30,30))$
\end{verbatim}

\begin{verbatim}
(%i3) draw2d(palette = gray, image(im,0,0,30,30))$
\end{verbatim}
(%i4) draw2d(palette = [15,20,-4],
            colorbox=false,
            image(im,0,0,30,30))$

See also colorbox.

If \( im \) is a matrix of vectors of length three, they are interpreted as red, green and blue color components.

(%i1) im: apply(
    'matrix,
    makelist(
        makelist([[random(300),
            random(300),
            random(300)],i,1,30],i,1,30))$

(%i2) draw2d(image(im,0,0,30,30))$
Package `draw` automatically loads package `picture`. In this example, a level picture object is built by hand and then rendered.

```lisp
(%i1) im: make_level_picture([45,87,2,134,204,16],3,2);
(%o1) picture(level, 3, 2, {Array: #(45 87 2 134 204 16)})
(%i2) /* default color palette */
   draw2d(image(im,0,0,30,30))$
```

```lisp
(%i3) /* gray palette */
   draw2d(palette = gray,
         image(im,0,0,30,30))$
```
An xpm file is read and then rendered.

```
(%i1) load(draw)$
(%i2) im: read_xpm("myfile.xpm")$
(%i3) draw2d(image(im,0,0,10,7))$
```

See also `make_level_picture`, `make_rgb_picture` and `read_xpm`.

http://www.telefonica.net/web2/biomates/maxima/gpdraw/image
contains more elaborated examples.

**implicit**

```
implicit (fcn,x,xmin,xmax,y,ymin,ymax)
implicit (fcn,x,xmin,xmax,y,ymin,ymax,z,zmin,zmax)
```

Draws implicit functions in 2D and 3D.

**2D**

```
implicit(fcn,x,xmin,xmax,y,ymin,ymax) plots the implicit function defined by fcn, with variable x taking values from xmin to xmax, and variable y taking values from ymin to ymax.
```

This object is affected by the following graphic options: `ip_grid`, `ip_grid_in`, `line_width`, `line_type`, `key` and `color`.

Example:

```
(%i1) draw2d(grid = true,
       line_type = solid,
       key = "y^2=x^3-2*x+1",
       implicit(y^2=x^3-2*x+1, x, -4,4, y, -4,4),
       line_type = dots,
       key = "x^3+y^3 = 3*x*y^2-x-1",
       implicit(x^3+y^3 = 3*x*y^2-x-1, x,-4,4, y,-4,4),
       title = "Two implicit functions"
)
```
Two implicit functions

\[ y^2 = x^3 - 2x + 1 \]
\[ x^3 + y^3 = 3xy^2 - x - 1 \]

3D

`implicit (fcn, x, xmin, xmax, y, ymin, ymax, z, zmin, zmax)` plots the implicit surface defined by `fcn`, with variable `x` taking values from `xmin` to `xmax`, variable `y` taking values from `ymin` to `ymax` and variable `z` taking values from `zmin` to `zmax`. This object implements the marching cubes algorithm.

This object is affected by the following graphic options: `x_voxel`, `y_voxel`, `z_voxel`, `line_width`, `line_type`, `key`, `wired_surface`, `enhanced3d` and `color`.

Example:

```plaintext
(%i1) draw3d(
    color=blue,
    implicit((x^2+y^2+z^2-1)*(x^2+(y-1.5)^2+z^2-0.5)=0.015, 
        x,-1,1,y,-1.2,2.3,z,-1,1),
    surface_hide=true);
```

\[ -1 -0.5 \] \[ 0 \] \[ 0.5 \] \[ 1 \]
\[ -1 \] \[ 0.5 \] \[ 1.5 \] \[ 2 \]

```
```

```plaintext
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**label**

```
label ([string,x,y],...)
label ([string,x,y,z],...)
```

Writes labels in 2D and 3D.

Colored labels work only with Gnuplot 4.3 and up.

This object is affected by the following **graphic options**: `label_alignment`, `label_orientation` and `color`.

**2D**

`label([string,x,y])` writes the `string` at point `[x,y]`

Example:

```
(%i1) draw2d(yrange = [0.1,1.4],
    color = red,
    label(["Label in red",0,0.3]),
    color = "#0000ff",
    label(["Label in blue",0,0.6]),
    color = light_blue,
    label(["Label in light-blue",0,0.9],
          ["Another light-blue",0,1.2]) )$
```

**3D**

`label([string,x,y,z])` writes the `string` at point `[x,y,z]`

Example:

```
(%i1) draw3d(explicit(exp(sin(x)+cos(x^2)),x,-3,3,y,-3,3),
    color = red,
    label(["UP 1",-2,0,3], ["UP 2",1.5,0,4]),
    color = blue,
    label(["DOWN 1",2,0,-3]) )$
```


mesh \((row_1,row_2,...)\) 

[Graphic object]

Draws a quadrangular mesh in 3D.

3D

Argument \(row_i\) is a list of \(n\) 3D points of the form \([x_{i1},y_{i1},z_{i1}], ..., [x_{in},y_{in},z_{in}]\), and all rows are of equal length. All these points define an arbitrary surface in 3D and in some sense it's a generalization of the elevation_grid object.

This object is affected by the following graphic options: line_type, line_width, color, key, wired_surface, enhanced3d and transform.

Examples:

A simple example.

\[
\%i1) \text{draw3d}(\\n\hspace{1em} \text{mesh}([[1,1,3], [7,3,1],[12,-2,4],[15,0,5]], \\
\hspace{1em} \hspace{1em} [[2,7,8], [4,3,1],[10,5,8],[12,7,1]], \\
\hspace{1em} \hspace{1em} \hspace{1em} [[-2,11,10],[6,9,5],[6,15,1],[20,15,2]])) \$
\]

Plotting a triangle in 3D.
(%i1) draw3d(
   line_width = 2,
   mesh([[1,0,0],[0,1,0]],
        [[0,0,1],[0,0,1]]))$

Two quadrilaterals.

(%i1) draw3d(
   surface_hide = true,
   line_width = 3,
   color = red,
   mesh([[0,0,0], [0,1,0]],
        [[2,0,2], [2,2,2]]),
   color = blue,
   mesh([[0,0,2], [0,1,2]],
        [[2,0,4], [2,2,4]]))$
parametric

\[ \text{parametric\ (xfun,yfun,par,parmin,parmax)} \]
\[ \text{parametric\ (xfun,yfun,zfun,par,parmin,parmax)} \]

Draws parametric functions in 2D and 3D.

This object is affected by the following graphic options: nticks, line_width, line_type, key, color and enhanced3d.

2D

The command \text{parametric(xfun, yfun, par, parmin, parmax)} plots the parametric function \([xfun, yfun]\), with parameter \(par\) taking values from \(parmin\) to \(parmax\).

Example:

\[
\text{(\%i1) draw2d(explicit(exp(x),x,-1,3),}
\text{\hspace{1cm}color = red,}
\text{\hspace{1cm}key = "This is the parametric one!!",}
\text{\hspace{1cm}parametric(2*cos(rrr),rrr^2,rrr,0,2*%pi))}$
\]

3D

\[ \text{parametric(xfun, yfun, zfun, par, parmin, parmax)} \]
plots the parametric curve \([xfun, yfun, zfun]\), with parameter \(par\) taking values from \(parmin\) to \(parmax\).

Example:

\[
\text{(\%i1) draw3d(explicit(exp(sin(x)+cos(x^2)),x,-3,3,y,-3,3),}
\text{\hspace{1cm}color = royalblue,}
\text{\hspace{1cm}parametric(cos(5*u)^2,sin(7*u),u-2,u,0,2*%pi),}
\text{\hspace{1cm}color = turquoise,}
\text{\hspace{1cm}line_width = 2,}
\text{\hspace{1cm}parametric(t^2,sin(t),2+t,t,0,2),}
\text{\hspace{1cm}surface_hide = true,}
\text{\hspace{1cm}title = "Surface & curves" })}$
\]
parametric_surface (xfun, yfun, zfun, par1, par1min, par1max, par2, par2min, par2max)  
Draws parametric surfaces in 3D.

3D

The command `parametric_surface(xfun, yfun, zfun, par1, par1min, par1max, par2, par2min, par2max)` plots the parametric surface `[xfun, yfun, zfun]`, with parameter `par1` taking values from `par1min` to `par1max` and parameter `par2` taking values from `par2min` to `par2max`.

This object is affected by the following graphic options: `draw_realpart`, `xu_grid`, `yv_grid`, `line_type`, `line_width`, `key`, `wired_surface`, `enhanced3d` and `color`.

Example:

(%i1) draw3d(title = "Sea shell",
            xu_grid = 100,
            yv_grid = 25,
            view = [100, 20],
            surface_hide = true,
            parametric_surface(0.5*u*cos(u)*(cos(v)+1),
                                0.5*u*sin(u)*(cos(v)+1),
                                u*sin(v) - ((u+3)/8*%pi)^2 - 20,
                                u, 0, 13*%pi, v, -%pi, %pi))$
points

points (\([[x_1,y_1], [x_2,y_2],\ldots]\])
points (\([x_1,x_2,\ldots], [y_1,y_2,\ldots]\))
points (\([y_1,y_2,\ldots]\))
points (\([[x_1,y_1,z_1], [x_2,y_2,z_2],\ldots]\))
points (\([x_1,x_2,\ldots], [y_1,y_2,\ldots], [z_1,z_2,\ldots]\))
points (\(\text{matrix}\))
points (\(1d_y\_array\))
points (\(1d_x\_array, 1d_y\_array\))
points (\(1d_x\_array, 1d_y\_array, 1d_z\_array\))
points (\(2d\_xy\_array\))
points (\(2d\_xyz\_array\))

Draws points in 2D and 3D.

This object is affected by the following graphic options: \text{point_size}, \text{point_type}, \text{points\_joined}, \text{line_width}, \text{key}, \text{line\_type} and \text{color}. In 3D mode, it is also affected by \text{enhanced3d}

2D

points (\([[x_1,y_1], [x_2,y_2],\ldots]\)) or points (\([x_1,x_2,\ldots], [y_1,y_2,\ldots]\)) plots points \([x_1,y_1], [x_2,y_2], \ldots\). If abscissas are not given, they are set to consecutive positive integers, so that points (\([y_1,y_2,\ldots]\)) draws points \([1,y_1], [2,y_2], \ldots\), etc. If \(\text{matrix}\) is a two-column or two-row matrix, points (\(\text{matrix}\)) draws the associated points. If \(\text{matrix}\) is a one-column or one-row matrix, abscissas are assigned automatically.

If \(1d_y\_array\) is a 1D lisp array of numbers, points (\(1d_y\_array\)) plots them setting abscissas to consecutive positive integers. points (\(1d_x\_array, 1d_y\_array\)) plots points with their coordinates taken from the two arrays passed as arguments. If \(2d\_xy\_array\) is a 2D array with two columns, or with two rows, points (\(2d\_xy\_array\)) plots the corresponding points on the plane.

Examples:

Two types of arguments for \text{points}, a list of pairs and two lists of separate coordinates.

(\%i1) \text{draw2d}(\ldots)
key = "Small points",
points(makelist([random(20),random(50)],k,1,10)),
point_type  = circle,
point_size  = 3,
pointsJoined = true,
key        = "Great points",
points(makelist(k,k,1,20),makelist(random(30),k,1,20)),
point_type  = filled_down_triangle,
key        = "Automatic abscissas",
color     = red,
points([2,12,8]))$

Drawing impulses.

(%i11) draw2d(
    pointsJoined = impulses,
    line_width    = 2,
    color         = red,
    points(makelist([random(20),random(50)],k,1,10))))$
Array with ordinates.

```lisp
(%i1) a: make_array (flonum, 100) $
(%i2) for i:0 thru 99 do a[i]: random(1.0) $
(%i3) draw2d(points(a)) $
```

Two arrays with separate coordinates.

```lisp
(%i1) x: make_array (flonum, 100) $
(%i2) y: make_array (fixnum, 100) $
(%i3) for i:0 thru 99 do ( 
   x[i]: float(i/100),
   y[i]: random(10) ) $
(%i4) draw2d(points(x, y)) $
```
A two-column 2D array.

```
(%i1) xy: make_array(flonum, 100, 2) 
(%i2) for i:0 thru 99 do ( 
   xy[i, 0]: float(i/100),
   xy[i, 1]: random(10) ) $
(%i3) draw2d(points(xy)) $
```

Drawing an array filled with function read_array.

```
(%i1) a: make_array(flonum,100) $
(%i2) read_array (file_search ("pidigits.data"), a) $
(%i3) draw2d(points(a)) $
```

3D

points([[x1, y1, z1], [x2, y2, z2], ...]) or points([[x1, x2, ...], [y1, y2, ...], [z1, z2, ...]) plots points [x1, y1, z1], [x2, y2, z2], etc. If matrix is a three-column or three-row matrix, points (matrix) draws the associated points.
When arguments are lisp arrays, points (1d_x_array, 1d_y_array, 1d_z_array) takes coordinates from the three 1D arrays. If 2d_xyz_array is a 2D array with three columns, or with three rows, points (2d_xyz_array) plots the corresponding points.

Examples:

One tridimensional sample,

```lisp
(%i1) load ("numericalio")
(%i2) s2 : read_matrix (file_search ("wind.data"))
(%i3) draw3d(title = "Daily average wind speeds",
    point_size = 2,
    points(args(submatrix (s2, 4, 5))) )
```

Two tridimensional samples,

```lisp
(%i1) load ("numericalio")
(%i2) s2 : read_matrix (file_search ("wind.data"))
(%i3) draw3d(
    title = "Daily average wind speeds. Two data sets",
    point_size = 2,
    key = "Sample from stations 1, 2 and 3",
    points(args(submatrix (s2, 4, 5))),
    point_type = 4,
    key = "Sample from stations 1, 4 and 5",
    points(args(submatrix (s2, 2, 3))) )
```

Unidimensional arrays,

```lisp
(%i1) x: make_array (fixnum, 10) $
(%i2) y: make_array (fixnum, 10) $
(%i3) z: make_array (fixnum, 10) $
(%i4) for i:0 thru 9 do ( 
    x[i]: random(10),
    y[i]: random(10),
    z[i]: random(10) ) $
(%i5) draw3d(points(x,y,z)) $
```
Bidimensional colored array,

(%i1) xyz: make_array(fixnum, 10, 3) $
(%i2) for i:0 thru 9 do (  
    xyz[i, 0]: random(10),
    xyz[i, 1]: random(10),
    xyz[i, 2]: random(10) ) $
(%i3) draw3d(  
    enhanced3d = true,
    points_joined = true,
    points(xyz)) $

Color numbers explicitly specified by the user.

(%i1) pts: makelist([t,t^2,cos(t)], t, 0, 15)$
(%i2) col_num: makelist(k, k, 1, length(pts))$
(%i3) draw3d(  
    enhanced3d = ['part(col_num,k),k],
    point_size = 3,
    point_type = filled_circle,
    points(pts))$
Chapter 52: draw

polar \((radius, ang, minang, maxang)\) \[\text{[Graphic object]}\]

Draws 2D functions defined in polar coordinates.

2D

polar \((radius, ang, minang, maxang)\) plots function \(radius(\text{ang})\) defined in polar coordinates, with variable \(\text{ang}\) taking values from \(\text{minang}\) to \(\text{maxang}\).

This object is affected by the following graphic options: \(\text{nticks}, \text{line_width}, \text{line_type}, \text{key}\) and \(\text{color}\).

Example:

\[
(\%i1) \text{draw2d(user_preamble = "set grid polar"},
\text{nticks} = 200,
\text{xrange} = [-5,5],
\text{yrange} = [-5,5],
\text{color} = \text{blue},
\text{line_width} = 3,
\text{title} = "Hyperbolic Spiral",
\text{polar}(10/\text{theta}, \text{theta}, 1, 10*%pi) )$
\]

Hyperbolic Spiral
polygon

polygon([[x1, y1], [x2, y2], ...])
polygon([x1, y1, x2, y2, ...])

Draws polygons in 2D.

2D

The commands polygon([[x1, y1], [x2, y2], ...]) or polygon([x1, y1, x2, y2, ...]) plot on the plane a polygon with vertices [x1, y1], [x2, y2], etc. This object is affected by the following graphic options: transparent, fill_color, border, line_width, key, line_type and color.

Example:

(%i1) draw2d(color = "#e245f0",
           line_width = 8,
           polygon([[3,2],[7,2],[5,5]]),
           border = false,
           fill_color = yellow,
           polygon([[5,2],[9,2],[7,5]]))$

quadrilateral(point_1, point_2, point_3, point_4)

Draws a quadrilateral.

2D

quadrilateral([x1, y1], [x2, y2], [x3, y3], [x4, y4]) draws a quadrilateral with vertices [x1, y1], [x2, y2], [x3, y3], and [x4, y4].

This object is affected by the following graphic options: transparent, fill_color, border, line_width, key, xaxis_secondary, yaxis_secondary, line_type, transform and color.

Example:

(%i1) draw2d(
           quadrilateral([1,1],[2,2],[3,-1],[2,-2]))$
Chapter 52: draw

3D

quadrilateral([x1, y1, z1], [x2, y2, z2], [x3, y3, z3], [x4, y4, z4]) draws a quadrilateral with vertices [x1, y1, z1], [x2, y2, z2], [x3, y3, z3], and [x4, y4, z4].

This object is affected by the following graphic options: line_type, line_width, color, key, enhanced3d and transform.

rectangle ([x1,y1], [x2,y2])
[Graphic object]
Draws rectangles in 2D.

2D

rectangle ([x1,y1], [x2,y2]) draws a rectangle with opposite vertices [x1,y1] and [x2,y2].

This object is affected by the following graphic options: transparent, fill_color, border, line_width, key, line_type and color.

Example:

(%i1) draw2d(fill_color = red, line_width = 6, line_type = dots, transparent = false, fill_color = blue, rectangle([-2,-2],[8,-1]), /* opposite vertices */ transparent = true, line_type = solid, line_width = 1, rectangle([9,4],[2,-1.5]), xrange = [-3,10], yrange = [-3,4.5])$
region (expr, var1, minval1, maxval1, var2, minval2, maxval2)        [Graphic object]
Plots a region on the plane defined by inequalities.

2D expr is an expression formed by inequalities and boolean operators \texttt{and}, \texttt{or}, and \texttt{not}. The region is bounded by the rectangle defined by \([\text{minval1, maxval1}]\) and \([\text{minval2, maxval2}]\).

This object is affected by the following \textit{graphic options}: \texttt{fill\_color}, \texttt{key}, \texttt{x\_voxel} and \texttt{y\_voxel}.

Example:

\begin{verbatim}
(%i1) draw2d(
   x_voxel = 30,
   y_voxel = 30,
   region(x^2+y^2<1 and x^2+y^2 > 1/2,
       x, -1.5, 1.5, y, -1.5, 1.5));
\end{verbatim}
**spherical** \((radius, azi, minazi, maxazi, zen, minzen, maxzen)\)  
Draws 3D functions defined in spherical coordinates.

3D

\(\text{spherical}(radius, azi, minazi, maxazi, zen, minzen, maxzen)\) plots the function \(radius(azi, zen)\) defined in spherical coordinates, with azimuth \(azi\) taking values from \(minazi\) to \(maxazi\) and zenith \(zen\) taking values from \(minzen\) to \(maxzen\).

This object is affected by the following **graphic options**: \(\text{xu_grid}, \text{yv_grid}, \text{line_type}, \text{key}, \text{wired_surface}, \text{enhanced3d}\) and \(\text{color}\).

Example:

\((%i1) \text{draw3d(spherical(1,a,0,2*%pi,z,0,%pi))}\)$

---

**triangle** \((point_1, point_2, point_3)\)  
Draws a triangle.

2D

\(\text{triangle}([x1,y1], [x2,y2], [x3,y3])\) draws a triangle with vertices \([x1,y1]\), \([x2,y2]\), and \([x3,y3]\).

This object is affected by the following **graphic options**: \(\text{transparent}, \text{fill_color}, \text{border}, \text{line_width}, \text{key}, \text{xaxis_secondary}, \text{yaxis_secondary}, \text{line_type}, \text{transform}\) and \(\text{color}\).

Example:

\((%i1) \text{draw2d(}
    \text{triangle([1,1],[2,2],[3,-1]))}\)$
3D

triangle ([x1,y1,z1], [x2,y2,z2], [x3,y3,z3]) draws a triangle with vertices [x1,y1,z1], [x2,y2,z2], and [x3,y3,z3].

This object is affected by the following graphic options: line_type, line_width, color, key, enhanced3d and transform.

tube (xfun,yfun,zfun,rfun,p,pmin,pmax)

Draws a tube in 3D with varying diameter.

3D

[xfun,yfun,zfun] is the parametric curve with parameter \( p \) taking values from \( pmin \) to \( pmax \). Circles of radius \( rfun \) are placed with their centers on the parametric curve and perpendicular to it.

This object is affected by the following graphic options: xu_grid, yv_grid, line_type, line_width, key, wired_surface, enhanced3d, color and capping.

Example:

(%i1) draw3d(
    enhanced3d = true,
    xu_grid = 50,
    tube(cos(a), a, 0, cos(a/10)^2,
         a, 0, 4*%pi))$
vector

\begin{align*}
\text{vector} &\left( [x,y], [dx,dy] \right) \\
\text{vector} &\left( [x,y,z], [dx,dy,dz] \right)
\end{align*}

Draws vectors in 2D and 3D.

This object is affected by the following \textit{graphic options}: head\_both, head\_length, head\_angle, head\_type, line\_width, line\_type, key and color.

\textbf{2D}

\texttt{vector}([x, y], [dx, dy]) plots vector [dx, dy] with origin in [x, y].

Example:

\begin{verbatim}
(%i1) draw2d(xrange = [0,12],
yrange = [0,10],
head_length = 1,
vector([0,1],[5,5]), /* default type */
head_type = 'empty,
vector([3,1],[5,5]),
head_both = true,
head_type = 'nofilled,
line_type = dots,
vector([6,1],[5,5]))$
\end{verbatim}
3D

`vector([x,y,z], [dx,dy,dz])` plots vector `[dx,dy,dz]` with origin in `[x,y,z]`.

Example:

```
(%i1) draw3d(color = cyan,
   vector([0,0,0],[1,1,1]/sqrt(3)),
   vector([0,0,0],[1,-1,0]/sqrt(2)),
   vector([0,0,0],[1,1,-2]/sqrt(6)) )$
```

52.3 Functions and Variables for pictures

`get_pixel (pic,x,y)`
Returns pixel from picture. Coordinates x and y range from 0 to `width-1` and `height-1`, respectively.
make_level_picture

make_level_picture (data)
make_level_picture (data,width,height)

Returns a levels picture object. make_level_picture (data) builds the picture object from matrix data. make_level_picture (data,width,height) builds the object from a list of numbers; in this case, both the width and the height must be given.

The returned picture object contains the following four parts:

1. symbol level
2. image width
3. image height
4. an integer array with pixel data ranging from 0 to 255. Argument data must contain only numbers ranged from 0 to 255; negative numbers are substituted by 0, and those which are greater than 255 are set to 255.

Example:

Level picture from matrix.

(%i1) make_level_picture(matrix([3,2,5],[7,-9,3000]));
(%o1)  picture(level, 3, 2, {Array: #(3 2 5 7 0 255)})

Level picture from numeric list.

(%i1) make_level_picture([-2,0,54,%pi],2,2);
(%o1) picture(level, 2, 2, {Array: #(0 0 54 3)})

make_rgb_picture (redlevel,greenlevel,bluelevel)

Returns an rgb-coloured picture object. All three arguments must be levels picture; with red, green and blue levels.

The returned picture object contains the following four parts:

1. symbol rgb
2. image width
3. image height
4. an integer array of length 3*width*height with pixel data ranging from 0 to 255. Each pixel is represented by three consecutive numbers (red, green, blue).

Example:

(%i1) red: make_level_picture(matrix([3,2],[7,260]));
(%o1) picture(level, 2, 2, {Array: #(3 2 7 255)})

(%i2) green: make_level_picture(matrix([54,23],[73,-9]));
(%o2) picture(level, 2, 2, {Array: #(54 23 73 0)})

(%i3) blue: make_level_picture(matrix([123,82],[45,32.5698]));
(%o3) picture(level, 2, 2, {Array: #(123 82 45 33)})

(%i4) make_rgb_picture(red,green,blue);
(%o4) picture(rgb, 2, 2,
{Array: #(3 54 123 23 82 7 73 45 255 0 33)})

negative_picture (pic)

Returns the negative of a (level or rgb) picture.
picture_equalp (x, y)
  Returns true in case of equal pictures, and false otherwise.

picturep (x)
  Returns true if the argument is a well formed image, and false otherwise.

read_xpm (xpm_file)
  Reads a file in xpm and returns a picture object.

rgb2level (pic)
  Transforms an rgb picture into a level one by averaging the red, green and blue channels.

take_channel (im, color)
  If argument color is red, green or blue, function take_channel returns the corresponding color channel of picture im. Example:

  (%i1) red: make_level_picture(matrix([3,2],[7,260]));
  (%o1) picture(level, 2, 2, {Array: #(3 2 7 255)})
  (%i2) green: make_level_picture(matrix([54,23],[73,-9]));
  (%o2) picture(level, 2, 2, {Array: #(54 23 73 0)})
  (%i3) blue: make_level_picture(matrix([123,82],[45,32.5698]));
  (%o3) picture(level, 2, 2, {Array: #(123 82 45 33)})
  (%i4) make_rgb_picture(red,green,blue);
  (%o4) picture(rgb, 2, 2,
           {Array: #(3 54 123 2 23 82 7 73 45 255 0 33)})
  (%i5) take_channel('%,'green); /* simple quote!!! */
  (%o5) picture(level, 2, 2, {Array: #(54 23 73 0)})

52.4 Functions and Variables for worldmap
This package automatically loads package draw.

52.4.1 Variables and Functions

boundaries_array
  Default value: false
  boundaries_array is where the graphic object geomap looks for boundaries coordinates.
  Each component of boundaries_array is an array of floating point quantities, the coordinates of a polygonal segment or map boundary.
  See also geomap.

numbered_boundaries (nlist)
  Draws a list of polygonal segments (boundaries), labeled by its numbers (boundaries_array coordinates). This is of great help when building new geographical entities.
  Example:
  Map of Europe labeling borders with their component number in boundaries_array.

  (%i1) load(worldmap)$
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(%i2) european_borders:
    region_boundaries(-31.81,74.92,49.84,32.06)$
(%i3) numbered_boundaries(european_borders)$

make_poly_continent
    make_poly_continent (continent_name)
    make_poly_continent (country_list)

Makes the necessary polygons to draw a colored continent or a list of countries.

Example:

(%i1) load(worldmap)$
(%i2) /* A continent */
    make_poly_continent(Africa)$
(%i3) apply(draw2d, %)$

(%i4) /* A list of countries */
    make_poly_continent([[Germany,Denmark,Poland]])$
(%i5) apply(draw2d, %)$
**make_poly_country (country_name)**

Makes the necessary polygons to draw a colored country. If islands exist, one country
can be defined with more than just one polygon.

Example:

```maxima
(%i1) load(worldmap)$
(%i2) make_poly_country(India)$
(%i3) apply(draw2d, %)$
```

**make_polygon (nlist)**

Returns a polygon object from boundary indices. Argument nlist is a list of compo-
nents of boundaries_array.

Example:

Bhutan is defined by boundary numbers 171, 173 and 1143, so that make_polygon([171,173,1143]) appends arrays of coordinates boundaries_array[171], boundaries_array[173] and boundaries_array[1143] and returns a polygon
object suited to be plotted by draw. To avoid an error message, arrays must be
compatible in the sense that any two consecutive arrays have two coordinates in the extremes in common. In this example, the two first components of `boundaries_array[171]` are equal to the last two coordinates of `boundaries_array[173]`, and the two first of `boundaries_array[173]` are equal to the two first of `boundaries_array[1143]`; in conclusion, boundary numbers 171, 173 and 1143 (in this order) are compatible and the colored polygon can be drawn.

```lisp
(%i1) load(worldmap)$
(%i2) Bhutan;
(%o2) [[171, 173, 1143]]
(%i3) boundaries_array[171];
(%o3) {Array: 
  #(88.750549 27.14727 88.806351 27.25305 88.901367 27.282221 
   88.917877 27.321039)}
(%i4) boundaries_array[173];
(%o4) {Array: 
  #(91.659554 27.76511 91.6008 27.66666 91.598022 27.62499 
   91.631348 27.536381 91.765533 27.45694 91.775253 27.4161 
   92.007751 27.471939 92.11441 27.28583 92.015259 27.168051 
   92.015533 27.08083 92.083313 27.02277 92.112183 26.920271 
   92.069977 26.86194 91.997192 26.85194 91.915253 26.893881 
   91.916924 26.85416 91.8358 26.863331 91.712479 26.799999 
   91.542191 26.80444 91.492188 26.87472 91.418854 26.873329 
   91.371353 26.80083 91.307457 26.778049 90.682457 26.77417 
   90.392197 26.903601 90.344131 26.894159 90.143044 26.75333 
   89.98996 26.73583 89.841919 26.70138 89.618301 26.72694 
   89.636093 26.771111 89.360786 26.859989 89.22081 26.81472 
   89.110237 26.829161 88.921631 26.98777 88.873016 26.95499 
   88.867737 27.080549 88.843307 27.108601 88.750549 
   27.14727)}
(%i5) boundaries_array[1143];
(%o5) {Array: 
  #(91.659554 27.76511 91.6008 27.66666 91.598022 27.62499 
   91.631348 27.536381 91.765533 27.45694 91.775253 27.4161 
   92.007751 27.471939 92.11441 27.28583 92.015259 27.168051 
   92.015533 27.08083 92.083313 27.02277 92.112183 26.920271 
   92.069977 26.86194 91.997192 26.85194 91.915253 26.893881 
   91.916924 26.85416 91.8358 26.863331 91.712479 26.799999 
   91.542191 26.80444 91.492188 26.87472 91.418854 26.873329 
   91.371353 26.80083 91.307457 26.778049 90.682457 26.77417 
   90.392197 26.903601 90.344131 26.894159 90.143044 26.75333 
   89.98996 26.73583 89.841919 26.70138 89.618301 26.72694 
   89.636093 26.771111 89.360786 26.859989 89.22081 26.81472 
   89.110237 26.829161 88.921631 26.98777 88.873016 26.95499 
   88.867737 27.080549 88.843307 27.108601 88.750549 
   27.14727)}
(%i6) Bhutan_polygon: make_polygon([171,173,1143])$
(%i7) draw2d(Bhutan_polygon)$
region_boundaries (x1,y1,x2,y2)               [Function]
Detects polygonal segments of global variable boundaries_array fully contained in
the rectangle with vertices (x1,y1) -upper left- and (x2,y2) -bottom right-.
Example:
Returns segment numbers for plotting southern Italy.

(%i1) load(worldmap)$
(%i2) region_boundaries(10.4,41.5,20.7,35.4);
(%o2) [1846, 1863, 1864, 1881, 1888, 1894]
(%i3) draw2d(geomap(%))$

region_boundaries_plus (x1,y1,x2,y2)          [Function]
Detects polygonal segments of global variable boundaries_array containing at least
one vertex in the rectangle defined by vertices (x1,y1) -upper left- and (x2,y2) -bottom
right-.
Example:

(%i11) load(worldmap)$
52.4.2 Graphic objects

**geomap**

*geomap (numlist)*

*geomap (numlist, 3Dprojection)*

Draws cartographic maps in 2D and 3D.

**2D**

This function works together with global variable `boundaries_array`.

Argument `numlist` is a list containing numbers or lists of numbers. All these numbers must be integers greater or equal than zero, representing the components of global array `boundaries_array`.

Each component of `boundaries_array` is an array of floating point quantities, the coordinates of a polygonal segment or map boundary.

`geomap (numlist)` flattens its arguments and draws the associated boundaries in `boundaries_array`.

This object is affected by the following graphic options: `line_width`, `line_type` and `color`.

**Examples:**

A simple map defined by hand:

```lisp
(%i1) load(worldmap)$
(%i2) /* Vertices of boundary #0: {(1,1),(2,5),(4,3)} */
   ( bnd0: make_array(flonum,6),
     bnd0[0]:1.0, bnd0[1]:1.0, bnd0[2]:2.0,
     bnd0[3]:5.0, bnd0[4]:4.0, bnd0[5]:3.0 )$
```
The auxiliary package worldmap sets the global variable boundaries_array to real world boundaries in (longitude, latitude) coordinates. These data are in the public domain and come from https://web.archive.org/web/20100310124019/http://www-cger.nies.go.jp/grid-e/gridtxt/grid19.html. Package worldmap defines also boundaries for countries, continents and coastlines as lists with the necessary components of boundaries_array (see file share/draw/worldmap.mac for more information). Package worldmap automatically loads package worldmap.

(%i1) load(worldmap)$
(%i2) c1: gr2d(geomap([Canada,United_States, Mexico,Cuba]))$
Package `worldmap` is also useful for plotting countries as polygons. In this case, graphic object `geomap` is no longer necessary and the `polygon` object is used instead. Since lists are now used and not arrays, maps rendering will be slower. See also `make_poly_country` and `make_poly_continent` to understand the following code.
3D

gemap (numlist) projects map boundaries on the sphere of radius 1 centered at (0,0,0). It is possible to change the sphere or the projection type by using geomap (numlist,3Dprojection).

Available 3D projections:

- [spherical_projection,x,y,z,r]: projects map boundaries on the sphere of radius r centered at (x,y,z).

(%i1) load(worldmap)$
(%i2) draw3d(geomap(Australia), /* default projection */
    geomap(Australia,
        [spherical_projection,2,2,2,3]))$

- [cylindrical_projection,x,y,z,r,rc]: re-projects spherical map boundaries on the cylinder of radius rc and axis passing through the poles of the globe of radius r centered at (x,y,z).

(%i1) load(worldmap)$
(%i2) draw3d(geomap([America_coastlines,Eurasia_coastlines],
    cylindrical_projection,2,2,2,2,2))$
- [conic_projection, \(x, y, z, r, \alpha\)]: re-projects spherical map boundaries on the cones of angle \(\alpha\), with axis passing through the poles of the globe of radius \(r\) centered at \((x, y, z)\). Both the northern and southern cones are tangent to sphere.

\begin{verbatim}
(%i1) load(worldmap)$
(%i2) draw3d(geomap(World_coastlines,
                 [conic_projection,0,0,0,1,90]))$
\end{verbatim}

See also [http://riotorto.users.sf.net/gnuplot/geomap](http://riotorto.users.sf.net/gnuplot/geomap) for more elaborated examples.
53 drawdf

53.1 Introduction to drawdf

The function `drawdf` draws the direction field of a first-order Ordinary Differential Equation (ODE) or a system of two autonomous first-order ODE’s.

Since this is an additional package, in order to use it you must first load it with `load(drawdf)`. Drawdf is built upon the `draw` package, which requires Gnuplot 4.2.

To plot the direction field of a single ODE, the ODE must be written in the form:

\[
\frac{dy}{dx} = F(x, y)
\]

and the function \(F\) should be given as the argument for `drawdf`. If the independent and dependent variables are not \(x\) and \(y\), as in the equation above, then those two variables should be named explicitly in a list given as an argument to the `drawdf` command (see the examples).

To plot the direction field of a set of two autonomous ODE’s, they must be written in the form

\[
\frac{dx}{dt} = G(x, y) \quad \frac{dy}{dt} = F(x, y)
\]

and the argument for `drawdf` should be a list with the two functions \(G\) and \(F\), in that order; namely, the first expression in the list will be taken to be the time derivative of the variable represented on the horizontal axis, and the second expression will be the time derivative of the variable represented on the vertical axis. Those two variables do not have to be \(x\) and \(y\), but if they are not, then the second argument given to `drawdf` must be another list naming the two variables, first the one on the horizontal axis and then the one on the vertical axis.

If only one ODE is given, `drawdf` will implicitly admit \(x=t\), and \(G(x,y)=1\), transforming the non-autonomous equation into a system of two autonomous equations.

53.2 Functions and Variables for drawdf

53.2.1 Functions

`drawdf` (Function)

\[
\text{drawdf} (\text{dydx}, \ldots \text{options and objects} \ldots)
\]

\[
\text{drawdf} (\text{dvdu}, [u,v], \ldots \text{options and objects} \ldots)
\]

\[
\text{drawdf} (\text{dvdu}, [u,umin,umax], [v,vmin,vmax], \ldots \text{options and objects} \ldots)
\]

\[
\text{drawdf} ([\text{dxdt,dydt}], \ldots \text{options and objects} \ldots)
\]

\[
\text{drawdf} ([\text{dudt,dvdt}], [u,v], \ldots \text{options and objects} \ldots)
\]

\[
\text{drawdf} ([\text{dudt,dvdt}], [u,umin,umax], [v,vmin,vmax], \ldots \text{options and objects} \ldots)
\]

Function `drawdf` draws a 2D direction field with optional solution curves and other graphics using the `draw` package.
The first argument specifies the derivative(s), and must be either an expression or a list of two expressions. \(dy/dx, dx/dt\) and \(dy/dt\) are expressions that depend on \(x\) and \(y\). \(dv/du, du/dt\) and \(dv/dt\) are expressions that depend on \(u\) and \(v\).

If the independent and dependent variables are not \(x\) and \(y\), then their names must be specified immediately following the derivative(s), either as a list of two names \([u,v]\), or as two lists of the form \([u,umin,umax]\) and \([v,vmin,vmax]\).

The remaining arguments are graphic options, graphic objects, or lists containing graphic options and objects, nested to arbitrary depth. The set of graphic options and objects supported by \texttt{drawdf} is a superset of those supported by \texttt{draw2d} and \texttt{gr2d} from the \texttt{draw} package.

The arguments are interpreted sequentially: graphic options affect all following graphic objects. Furthermore, graphic objects are drawn on the canvas in order specified, and may obscure graphics drawn earlier. Some graphic options affect the global appearance of the scene.

The additional graphic objects supported by \texttt{drawdf} include: \texttt{solns_at}, \texttt{points_at}, \texttt{saddles_at}, \texttt{soln_at}, \texttt{point_at}, and \texttt{saddle_at}.

The additional graphic options supported by \texttt{drawdf} include: \texttt{field_degree}, \texttt{soln_arrows}, \texttt{field_arrows}, \texttt{field_grid}, \texttt{field_color}, \texttt{show_field}, \texttt{tstep}, \texttt{nsteps}, \texttt{duration}, \texttt{direction}, \texttt{field_tstep}, \texttt{field NSTeps}, and \texttt{field_duration}.

Commonly used graphic objects inherited from the \texttt{draw} package include: \texttt{explicit}, \texttt{implicit}, \texttt{parametric}, \texttt{polygon}, \texttt{points}, \texttt{vector}, \texttt{label}, and all others supported by \texttt{draw2d} and \texttt{gr2d}.

Commonly used graphic options inherited from the \texttt{draw} package include: \texttt{points_joined}, \texttt{color}, \texttt{point_type}, \texttt{point_size}, \texttt{line_width}, \texttt{line_type}, \texttt{key}, \texttt{title}, \texttt{xlabel}, \texttt{ylabel}, \texttt{user_preamble}, \texttt{terminal}, \texttt{dimensions}, \texttt{file_name}, and all others supported by \texttt{draw2d} and \texttt{gr2d}.

See also \texttt{draw2d}.

Users of wxMaxima or Imaxima may optionally use \texttt{wxdrawdf}, which is identical to \texttt{drawdf} except that the graphics are drawn within the notebook using \texttt{wxdraw}.

To make use of this function, write first \texttt{load(drawdf)}.

Examples:

\begin{verbatim}
(%i1) load(drawdf)$
(%i2) drawdf(\texttt{exp(-x)+y}) /* default vars: x,y */
(%i3) drawdf(\texttt{exp(-t)+y}, [t,y]) /* default range: \([-10,10]\] */
(%i4) drawdf([y,-9*\texttt{sin}(x)-y/5], [x,1,5], [y,-2,2])$
\end{verbatim}

For backward compatibility, \texttt{drawdf} accepts most of the parameters supported by \texttt{plotdf}.

\begin{verbatim}
(%i5) drawdf(2*\texttt{cos}(t)-1+y, [t,y], [t,-5,10], [y,-4,9],
          [\texttt{trajectory_at},0,0])$
\end{verbatim}

\texttt{soln_at} and \texttt{solns_at} draw solution curves passing through the specified points, using a slightly enhanced 4th-order Runge Kutta numerical integrator.

\begin{verbatim}
(%i6) drawdf(2*\texttt{cos}(t)-1+y, [t,-5,10], [y,-4,9],
          \texttt{solns_at}([0,0.1],[0,-0.1]))
\end{verbatim}
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\[
\text{color=blue, soln_at(0,0))}
\]

\[
\text{field_degree=2}\text{ causes the field to be composed of quadratic splines, based on the first and second derivatives at each grid point. field_grid=[COLS,ROWS] specifies the number of columns and rows in the grid.}
\]

\[
\text{(%i7) drawdf(2*cos(t)-1+y, [t,-5,10], [y,-4,9], field_degree=2, field_grid=[20,15], solns_at([0,0.1],[0,-0.1]), color=blue, soln_at(0,0))}
\]

\[
\text{soln_arrows=true adds arrows to the solution curves, and (by default) removes them from the direction field. It also changes the default colors to emphasize the solution curves.}
\]

\[
\text{(%i8) drawdf(2*cos(t)-1+y, [t,-5,10], [y,-4,9], soln_arrows=true, solns_at([0,0.1],[0,-0.1],[0,0]), duration=40}\
\text{specifies the time duration of numerical integration (default 10). Integration will also stop automatically if the solution moves too far away from the plotted region, or if the derivative becomes complex or infinite. Here we also specify field_degree=2 to plot quadratic splines. The equations below model a predator-prey system.}
\]

\[
\text{(%i9) drawdf([x*(1-x-y), y*(3/4-y-x/2)], [x,0,1.1], [y,0,1], field_degree=2, duration=40, soln_arrows=true, point_at(1/2,1/2), solns_at([0.1,0.2],[0.2,0.1],[1,0.8],[0.8,1],[0.1,0.1],[0.6,0.05],[0.05,0.4],[1,0.01],[0.01,0.75])}
\]

\[
\text{field_degree=’solns causes the field to be composed of many small solution curves computed by 4th-order Runge Kutta, with better results in this case.}
\]

\[
\text{(%i10) drawdf([x*(1-x-y), y*(3/4-y-x/2)], [x,0,1.1], [y,0,1], field_degree=’solns, duration=40, soln_arrows=true, point_at(1/2,1/2), solns_at([0.1,0.2],[0.2,0.1],[1,0.8],[0.8,1],[0.1,0.1],[0.6,0.05],[0.05,0.4],[1,0.01],[0.01,0.75])}
\]

\[
\text{saddles_at attempts to automatically linearize the equation at each saddle, and to plot a numerical solution corresponding to each eigenvector, including the separatrices. tstep=0.05 specifies the maximum time step for the numerical integrator (the default is 0.1). Note that smaller time steps will sometimes be used in order to keep the x and y steps small. The equations below model a damped pendulum.}
\]

\[
\text{(%i11) drawdf([y,-9*sin(x)-y/5], tstep=0.05, soln_arrows=true, point_size=0.5, points_at([0,0],[2*pi,0],[-2*pi,0]), field_degree=’solns, saddles_at([pi,0],[-pi,0])}
\]

\[
\text{show_field=false suppresses the field entirely.}
\]

\[
\text{(%i12) drawdf([y,-9*sin(x)-y/5], tstep=0.05,}
\]
show_field=false, soln_arrows=true, point_size=0.5, points_at([0,0], [2*%pi,0], [-2*%pi,0]), saddles_at([3*%pi,0], [-3*%pi,0], [%pi,0], [-%pi,0])$

`drawdf` passes all unrecognized parameters to `draw2d` or `gr2d`, allowing you to combine the full power of the `draw` package with `drawdf`.

(%i13) drawdf(x^2+y^2, [x,-2,2], [y,-2,2], field_color=gray, key="soln 1", color=black, soln_at(0,0), key="soln 2", color=red, soln_at(0,1), key="isocline", color=green, line_width=2, nticks=100, parametric(cos(t),sin(t),t,0,2*%pi))$

`drawdf` accepts nested lists of graphic options and objects, allowing convenient use of `makelist` and other function calls to generate graphics.

(%i14) colors : ['red,'blue,'purple,'orange,'green]$
(%i15) drawdf([x-x*y/2, (x*y - 3*y)/4], [x,2.5,3.5], [y,1.5,2.5], field_color = gray, makelist([ key = concat("soln",k), color = colors[k], soln_at(3, 2 + k/20) ], k,1,5))$
54 dynamics

54.1 The dynamics package

Package *dynamics* includes functions for 3D visualization, animations, graphical analysis of differential and difference equations and numerical solution of differential equations. The functions for differential equations are described in the section on Chapter 22 [Numerical], page 365, and the functions to plot the Mandelbrot and Julia sets are described in the section on Plotting.

All the functions in this package will be loaded automatically the first time they are used.

54.2 Graphical analysis of discrete dynamical systems

`chaosgame ([[x1, y1], ..., [xm, ym]], [x0, y0], b, n, options, ...);`  
[Function]

Implements the so-called chaos game: the initial point \((x_0, y_0)\) is plotted and then one of the \(m\) points \([x_1, y_1], ..., [xm, ym]\) will be selected at random. The next point plotted will be on the segment from the previous point plotted to the point chosen randomly, at a distance from the random point which will be \(b\) times that segment’s length. The procedure is repeated \(n\) times. The options are the same as for `plot2d`.

**Example.** A plot of Sierpinsky’s triangle:

```
(%i1) chaosgame([[0, 0], [1, 0], [0.5, sqrt(3)/2]], [0.1, 0.1], 1/2, 30000, [style, dots]);
```

```
0
 0.1
 0.2
 0.3
 0.4
 0.5
 0.6
 0.7
 0.8
 0.9
 0
 0.1
 0.2
 0.3
 0.4
 0.5
 0.6
 0.7
 0.8
 0.9
1
contraction factor: 0.5
The chaos game with 3 points
```

`evolution (F, y0, n, ..., options, ...);`  
[Function]

Draws \(n+1\) points in a two-dimensional graph, where the horizontal coordinates of the points are the integers 0, 1, 2, ..., \(n\), and the vertical coordinates are the corresponding values \(y(n)\) of the sequence defined by the recurrence relation

\[
y_{n+1} = F(y_n)
\]
With initial value $y(0)$ equal to $y_0$. $F$ must be an expression that depends only on one variable (in the example, it depend on $y$, but any other variable can be used), $y_0$ must be a real number and $n$ must be a positive integer. This function accepts the same options as plot2d.

**Example.**

```maxima
(%i1) evolution(cos(y), 2, 11);
```

![Graph of $y(n)$ vs $n$ showing a sequence of points defined by the dynamical system $u_{n+1} = F(u_n, v_n)$ and $v_{n+1} = G(u_n, v_n)$ with initial values $u_0$ and $v_0$.]

```maxima
evolution2d ([F, G], [u, v], [u0, y0], n, options, ...); [Function]
```

Shows, in a two-dimensional plot, the first $n+1$ points in the sequence of points defined by the two-dimensional discrete dynamical system with recurrence relations

\[
\begin{align*}
  u_{n+1} &= F(u_n, v_n) \\
  v_{n+1} &= G(u_n, v_n)
\end{align*}
\]

With initial values $u_0$ and $v_0$. $F$ and $G$ must be two expressions that depend only on two variables, $u$ and $v$, which must be named explicitly in a list. The options are the same as for plot2d.

**Example.** Evolution of a two-dimensional discrete dynamical system:

```maxima
(%i1) f: 0.6*x*(1+2*x)+0.8*y*(x-1)-y^2-0.9$
(%i2) g: 0.1*x*(1-6*x+4*y)+0.1*y*(1+9*y)-0.4$
(%i3) evolution2d([f,g], [x,y], [-0.5,0], 50000, [style,dots]);
```
And an enlargement of a small region in that fractal:

\[
\text{(\%i19) evolution2d([f, g], [x, y], [-0.5, 0], 300000, [x, -0.8, -0.6], [y, -0.4, -0.2], [style, dots]);}
\]

\text{ifs } ([r1, \ldots, rm], [A1, \ldots, Am], [[x1, y1], \ldots, [xm, ym]], [x0, y0], n, \quad \text{[Function]} \quad \text{options, \ldots});

Implements the Iterated Function System method. This method is similar to the method described in the function \text{chaosgame}. but instead of shrinking the segment from the current point to the randomly chosen point, the 2 components of that segment will be multiplied by the 2 by 2 matrix \text{ Ai} that corresponds to the point chosen randomly.

The random choice of one of the \(m\) attractive points can be made with a non-uniform probability distribution defined by the weights \(r1, \ldots, rm\). Those weights are given in cumulative form; for instance if there are 3 points with probabilities 0.2, 0.5 and 0.3, the weights \( r1, r2 \) and \( r3 \) could be 2, 7 and 10. The options are the same as for \text{plot2d}. 
**Example.** Barnsley’s fern, obtained with 4 matrices and 4 points:

(%i1) a1: matrix([0.85,0.04],[-0.04,0.85])
(%i2) a2: matrix([0.2,-0.26],[0.23,0.22])
(%i3) a3: matrix([-0.15,0.28],[0.26,0.24])
(%i4) a4: matrix([0,0],[0,0.16])
(%i5) p1: [0,1.6]
(%i6) p2: [0,1.6]
(%i7) p3: [0,0.44]
(%i8) p4: [0,0]
(%i9) w: [85,92,99,100]
(%i10) ifs(w, [a1,a2,a3,a4], [p1,p2,p3,p4], [5,0], 50000, [style,dots]);

**Example.** Orbits diagram of the quadratic map, with a parameter $a$:

(%i11) orbits(x^2+a, 0, 50, 200, [a, -2, 0.25], [style, dots]);
To enlarge the region around the lower bifurcation near $x = -1.25$ use:

```latex
(\%i2) orbits(x^2+a, 0, 100, 400, [a,-1,-1.53], [x,-1.6,-0.8], [nticks, 400], [style,dots]);
```

**staircase** ($F, y0, n,options,\ldots$);  
[Function]

Draws a staircase diagram for the sequence defined by the recurrence relation

$$y_{n+1} = F(y_n)$$

The interpretation and allowed values of the input parameters is the same as for the function **evolution**. A staircase diagram consists of a plot of the function $F(y)$, together with the line $G(y) = y$. A vertical segment is drawn from the point $(y0, y0)$ on that line until the point where it intersects the function $F$. From that point a horizontal segment is drawn until it reaches the point $(yI, yI)$ on the line, and the procedure is repeated $n$ times until the point $(y_n, y_n)$ is reached. The options are the same as for **plot2d**.
Example.

(%i1) staircase(cos(y), 1, 11, [y, 0, 1.2]);

54.3 Visualization with VTK

Function scene creates 3D images and animations using the Visualization ToolKit (VTK) software. In order to use that function, Xmaxima and VTK should be installed in your system (including the TCL bindings of VTK, which in some system might come in a separate package).

scene (objects, ..., options, ...): [Function]

Accepts an empty list or a list of several [scene_objects], page 873, and [scene_options], page 872. The program launches Xmaxima, which opens an external window representing the given objects in a 3-dimensional space and applying the options given. Each object must belong to one of the following 4 classes: sphere, cube, cylinder or cone (see [scene_objects], page 873). Objects are identified by giving their name or by a list in which the first element is the class name and the following elements are options for that object.

Example. A hexagonal pyramid with a blue background:

(%i1) scene(cone, [background, "#9980e5"])$
By holding down the left button of the mouse while it is moved on the graphics window, the camera can be rotated showing different views of the pyramid. The two plot options [scene_elevation], page 872, and [scene_azimuth], page 872, can also be used to change the initial orientation of the viewing camera. The camera can be moved by holding the middle mouse button while moving it and holding the right-side mouse button while moving it up or down will zoom in or out.

Each object option should be a list starting with the option name, followed by its value. The list of allowed options can be found in the [object_options], page 874, section.

**Example.** This will show a sphere falling to the ground and bouncing off without losing any energy. To start or pause the animation, press the play/pause button.

(ï1) p: makelist ([0,0,2.1- 9.8*t^2/2], t, 0, 0.64, 0.01)$

(ï2) p: append (p, reverse(p))$

(ï3) ball: [sphere, [radius,0.1], [thetaresolution,20], [phiresolution,20], [position,0,0,2.1], [color,red], [animate,position,p]]$

(ï4) ground: [cube, [xlength,2], [ylength,2], [zlength,0.2], [position,0,0,-0.1],[color,violet]]$

(ï5) scene (ball, ground, restart)$
The `restart` option was used to make the animation restart automatically every time the last point in the position list is reached. The accepted values for the colors are the same as for the `color` option of `plot2d`.

### 54.3.1 Scene options

#### azimuth `[azimuth, angle]`

Default value: 135

The rotation of the camera on the horizontal (x, y) plane. `angle` must be a real number; an angle of 0 means that the camera points in the direction of the y axis and the x axis will appear on the right.

#### background `[background, color]`

Default value: `black`

The color of the graphics window’s background. It accepts color names or hexadecimal red-green-blue strings (see the `color` option of `plot2d`).

#### elevation `[elevation, angle]`

Default value: 30

The vertical rotation of the camera. The `angle` must be a real number; an angle of 0 means that the camera points on the horizontal, and the default angle of 30 means that the camera is pointing 30 degrees down from the horizontal.

#### height `[height, pixels]`

Default value: 500
The height, in pixels, of the graphics window. \textit{pixels} must be a positive integer number.

\textbf{restart [restart, value]} \hspace{1cm} \text{[Scene option]}
\begin{itemize}
  \item Default value: \textit{false}
\end{itemize}
A true value means that animations will restart automatically when the end of the list is reached. Writing just “restart” is equivalent to [restart, true].

\textbf{tstep [tstep, time]} \hspace{1cm} \text{[Scene option]}
\begin{itemize}
  \item Default value: 10
\end{itemize}
The amount of time, in mili-seconds, between iterations among consecutive animation frames. \textit{time} must be a real number.

\textbf{width [width, pixels]} \hspace{1cm} \text{[Scene option]}
\begin{itemize}
  \item Default value: 500
\end{itemize}
The width, in pixels, of the graphics window. \textit{pixels} must be a positive integer number.

\textbf{windowname [windowtitle, name]} \hspace{1cm} \text{[Scene option]}
\begin{itemize}
  \item Default value: \textit{.scene}
\end{itemize}
\textit{name} must be a string that can be used as the name of the Tk window created by Xmaxima for the \textit{scene} graphics. The default value \textit{.scene} implies that a new top level window will be created.

\textbf{windowtitle [windowtitle, name]} \hspace{1cm} \text{[Scene option]}
\begin{itemize}
  \item Default value: \textit{Xmaxima: scene}
\end{itemize}
\textit{name} must be a string that will be written in the title of the window created by \textit{scene}.

\subsection{54.3.2 Scene objects}

\textbf{cone [cone, options]} \hspace{1cm} \text{[Scene object]}
Creates a regular pyramid with height equal to 1 and a hexagonal base with vertices 0.5 units away from the axis. Options \textit{[object_height], page 875, and [object_radius], page 876,} can be used to change those defaults and option \textit{[object_resolution], page 876,} can be used to change the number of edges of the base; higher values will make it look like a cone. By default, the axis will be along the x axis, the middle point of the axis will be at the origin and the vertex on the positive side of the x axis; use options \textit{[object_orientation], page 875, and [object_center], page 874,} to change those defaults.

\textbf{Example.} This shows a pyramid that starts rotating around the z axis when the play button is pressed.

(\%i1) scene([cone, [orientation,0,30,0], [tstep,100],
[animate,orientation,makelist([0,30,i],i,5,360,5)], restart])$

\textbf{cube [cube, options]} \hspace{1cm} \text{[Scene object]}
A cube with edges of 1 unit and faces parallel to the xy, xz and yz planes. The lengths of the three edges can be changed with options \textit{[object_xlength], page 877,},
cylinder [cylinder, options]  
[Scene object]  
Creates a regular prism with height equal to 1 and a hexagonal base with vertices 0.5 units away from the axis. Options [object_height], page 875, and [object_radius], page 876, can be used to change those defaults and option [object_resolution], page 876, can be used to change the number of edges of the base; higher values will make it look like a cylinder. The default height can be changed with the option [object_height], page 875,. By default, the axis will be along the x axis and the middle point of the axis will be at the origin; use options [object_orientation], page 875, and [object_center], page 874, to change those defaults.

sphere [sphere, options]  
[Scene object]  
A sphere with default radius of 0.5 units and center at the origin.

54.3.3 Scene object’s options

animation [animation, property, positions]  
[Object option]  
property should be one of the following 4 object’s properties: [object_origin], page 875,, [object_scale], page 876,, [object_position], page 876, or [object_orientation], page 875, and positions should be a list of points. When the play button is pressed, the object property will be changed sequentially through all the values in the list, at intervals of time given by the option [scene_tstep], page 873,. The rewind button can be used to point at the start of the sequence making the animation restart after the play button is pressed again.  
See also [object_track], page 877,.

capping [capping, number]  
[Object option]  
Default value: 1  
In a cone or a cylinder, it defines whether the base (or bases) will be shown. A value of 1 for number makes the base visible and a value of 0 makes it invisible.

center [center, point]  
[Object option]  
Default value: [0, 0, 0]  
The coordinates of the object’s geometric center, with respect to its [object_position], page 876,. point can be a list with 3 real numbers, or 3 real numbers separated by commas. In a cylinder, cone or cube it will be at half its height and in a sphere at its center.

color [color, colorname]  
[Object option]  
Default value: white  
The color of the object. It accepts color names or hexadecimal red-green-blue strings (see the color option of plot2d).
Chapter 54: dynamics

endphi [endphi, angle]  
Default value: 180  
In a sphere phi is the angle on the vertical plane that passes through the z axis, measured from the positive part of the z axis. angle must be a number between 0 and 180 that sets the final value of phi at which the surface will end. A value smaller than 180 will eliminate a part of the sphere’s surface.  
See also [object_startphi], page 876, and [object_phiresolution], page 875.

endtheta [endtheta, angle]  
Default value: 360  
In a sphere theta is the angle on the horizontal plane (longitude), measured from the positive part of the x axis. angle must be a number between 0 and 360 that sets the final value of theta at which the surface will end. A value smaller than 360 will eliminate a part of the sphere’s surface.  
See also [object_starttheta], page 877, and [object_thetaresolution], page 877.

height [height, value]  
Default value: 1  
value must be a positive number which sets the height of a cone or a cylinder.

linewidth [linewidth, value]  
Default value: 1  
The width of the lines, when option [object_wireframe], page 878, is used. value must be a positive number.

opacity [opacity, value]  
Default value: 1  
value must be a number between 0 and 1. The lower the number, the more transparent the object will become. The default value of 1 means a completely opaque object.

orientation [orientation, angles]  
Default value: [0, 0, 0]  
Three angles by which the object will be rotated with respect to the three axis. angles can be a list with 3 real numbers, or 3 real numbers separated by commas. Example: [0, 0, 90] rotates the x axis of the object to the y axis of the reference frame.

origin [origin, point]  
Default value: [0, 0, 0]  
The coordinates of the object’s origin, with respect to which its other dimensions are defined. point can be a list with 3 real numbers, or 3 real numbers separated by commas.

phiresolution [phiresolution, num]  
Default value:
The number of sub-intervals into which the phi angle interval from \([object\_startphi]\), page 876, to \([object\_endphi]\), page 875, will be divided. \(num\) must be a positive integer.

See also \([object\_startphi]\), page 876, and \([object\_endphi]\), page 875.

\textbf{points} \([\text{points}]\)

\begin{itemize}
\item \text{Object option}
\item Only the vertices of the triangulation used to render the surface will be shown. \textbf{Example:} \([\text{sphere, [points]}]\)
\item See also \([object\_surface]\), page 877, and \([object\_wireframe]\), page 878.
\end{itemize}

\textbf{pointsize} \(\text{[pointsize, value]}\)

\begin{itemize}
\item \text{Object option}
\item Default value: 1
\item The size of the points, when option \([object\_points]\), page 876, is used. \(value\) must be a positive number.
\end{itemize}

\textbf{position} \(\text{[position, point]}\)

\begin{itemize}
\item \text{Object option}
\item Default value: \([0, 0, 0]\)
\item The coordinates of the object’s position. \(point\) can be a list with 3 real numbers, or 3 real numbers separated by commas.
\end{itemize}

\textbf{radius} \(\text{[radius, value]}\)

\begin{itemize}
\item \text{Object option}
\item Default value: 0.5
\item The radius or a sphere or the distance from the axis to the base’s vertices in a cylinder or a cone. \(value\) must be a positive number.
\end{itemize}

\textbf{resolution} \(\text{[resolution, number]}\)

\begin{itemize}
\item \text{Object option}
\item Default value: 6
\item \(number\) must be a integer greater than 2 that sets the number of edges in the base of a cone or a cylinder.
\end{itemize}

\textbf{scale} \(\text{[scale, factors]}\)

\begin{itemize}
\item \text{Object option}
\item Default value: \([1, 1, 1]\)
\item Three numbers by which the object will be scaled with respect to the three axis. \(factors\) can be a list with 3 real numbers, or 3 real numbers separated by commas. \textbf{Example:} \([2, 0.5, 1]\) enlarges the object to twice its size in the x direction, reduces the dimensions in the y direction to half and leaves the z dimensions unchanged.
\end{itemize}

\textbf{startphi} \(\text{[startphi, angle]}\)

\begin{itemize}
\item \text{Object option}
\item Default value: 0
\item In a sphere phi is the angle on the vertical plane that passes through the z axis, measured from the positive part of the z axis. \(angle\) must be a number between 0 and 180 that sets the initial value of phi at which the surface will start. A value bigger than 0 will eliminate a part of the sphere’s surface.
\item See also \([object\_endphi]\), page 875, and \([object\_phiresolution]\), page 875.
\end{itemize}
starttheta [starttheta, angle]  [Object option]
  Default value: 0
  In a sphere theta is the angle on the horizontal plane (longitude), measured from the positive part of the x axis. angle must be a number between 0 and 360 that sets the initial value of theta at which the surface will start. A value bigger than 0 will eliminate a part of the sphere’s surface.
  See also [object_endtheta], page 875, and [object_thetaresolution], page 877.

surface [surface]  [Object option]
  The surfaces of the object will be rendered and the lines and points of the triangulation used to build the surface will not be shown. This is the default behavior, which can be changed using either the option [object_points], page 876, or [object_wireframe], page 878.

thetaresolution [thetaresolution, num]  [Object option]
  Default value:
  The number of sub-intervals into which the theta angle interval from [object_starttheta], page 877, to [object_endtheta], page 875, will be divided. num must be a positive integer.
  See also [object_starttheta], page 877, and [object_endtheta], page 875.

track [track, positions]  [Object option]
  positions should be a list of points. When the play button is pressed, the object position will be changed sequentially through all the points in the list, at intervals of time given by the option [scene_tstep], page 873, leaving behind a track of the object’s trajectory. The rewind button can be used to point at the start of the sequence making the animation restart after the play button is pressed again.

Example. This will show the trajectory of a ball thrown with speed of 5 m/s, at an angle of 45 degrees, when the air resistance can be neglected:

(%i1) p: makelist ([0,4*t,4*t- 9.8*t^2/2], t, 0, 0.82, 0.01)$
(%i2) ball: [sphere, [radius,0.1], [color,red], [track,p]]$
(%i3) ground: [cube, [xlength,2], [ylength,4], [zlength,0.2],
  [position,0,1.5,-0.2],[color,green]]$
(%i4) scene (ball, ground)$
  See also [object_animation], page 874.

xlength [xlength, length]  [Object option]
  Default value: 1
  The height of a cube in the x direction. length must be a positive number. See also [object_ylength], page 877, and [object_zlength], page 878.

ylength [ylength, length]  [Object option]
  Default value: 1
The height of a cube in the y direction. \textit{length} must be a positive number. See also \texttt{[object_xlength]}, page 877, and \texttt{[object_zlength]}, page 878.

\texttt{zlength} \[\texttt{zlength, length} \] 
\hspace{1em} [Object option]
\hspace{1em} Default value: 1
\hspace{1em} The height of a cube in z the direction. \textit{length} must be a positive number. See also \texttt{[object_xlength]}, page 877, and \texttt{[object_ylength]}, page 877.

\texttt{wireframe} \[\texttt{wireframe} \] 
\hspace{1em} [Object option]
\hspace{1em} Only the edges of the triangulation used to render the surface will be shown. \textbf{Example:}
\hspace{1em} \texttt{[cube, [wireframe]]}
\hspace{1em} See also \texttt{[object_surface]}, page 877, and \texttt{[object_points]}, page 876.
55 engineering-format

Engineering-format changes the way maxima outputs floating-point numbers to the notation engineers are used to: \( a \times 10^b \) with \( b \) dividable by three.

55.1 Functions and Variables for engineering-format

\texttt{engineering\_format\_floats} \hspace{1cm} \textit{[Option variable]}

Default value: \texttt{true}

This variable allows to temporarily switch off engineering-format.

(\texttt{\%i1}) \hspace{1cm} \texttt{load("engineering\_format");}

(\texttt{\%o1}) \hspace{1cm} \texttt{/maxima/share/contrib/engineering-format.lisp}

(\texttt{\%i2}) \hspace{1cm} \texttt{float(sin(10)/10000);}

(\texttt{\%o2}) \hspace{1cm} \texttt{- 54.40211108893698e-6}

(\texttt{\%i3}) \hspace{1cm} \texttt{engineering\_format\_floats: false$}

(\texttt{\%i4}) \hspace{1cm} \texttt{float(sin(10)/10000);}

(\texttt{\%o4}) \hspace{1cm} \texttt{- 5.440211108893698e-5}

See also \texttt{fpprintprec} and \texttt{float}.

55.2 Known Bugs

The output routine of SBCL 1.3.0 has a bug that sometimes causes the exponent not to be dividable by three. The value of the displayed number is still valid in this case.
56 ezunits

56.1 Introduction to ezunits

ezunits is a package for working with dimensional quantities, including some functions for dimensional analysis. ezunits can carry out arithmetic operations on dimensional quantities and unit conversions. The built-in units include Systeme Internationale (SI) and US customary units, and other units can be declared. See also physical_constants, a collection of physical constants.

load(ezunits) loads this package. demo(ezunits) displays several examples. The convenience function known_units returns a list of the built-in and user-declared units, while display_known_unit_conversions displays the set of known conversions in an easy-to-read format.

An expression $a'b$ represents a dimensional quantity, with $a$ indicating a nondimensional quantity and $b$ indicating the dimensional units. A symbol can be used as a unit without declaring it as such; unit symbols need not have any special properties. The quantity and unit of an expression $a'b$ can be extracted by the qty and units functions, respectively.

A symbol may be declared to be a dimensional quantity, with specified quantity or specified units or both.

An expression $a'b''c$ converts from unit $b$ to unit $c$. ezunits has built-in conversions for SI base units, SI derived units, and some non-SI units. Unit conversions not already known to ezunits can be declared. The unit conversions known to ezunits are specified by the global variable known_unit_conversions, which comprises built-in and user-defined conversions. Conversions for products, quotients, and powers of units are derived from the set of known unit conversions.

As Maxima generally prefers exact numbers (integers or rationals) to inexact (float or bigfloat), so ezunits preserves exact numbers when they appear in dimensional quantities. All built-in unit conversions are expressed in terms of exact numbers; inexact numbers in declared conversions are coerced to exact.

There is no preferred system for display of units; input units are not converted to other units unless conversion is explicitly indicated. ezunits recognizes the prefixes m-, k-, M, and G- (for milli-, kilo-, mega-, and giga-) as applied to SI base units and SI derived units, but such prefixes are applied only when indicated by an explicit conversion.

Arithmetic operations on dimensional quantities are carried out by conventional rules for such operations.

- $(x'a) \ast (y'b)$ is equal to $(x \ast y)'(a \ast b)$.
- $(x'a) + (y'a)$ is equal to $(x + y)'a$.
- $(x'a)^y$ is equal to $x'^{y\ast a^y}$ when $y$ is nondimensional.

ezunits does not require that units in a sum have the same dimensions; such terms are not added together, and no error is reported.

ezunits includes functions for elementary dimensional analysis, namely the fundamental dimensions and fundamental units of a dimensional quantity, and computation of dimensionless quantities and natural units. The functions for dimensional analysis were adapted from similar functions in another package, written by Barton Willis.
For the purpose of dimensional analysis, a list of fundamental dimensions and an associated list of fundamental units are maintained; by default the fundamental dimensions are length, mass, time, charge, temperature, and quantity, and the fundamental units are the associated SI units, but other fundamental dimensions and units can be declared.

56.2 Introduction to physical_constants

`physical_constants` is a collection of physical constants, copied from CODATA 2006 recommended values (http://physics.nist.gov/ constants). `load(physical_constants)` loads this package, and loads `ezunits also, if it is not already loaded.

A physical constant is represented as a symbol which has a property which is the constant value. The constant value is a dimensional quantity, as represented by `ezunits`. The function `constvalue` fetches the constant value; the constant value is not the ordinary value of the symbol, so symbols of physical constants persist in evaluated expressions until their values are fetched by `constvalue`.

`physical_constants` includes some auxiliary information, namely, a description string for each constant, an estimate of the error of its numerical value, and a property for TeX display. To identify physical constants, each symbol has the `physical_constant` property; `propvars(physical_constant)` therefore shows the list of all such symbols.

`physical_constants` comprises the following constants.

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>%c</td>
<td>speed of light in vacuum</td>
</tr>
<tr>
<td>%mu_0</td>
<td>magnetic constant</td>
</tr>
<tr>
<td>%e_0</td>
<td>electric constant</td>
</tr>
<tr>
<td>%Z_0</td>
<td>characteristic impedance of vacuum</td>
</tr>
<tr>
<td>%G</td>
<td>Newtonian constant of gravitation</td>
</tr>
<tr>
<td>%h</td>
<td>Planck constant</td>
</tr>
<tr>
<td>%h_bar</td>
<td>Planck constant</td>
</tr>
<tr>
<td>%m_P</td>
<td>Planck mass</td>
</tr>
<tr>
<td>%T_P</td>
<td>Planck temperature</td>
</tr>
<tr>
<td>%l_P</td>
<td>Planck length</td>
</tr>
<tr>
<td>%t_P</td>
<td>Planck time</td>
</tr>
<tr>
<td>%e</td>
<td>elementary charge</td>
</tr>
<tr>
<td>%Phi_0</td>
<td>magnetic flux quantum</td>
</tr>
<tr>
<td>%G_0</td>
<td>conductance quantum</td>
</tr>
<tr>
<td>%K_J</td>
<td>Josephson constant</td>
</tr>
<tr>
<td>%R_K</td>
<td>von Klitzing constant</td>
</tr>
<tr>
<td>%mu_B</td>
<td>Bohr magneton</td>
</tr>
<tr>
<td>%mu_N</td>
<td>nuclear magneton</td>
</tr>
</tbody>
</table>
%alpha  fine-structure constant
%R_inf  Rydberg constant
%a_0  Bohr radius
%E_h  Hartree energy
%ratio_h_me  quantum of circulation
%m_e  electron mass
%N_A  Avogadro constant
%m_u  atomic mass constant
%F  Faraday constant
%R  molar gas constant
%%k  Boltzmann constant
%V_m  molar volume of ideal gas
%n_0  Loschmidt constant
%ratio_S0_R  Sackur-Tetrode constant (absolute entropy constant)
%sigma  Stefan-Boltzmann constant
%c_1  first radiation constant
%c_1L  first radiation constant for spectral radiance
%c_2  second radiation constant
%b  Wien displacement law constant
%b_prime  Wien displacement law constant

Reference: http://physics.nist.gov/constants

Examples:
The list of all symbols which have the physical_constant property.

(%i1) load ("physical_constants")$
(%i2) propvars (physical_constant);
(%o2) [%c, %mu_0, %e_0, %Z_0, %G, %h, %h_bar, %m_P, %T_P, %l_P, %t_P, %e, %Phi_0, %G_0, %K_J, %R_K, %mu_B, %mu_N, %alpha, %R_inf, %a_0, %E_h, %ratio_h_me, %m_e, %N_A, %m_u, %F, %R, %%k, %V_m, %n_0, %ratio_S0_R, %sigma, %c_1, %c_1L, %c_2, %b, %b_prime]

Properties of the physical constant %c.

(%i1) load ("physical_constants")$
(%i2) constantp (%c);
(%o2) true
(%i3) get (%c, description);
(%o3) speed of light in vacuum
The energy equivalent of 1 pound-mass. The symbol \( \%c \) persists until its value is fetched by \texttt{constvalue}. 

\begin{verbatim}
(%i1) load ("physicalconstants")$
(%i2) m * \%c^2;
  2
(%o2) \%c m
(%i3) \%, m = 1 ' lbm;
  2
(%o3) \%c ' lbm
(%i4) constvalue (\%);
  2
(%o4) 89875517873681764 ' ------
2
(%i5) E : \% ' ' J;
Computing conversions to base units; may take a moment.
  3
(%o5) 366838848464007200 ' J
  9
(%i6) E ' ' GJ;
  4
(%o6) 458548560580009 ' GJ
  11250000
(%i7) float (\%);
(%o7) 4.0759872051556356e+7 ' GJ
\end{verbatim}

### 56.3 Functions and Variables for \texttt{ezunits}

\texttt{'}

The dimensional quantity operator. An expression \( a'\) represents a dimensional quantity, with \( a \) indicating a nondimensional quantity and \( b \) indicating the dimensional units. A symbol can be used as a unit without declaring it as such; unit symbols need not have any special properties. The quantity and unit of an expression \( a'\) can be extracted by the \texttt{qty} and \texttt{units} functions, respectively.

Arithmetic operations on dimensional quantities are carried out by conventional rules for such operations.
• \((x'a) \times (y'b)\) is equal to \((x \times y)'(a \times b)\).
• \((x'a) + (y'a)\) is equal to \((x + y)'a\).
• \((x'a)^y\) is equal to \(x^y'a^y\) when \(y\) is nondimensional.

ezunits does not require that units in a sum have the same dimensions; such terms are not added together, and no error is reported.

load(ezunits) enables this operator.

Examples:

**SI (Systeme Internationale) units.**

\[
\begin{align*}
\%i1 & \text{ load ("ezunits")$} \\
\%i2 & \text{ foo : 10 ' m;} \\
\%o2 & 10 ' m \\
\%i3 & \text{ qty (foo);} \\
\%o3 & 10 \\
\%i4 & \text{ units (foo);} \\
\%o4 & \text{ m} \\
\%i5 & \text{ dimensions (foo);} \\
\%o5 & \text{ length}
\end{align*}
\]

"Customary" units.

\[
\begin{align*}
\%i1 & \text{ load ("ezunits")$} \\
\%i2 & \text{ bar : x ' acre;} \\
\%o2 & \text{ x ' acre} \\
\%i3 & \text{ dimensions (bar);} \\
\%o3 & \text{ length} \\
\%i4 & \text{ fundamental_units (bar);} \\
\%o4 & \text{ m}
\end{align*}
\]

Units ad hoc.

\[
\begin{align*}
\%i1 & \text{ load ("ezunits")$} \\
\%i2 & \text{ baz : 3 ' sheep + 8 ' goat + 1 ' horse;} \\
\%o2 & \text{ 8 ' goat + 3 ' sheep + 1 ' horse} \\
\%i3 & \text{ subst ([sheep = 3*goat, horse = 10*goat], baz);} \\
\%o3 & \text{ 27 ' goat} \\
\%i4 & \text{ baz2 : 1000 ' gallon/fortnight;} \\
\%o4 & \text{ gallon} \\
\%i5 & \text{ subst (fortnight = 14*day, baz2);} \\
\%o5 & \text{ 500 gallon}
\end{align*}
\]

Arithmetic operations on dimensional quantities.

\[
\begin{align*}
\%i1 & \text{ load ("ezunits")$} \\
\%i2 & \text{ 100 ' kg + 200 ' kg;}
\end{align*}
\]
The unit conversion operator. An expression $a^b^c$ converts from unit $b$ to unit $c$. \texttt{ezunits} has built-in conversions for SI base units, SI derived units, and some non-SI units. Unit conversions not already known to \texttt{ezunits} can be declared. The unit conversions known to \texttt{ezunits} are specified by the global variable \texttt{known_unit_conversions}, which comprises built-in and user-defined conversions. Conversions for products, quotients, and powers of units are derived from the set of known unit conversions.

There is no preferred system for display of units; input units are not converted to other units unless conversion is explicitly indicated. \texttt{ezunits} does not attempt to simplify units by prefixes (milli-, centi-, deci-, etc) unless such conversion is explicitly indicated.

\texttt{load(ezunits)} enables this operator.

Examples:

The set of known unit conversions.

\begin{verbatim}
(\%i1) load ("ezunits")$
(\%i2) display2d : false$
(\%i3) known_unit_conversions;
(\%o3) {acre = 4840*yard^2,Btu = 1055*J,cfm = feet^3/minute,
     cm = m/100,day = 86400*s,feet = 381*m/1250,ft = feet,
     g = kg/1000,gallon = 757*l/200,GHz = 1000000000*Hz,
     GOhm = 1000000000*Ohm,GPa = 1000000000*Pa,
     GWb = 1000000000*Wb,Gg = 1000000*kg,Gm = 1000000000*m,
     Gmol = 1000000*mol,Gs = 1000000000*s,ha = hectare,
     hectare = 100*m^2,hour = 3600*s,Hz = 1/s,inch = feet/12,
     km = 1000*m,kmol = 1000*mol,ks = 1000*s,l = liter,
     lbf = pound_force,lbm = pound_mass,liter = m^3/1000,
     metric_ton = Mg,mg = kg/1000000,MHz = 1000000*Hz,
     microgram = kg/1000000000,micrometer = m/1000000},
\end{verbatim}
micron = micrometer, microsecond = s/1000000, 
mile = 5280*feet, minute = 60*s, mm = m/1000, 
mmol = mol/1000, month = 2629800*s, MOhm = 1000000*Ohm, 
MPa = 1000000*Pa, ms = s/1000, MWb = 1000000*Wb, 
Mg = 1000*kg, Mm = 1000000*m, Mmol = 1000000000*mol, 
Ms = 1000000*s, ns = s/1000000000, ounce = pound_mass/16, 
oz = ounce, Ohm = s*J/C^2, 
pound_force = 32*ft*pound_mass/s^2, 
pound_mass = 200*kg/441, psi = pound_force/inch^2, 
Pa = N/m^2, week = 604800*s, Wb = J/A, yard = 3*feet, 
year = 31557600*s, C = s*A, F = C^2/J, GA = 1000000000*A, 
GC = 1000000000*C, GF = 1000000000*F, GH = 1000000000*H, 
GJ = 1000000000*J, GK = 1000000000*K, GN = 1000000000*N, 
GS = 1000000000*S, GT = 1000000000*T, GV = 1000000000*V, 
GW = 1000000000*W, H = J/A^2, J = m*N, kA = 1000*A, 
kC = 1000*C, kF = 1000*F, kH = 1000*H, kHz = 1000*Hz, 
kj = 1000*J, kK = 1000*K, kN = 1000*N, kOhm = 1000*Ohm, 
kPa = 1000*Pa, kS = 1000*S, kT = 1000*T, kV = 1000*V, 
kW = 1000*W, kwb = 1000*Wb, mA = A/1000, mC = C/1000, 
mF = F/1000, mHz = Hz/1000, mJ = J/1000, 
mK = K/1000, mN = N/1000, mOhm = Ohm/1000, mPa = Pa/1000, 
mS = S/1000, mT = T/1000, mV = V/1000, mW = W/1000, 
mWb = Wb/1000, MA = 1000000*A, MC = 1000000*C, 
MF = 1000000*F, MH = 1000000*H, MJ = 1000000*J, 
MK = 1000000*K, MN = 1000000*N, MS = 1000000*S, 
MT = 1000000*T, MV = 1000000*V, MW = 1000000*W, 
N = kg*m/s^2, R = 5*K/9, S = 1/Ohm, T = J/(m^2*A), V = J/C, 
W = J/s

Elementary unit conversions.

(%i1) load ("ezunits")$
(%i2) 1 ' ft ' ' m;
   Computing conversions to base units; may take a moment.

  381
(381) ---- ' m
   1250

(%i3) %, numer;

(0.3048) ' m

(%i4) 1 ' kg ' ' lbm;

(441)

(%o4) ---- ' lbm
   200

(%i5) %, numer;

(2.205) ' lbm

(%i6) 1 ' W ' ' Btu/hour;

(720) Btu

(%o6) ----'
211 hour

(%i7) %, numer;
3.412322274881517 \text{ Btu} \quad \text{hour}

(%o7)

(%i8) 100 ' degC ' degF;
212 ' degF

(%o8)

(%i9) -40 ' degF ' degC;
(- 40) ' degC

(%o9)

(%i10) 1 ' acre*ft ' m^3;
60228605349 3

(%o10) 

(%i11) %, numer;
3

(%o11) 1233.48183754752 ' m

Coercing quantities in feet and meters to one or the other.

(%i1) load ("ezunits")$

(%i2) 100 ' m + 100 ' ft;

(%o2) 100 ' m + 100 ' ft

(%i3) (100 ' m + 100 ' ft) ' ft;

Computing conversions to base units; may take a moment.

163100

(%o3) 

(%i4) %, numer;
381

(%o4) 428.0839895013123 ' ft

(%i5) (100 ' m + 100 ' ft) ' m;

3262

(%o5) 

(%i6) %, numer;
25

(%o6) 130.48 ' m

Dimensional analysis to find fundamental dimensions and fundamental units.

(%i1) load ("ezunits")$

(%i2) foo : 1 ' acre * ft;

(%o2) 1 ' acre ft

(%i3) dimensions (foo);

3

(%o3) length

(%i4) fundamental_units (foo);

3

(%o4) m

(%i5) foo ' m^3;

Computing conversions to base units; may take a moment.

60228605349 3
%o5) ----------- ' m
        48828125

%o6) 3, numer;

%o6) 1233.48183754752 ' m

Declared unit conversions.

%i1) load ("ezunits")$
%i2) declare_unit_conversion (MMBtu = 10^6*Btu, kW = 1000*W);
%o2) done
%i3) declare_unit_conversion (kWh = kW*hour, MWh = 1000*kWh,
        bell = 1800*s);
%o3) done
%i4) 1 ' kW*s '' MWh;
Computing conversions to base units; may take a moment.

%o4) ------- ' MWh
        3600000

%i5) 1 ' kW/m^2 '' MMBtu/bell/ft^2;

%o5) ----------- ' --------
        8242187500 2
        bell ft

constvalue (x) [Function]

Shows the value and the units of one of the constants declared by package physical_
constants, which includes a list of physical constants, or of a new constant declared
in package ezunits (see declare_constvalue).

Note that constant values as recognized by constvalue are separate from values
declared by numerval and recognized by constantp.

Example:

%i1) load ("physical_constants")$
%i2) constvalue (%G);

%o2) 6.67428 ' -----
        2
        kg s

%i3) get ('%G, 'description);
%o3) Newtonian constant of gravitation

declare_constvalue (a, x) [Function]

Declares the value of a constant to be used in package ezunits. This function should
be loaded with load(ezunits).

Example:

%i1) load ("ezunits")$
%i2) declare_constvalue (FOO, 100 ' lbm / acre);
(\%o2) \quad \text{lbm} \\
(\%o2) \quad 100 \ ' \text{ acre} \\
(\%i3) \quad \text{FOO} \times (50 \ ' \text{ acre}); \\
(\%o3) \quad 50 \ \text{FOO} \ ' \text{ acre} \\
(\%i4) \quad \text{constvalue (\%)}; \\
(\%o4) \quad 5000 \ ' \text{ lbm} \\

\text{remove\_constvalue (a)} \quad \text{[Function]} \\
\text{Reverts the effect of declare\_constvalue. This function should be loaded with load(ezunits).} \\

\text{units (x)} \quad \text{[Function]} \\
\text{Returns the units of a dimensional quantity x, or returns 1 if x is nondimensional.} \\
x \text{ may be a literal dimensional expression } a' b, \text{ a symbol with declared units via declare\_units, or an expression containing either or both of those.} \\
\text{This function should be loaded with load(ezunits).} \\
\text{Example:} \\
(\%i1) \quad \text{load ("ezunits")}$ \\
(\%i2) \quad \text{foo : 100 ' kg}; \\
(\%o2) \quad 100 \ ' \text{ kg} \\
(\%i3) \quad \text{bar : x ' m/s}; \\
(\%o3) \quad x \ ' \text{ m/s} \\
(\%i4) \quad \text{units (foo)}; \\
(\%o4) \quad \text{kg} \\
(\%i5) \quad \text{units (bar)}; \\
(\%o5) \quad \text{m} \\
(\%i6) \quad \text{units (foo * bar)}; \\
(\%o6) \quad \text{kg m} \\
(\%i7) \quad \text{units (foo / bar)}; \\
(\%o7) \quad \text{kg s} \\
(\%i8) \quad \text{units (foo^2)}; \\
(\%o8) \quad 2 \\

\text{declare\_units (a, u)} \quad \text{[Function]} \\
\text{Declares that units should return units u for a, where u is an expression. This function should be loaded with load(ezunits).}
Example:

```lisp
(%i1) load ("ezunits")$
(%i2) units (aa);
(%o2) 1
(%i3) declare_units (aa, J);
(%o3) J
(%i4) units (aa);
(%o4) J
(%i5) units (aa^2);
(%o5) J
(%i6) foo : 100 ' kg;
(%o6) 100 ' kg
(%i7) units (aa * foo);
(%o7) kg J
```

**qty (x)**

Returns the nondimensional part of a dimensional quantity x, or returns x if x is nondimensional. x may be a literal dimensional expression a'b, a symbol with declared quantity, or an expression containing either or both of those.

This function should be loaded with load(ezunits).

Example:

```lisp
(%i1) load ("ezunits")$
(%i2) foo : 100 ' kg;
(%o2) 100 ' kg
(%i3) qty (foo);
(%o3) 100
(%i4) bar : v ' m/s;
(%o4) m
(%i5) foo * bar;
(%o5) kg m
(%i6) qty (foo * bar);
(%o6) kg m
```

**declare_qty (a, x)**

Declares that qty should return x for symbol a, where x is a nondimensional quantity.

This function should be loaded with load(ezunits).

Example:

```lisp
(%i1) load ("ezunits")$
(%i2) declare_qty (aa, xx);
(%o2) xx
(%i3) qty (aa);
```
(\%o3) \quad \text{xx}
(\%i4) \text{qty (aa^2);} \quad 2
(\%o4) \quad \text{xx}
(\%i5) \text{foo : 100 ' kg;}
(\%o5) \quad 100 \ ' \text{kg}
(\%i6) \text{qty (aa * foo);} \quad 100 \ \text{xx}

\text{unitp (x)} \quad \text{[Function]}

Returns \text{true} if \(x\) is a literal dimensional expression, a symbol declared dimensional, or an expression in which the main operator is declared dimensional. \text{unitp} returns \text{false} otherwise.

\text{load(ezunits)} loads this function.

\text{Examples:}

\text{unitp} applied to a literal dimensional expression.

(\%i1) \text{load ("ezunits")}$
(\%i2) \text{unitp (100 ' kg);}$
(\%o2) \quad \text{true}

\text{unitp} applied to a symbol declared dimensional.

(\%i1) \text{load ("ezunits")}$
(\%i2) \text{unitp (foo);}$
(\%o2) \quad \text{false}
(\%i3) \text{declare (foo, dimensional);}$
(\%o3) \quad \text{done}
(\%i4) \text{unitp (foo);}$
(\%o4) \quad \text{true}

\text{unitp} applied to an expression in which the main operator is declared dimensional.

(\%i1) \text{load ("ezunits")}$
(\%i2) \text{unitp (bar (x, y, z));}$
(\%o2) \quad \text{false}
(\%i3) \text{declare (bar, dimensional);}$
(\%o3) \quad \text{done}
(\%i4) \text{unitp (bar (x, y, z));}$
(\%o4) \quad \text{true}

\text{declare_unit_conversion (u = v, ...)} \quad \text{[Function]}

Appends equations \(u = v, \ldots\) to the list of unit conversions known to the unit conversion operator ". \(u\) and \(v\) are both multiplicative terms, in which any variables are units, or both literal dimensional expressions.

At present, it is necessary to express conversions such that the left-hand side of each equation is a simple unit (not a multiplicative expression) or a literal dimensional expression with the quantity equal to 1 and the unit being a simple unit. This limitation might be relaxed in future versions.

\text{known_unit_conversions} is the list of known unit conversions.
This function should be loaded with load(ezunits).

Examples:

Unit conversions expressed by equations of multiplicative terms.

```
(%i1) load ("ezunits")$
(%i2) declare_unit_conversion (nautical_mile = 1852 * m, 
   fortnight = 14 * day);
(%o2) done
(%i3) 100 ' nautical_mile / fortnight '' m/s;
   Computing conversions to base units; may take a moment.
   463    m
   ---- ' -
3024    s
```

Unit conversions expressed by equations of literal dimensional expressions.

```
(%i1) load ("ezunits")$
(%i2) declare_unit_conversion (1 ' fluid_ounce = 2 ' tablespoon);
(%o2) done
(%i3) declare_unit_conversion (1 ' tablespoon = 3 ' teaspoon);
(%o3) done
(%i4) 15 ' fluid_ounce '' teaspoon;
   Computing conversions to base units; may take a moment.
   90 ' teaspoon
```

Declare_dimensions (a_1, d_1, ..., a_n, d_n)  [Function]

Declares a_1, ..., a_n to have dimensions d_1, ..., d_n, respectively.

Each a_k is a symbol or a list of symbols. If it is a list, then every symbol in a_k is declared to have dimension d_k.

load(ezunits) loads these functions.

Examples:

```
(%i1) load ("ezunits") $
(%i2) declare_dimensions ([x, y, z], length, [t, u], time);
(%o2) done
(%i3) dimensions (y^2/u);
2   length
   ------
   time

(%i4) fundamental_units (y^2/u);
   0 errors, 0 warnings
     2
   m

(%o4) --
    s
```

Remove_dimensions (a_1, ..., a_n)  [Function]

Reverts the effect of declare_dimensions. This function should be loaded with load(ezunits).
declare_fundamental_dimensions (d_1, d_2, d_3, ...)     [Function]
remove_fundamental_dimensions (d_1, d_2, d_3, ...)     [Function]
fundamental_dimensions                                      [Global variable]
declare_fundamental_dimensions declares fundamental dimensions. Symbols d_1, d_2, d_3, ... are appended to the list of fundamental dimensions, if they are not already on the list.
remove_fundamental_dimensions reverts the effect of declare_fundamental_dimensions.
fundamental_dimensions is the list of fundamental dimensions. By default, the list comprises several physical dimensions.
load(ezunits) loads these functions.

Examples:
(\%i1) load ("ezunits") $
(\%i2) fundamental_dimensions;
(\%o2) [length, mass, time, current, temperature, quantity]
(\%i3) declare_fundamental_dimensions (money, cattle, happiness);
(\%o3) done
(\%i4) fundamental_dimensions;
(\%o4) [length, mass, time, current, temperature, quantity, money, cattle, happiness]
(\%i5) remove_fundamental_dimensions (cattle, happiness);
(\%o5) done
(\%i6) fundamental_dimensions;
(\%o6) [length, mass, time, current, temperature, quantity, money]

declare_fundamental_units (u_1, d_1, ..., u_n, d_n)     [Function]
remove_fundamental_units (u_1, ..., u_n)     [Function]
declare_fundamental_units declares u_1, ..., u_n to have dimensions d_1, ..., d_n, respectively. All arguments must be symbols.
After calling declare_fundamental_units, dimensions(u_k) returns d_k for each argument u_1, ..., u_n, and fundamental_units(d_k) returns u_k for each argument d_1, ..., d_n.
remove_fundamental_units reverts the effect of declare_fundamental_units.
load(ezunits) loads these functions.

Examples:
(\%i1) load ("ezunits") $
(\%i2) declare_fundamental_dimensions (money, cattle, happiness);
(\%o2) done
(\%i3) declare_fundamental_units (dollar, money, goat, cattle, smile, happiness);
(\%o3) [dollar, goat, smile]
(\%i4) dimensions (100 ' dollar/goat/km^-2);
       money
(\%o4) -------------------
        2
cattle length
(\%i5) dimensions \( (x \ ' \ smile/kg) \);
  \[ 0 \text{ errors, 0 warnings} \]
mass
(\%o6) dollar goat
  \[ \text{smile} \]

\textbf{dimensions \( (x) \) \hspace{1cm} \textbf{dimensions\_as\_list \( (x) \)]}

\textit{dimensions} returns the dimensions of the dimensional quantity \( x \) as an expression comprising products and powers of base dimensions.

\textit{dimensions\_as\_list} returns the dimensions of the dimensional quantity \( x \) as a list, in which each element is an integer which indicates the power of the corresponding base dimension in the dimensions of \( x \).

load(ezunits) loads these functions.

Examples:

(\%i1) load ("ezunits")$
(\%i2) dimensions (1000 \ ' \ \text{kg} \cdot \text{m}^2/\text{s}^3);
  \[ 2 \text{ length mass} \]
2
(\%o2) \[ 3 \text{ time} \]
3
(\%i3) declare\_units (foo, \text{acre ft/hour});
  \[ \text{acre ft} \]
\[ \text{hour} \]
(\%o3)
(\%i4) dimensions (foo);
  \[ 3 \text{ length} \]
3
(\%o4) \[ \text{time} \]
(\%i1) load ("ezunits")$
(\%i2) fundamental\_dimensions;
(\%o2) \[ \text{length, mass, time, charge, temperature, quantity} \]
(\%i3) dimensions\_as\_list (1000 \ ' \ \text{kg} \cdot \text{m}^2/\text{s}^3);
(\%o3) \[ 2, 1, - 3, 0, 0, 0 \]
(\%i4) declare\_units (foo, \text{acre ft/hour});
  \[ \text{acre ft} \]
\[ \text{hour} \]
(\%o4)
(\%i5) dimensions\_as\_list (foo);
fundamental_units

fundamental_units(x)

fundamental_units() returns the units associated with the fundamental dimensions of x, as determined by dimensions(x).

x may be a literal dimensional expression $a'b$, a symbol with declared units via declare_units, or an expression containing either or both of those.

fundamental_units() returns the list of all known fundamental units, as declared by declare_fundamental_units.

load(ezunits) loads this function.

Examples:

(%i1) load("ezunits")$
(%i2) fundamental_units();
(%o2) [m, kg, s, A, K, mol]
(%i3) fundamental_units (100 ' mile/hour);

m
(%o3) —
s
(%i4) declare_units (aa, g/foot^2);

g
(%o4) ----- 2
     foot
(%i5) fundamental_units (aa);

kg
(%o5) --
     2
     m
dimensionless (L)

Returns a basis for the dimensionless quantities which can be formed from a list L of dimensional quantities.

load(ezunits) loads this function.

Examples:

(%i1) load("ezunits")$
(%i2) dimensionless ([x ' m, y ' m/s, z ' s]);

0 errors, 0 warnings
0 errors, 0 warnings

y z
(%o2) [---]
x

Dimensionless quantities derived from fundamental physical quantities. Note that the first element on the list is proportional to the fine-structure constant.
Chapter 56: ezunits

(%i2) load ("physical_constants")$
(%i3) dimensionless([%h_bar, %m_e, %m_P, %%e, %c, %e_0]);
0 errors, 0 warnings
0 errors, 0 warnings

(%o3) [--------------, ----]
      %c %e_0 %h_bar %m_P

natural_unit (expr, [v_1, ..., v_n]) [Function]

Finds exponents e_1, ..., e_n such that dimension(expr) = dimension(v_1^e_1 \ldots v_n^e_n).
load(ezunits) loads this function.

Examples:
57 f90

57.1 Functions and Variables for f90

f90 (expr_1, ..., expr_n) [Function]
Prints one or more expressions expr_1, ..., expr_n as a Fortran 90 program. Output is printed to the standard output.

f90 prints output in the so-called "free form" input format for Fortran 90: there is no special attention to column positions. Long lines are split at a fixed width with the ampersand & continuation character.

load(f90) loads this function. See also the function fortran.

Examples:

```
(%i1) load ("f90")$
(%i2) foo : expand ((xxx + yyy + 7)^4);
   4  3  2
(%o2) yyy + 4 xxx yyy + 28 yyy + 6 xxx yyy + 84 xxx yyy
   2  3  2
  + 294 yyy + 4 xxx yyy + 84 xxx yyy + 588 xxx yyy + 1372 yyy
  4  3  2
 + xxx + 28 xxx + 294 xxx + 1372 xxx + 2401
(%i3) f90 ('foo = foo);
foo = yyy**4+4*xxx*yyy**3+28*yyy**3+6*xxx**2*yyy**2+84*xxx*yyy**2&
 +294*yyy**2+4*xxx**3*yyy+84*xxx**2*yyy+588*xxx*yyy+1372*yyy+xxx**&
 4+28*xxx**3+294*xxx**2+1372*xxx+2401
(%o3) false
```

Multiple expressions. Capture standard output into a file via the with_stdout function.

```
(%i1) load ("f90")$
(%i2) foo : sin (3*x + 1) - cos (7*x - 2);
(%o2) sin(3 x + 1) - cos(7 x - 2)
(%i3) with_stdout ("foo.f90",
  f90 (x=0.25, y=0.625, 'foo=foo, 'stop, 'end));
(%o3) false
(%i4) printfile ("foo.f90");
  x = 0.25
  y = 0.625
  foo = sin(3*x+1)-cos(7*x-2)
  stop
  end
(%o4) foo.f90
```
58 finance

58.1 Introduction to finance

This is the Finance Package (Ver 0.1).

In all the functions, rate is the compound interest rate, num is the number of periods and must be positive and flow refers to cash flow so if you have an Output the flow is negative and positive for Inputs.

Note that before using the functions defined in this package, you have to load it writing load(finance)$.

Author: Nicolas Guarin Zapata.

58.2 Functions and Variables for finance

days360 (year1,month1,day1,year2,month2,day2) [Function]
Calculates the distance between 2 dates, assuming 360 days years, 30 days months.
Example:
(%i1) load(finance)$
(%i2) days360(2008,12,16,2007,3,25);
(%o2) - 621

fv (rate,PV,num) [Function]
We can calculate the future value of a Present one given a certain interest rate. rate is the interest rate, PV is the present value and num is the number of periods.
Example:
(%i1) load(finance)$
(%i2) fv(0.12,1000,3);
(%o2) 1404.928

pv (rate,FV,num) [Function]
We can calculate the present value of a Future one given a certain interest rate. rate is the interest rate, FV is the future value and num is the number of periods.
Example:
(%i1) load(finance)$
(%i2) pv(0.12,1000,3);
(%o2) 711.7802478134108

graph_flow (val) [Function]
Plots the money flow in a time line, the positive values are in blue and upside; the negative ones are in red and downside. The direction of the flow is given by the sign of the value. val is a list of flow values.
Example:
(%i1) load(finance)$
(%i2) graph_flow([-5000,-3000,800,1300,1500,2000])$
annuity_pv (rate, PV, num)  
We can calculate the annuity knowing the present value (like an amount), it is a constant and periodic payment. rate is the interest rate, PV is the present value and num is the number of periods.

Example:
(%i1) load(finance)$
(%i2) annuity_pv(0.12, 5000, 10);
(%o2) 884.9208207992202

annuity_fv (rate, FV, num)  
We can calculate the annuity knowing the desired value (future value), it is a constant and periodic payment. rate is the interest rate, FV is the future value and num is the number of periods.

Example:
(%i1) load(finance)$
(%i2) annuity_fv(0.12, 65000, 10);
(%o2) 3703.970670389863

gEO_annuity_pv (rate, growing_rate, PV, num)  
We can calculate the annuity knowing the present value (like an amount), in a growing periodic payment. rate is the interest rate, growing_rate is the growing rate, PV is the present value and num is the number of periods.

Example:
(%i1) load(finance)$
(%i2) geo_annuity_pv(0.14, 0.05, 5000, 10);
(%o2) 802.6888176505123

gEO_annuity_fv (rate, growing_rate, FV, num)  
We can calculate the annuity knowing the desired value (future value), in a growing periodic payment. rate is the interest rate, growing_rate is the growing rate, FV is the future value and num is the number of periods.

Example:
(%i1) load(finance)$
(%i2) geo_annuity_fv(0.14, 0.05, 5000, 10);
(%o2) 216.5203395312695

amortization (rate, amount, num)  
Amortization table determined by a specific rate. rate is the interest rate, amount is the amount value, and num is the number of periods.

Example:
(%i1) load(finance)$
(%i2) amortization(0.05, 56000, 12)$

<table>
<thead>
<tr>
<th>n</th>
<th>Balance</th>
<th>Interest</th>
<th>Amortization</th>
<th>Payment</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.000</td>
<td>56000.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
</tr>
<tr>
<td>1.000</td>
<td>52481.777</td>
<td>2800.000</td>
<td>3518.223</td>
<td>6318.223</td>
</tr>
<tr>
<td>2.000</td>
<td>48787.643</td>
<td>2624.089</td>
<td>3694.134</td>
<td>6318.223</td>
</tr>
</tbody>
</table>
### Chapter 58: finance

#### 3.000
3.000 44908.802 2439.382 3878.841 6318.223

#### 4.000
4.000 40836.019 2245.440 4072.783 6318.223

#### 5.000
5.000 36559.597 2041.801 4276.422 6318.223

#### 6.000
6.000 32069.354 1827.980 4490.243 6318.223

#### 7.000
7.000 27354.599 1603.468 4714.755 6318.223

#### 8.000
8.000 22404.106 1367.730 4950.493 6318.223

#### 9.000
9.000 17206.088 1120.205 5198.018 6318.223

#### 10.000
10.000 11748.170 860.304 5457.919 6318.223

#### 11.000
11.000 6017.355 587.408 5730.814 6318.223

#### 12.000
12.000 0.000 300.868 6017.355 6318.223

#### arit_amortization (rate,increment,amount,num)

The amortization table determined by a specific rate and with growing payment can be calculated by `arit_amortization`. Notice that the payment is not constant, it presents an arithmetic growing, increment is then the difference between two consecutive rows in the "Payment" column. `rate` is the interest rate, `increment` is the increment, `amount` is the amount value, and `num` is the number of periods.

Example:

```lisp
(%i1) load(finance)$
(%i2) arit_amortization(0.05,1000,56000,12)$
```

<table>
<thead>
<tr>
<th>n</th>
<th>Balance</th>
<th>Interest</th>
<th>Amortization</th>
<th>Payment</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.000</td>
<td>56000.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
</tr>
<tr>
<td>1.000</td>
<td>57403.679</td>
<td>2800.000</td>
<td>-1403.679</td>
<td>1396.321</td>
</tr>
<tr>
<td>2.000</td>
<td>57877.541</td>
<td>2870.184</td>
<td>-473.863</td>
<td>2396.321</td>
</tr>
<tr>
<td>3.000</td>
<td>57375.097</td>
<td>2893.877</td>
<td>502.444</td>
<td>3396.321</td>
</tr>
<tr>
<td>4.000</td>
<td>55847.530</td>
<td>2868.755</td>
<td>1527.567</td>
<td>4396.321</td>
</tr>
<tr>
<td>5.000</td>
<td>53243.586</td>
<td>2792.177</td>
<td>2603.945</td>
<td>5396.321</td>
</tr>
<tr>
<td>6.000</td>
<td>50590.443</td>
<td>2662.179</td>
<td>3734.142</td>
<td>6396.321</td>
</tr>
<tr>
<td>7.000</td>
<td>49381.594</td>
<td>2475.472</td>
<td>4920.849</td>
<td>7396.321</td>
</tr>
<tr>
<td>8.000</td>
<td>43821.703</td>
<td>2229.430</td>
<td>6166.892</td>
<td>8396.321</td>
</tr>
<tr>
<td>9.000</td>
<td>39941.466</td>
<td>1921.085</td>
<td>7475.236</td>
<td>9396.321</td>
</tr>
<tr>
<td>10.000</td>
<td>22097.468</td>
<td>1547.323</td>
<td>8848.998</td>
<td>10396.321</td>
</tr>
<tr>
<td>11.000</td>
<td>11806.020</td>
<td>1104.873</td>
<td>10291.448</td>
<td>11396.321</td>
</tr>
<tr>
<td>12.000</td>
<td>0.000</td>
<td>590.301</td>
<td>11806.020</td>
<td>12396.321</td>
</tr>
</tbody>
</table>

#### geo_amortization (rate,growing_rate,amount,num)

The amortization table determined by rate, amount, and number of periods can be found by `geo_amortization`. Notice that the payment is not constant, it presents a geometric growing, `growing_rate` is then the quotient between two consecutive rows in the "Payment" column. `rate` is the interest rate, `amount` is the amount value, and `num` is the number of periods.

Example:

```lisp
(%i1) load(finance)$
(%i2) geo_amortization(0.05,0.03,56000,12)$
```

<table>
<thead>
<tr>
<th>n</th>
<th>Balance</th>
<th>Interest</th>
<th>Amortization</th>
<th>Payment</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.000</td>
<td>56000.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
</tr>
<tr>
<td>1.000</td>
<td>53365.296</td>
<td>2800.000</td>
<td>2634.704</td>
<td>5434.704</td>
</tr>
</tbody>
</table>
The table that represents the values in a constant and periodic saving can be found by `saving`. `amount` represents the desired quantity and `num` the number of periods to save.

Example:
```
(%i1) load(finance)$
(%i2) saving(0.15,12000,15)$
```

<table>
<thead>
<tr>
<th>n</th>
<th>Balance</th>
<th>Interest</th>
<th>Payment</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>1.00</td>
<td>252.205</td>
<td>0.00</td>
<td>252.205</td>
</tr>
<tr>
<td>2.00</td>
<td>542.240</td>
<td>37.831</td>
<td>252.205</td>
</tr>
<tr>
<td>3.00</td>
<td>875.781</td>
<td>81.336</td>
<td>252.205</td>
</tr>
<tr>
<td>4.00</td>
<td>1259.352</td>
<td>131.367</td>
<td>252.205</td>
</tr>
<tr>
<td>5.00</td>
<td>1700.460</td>
<td>188.903</td>
<td>252.205</td>
</tr>
<tr>
<td>6.00</td>
<td>2207.733</td>
<td>255.069</td>
<td>252.205</td>
</tr>
<tr>
<td>7.00</td>
<td>2791.098</td>
<td>331.160</td>
<td>252.205</td>
</tr>
<tr>
<td>8.00</td>
<td>3461.967</td>
<td>418.665</td>
<td>252.205</td>
</tr>
<tr>
<td>9.00</td>
<td>4233.467</td>
<td>519.295</td>
<td>252.205</td>
</tr>
<tr>
<td>10.00</td>
<td>5120.692</td>
<td>635.020</td>
<td>252.205</td>
</tr>
<tr>
<td>11.00</td>
<td>6141.000</td>
<td>768.104</td>
<td>252.205</td>
</tr>
<tr>
<td>12.00</td>
<td>7314.355</td>
<td>921.150</td>
<td>252.205</td>
</tr>
<tr>
<td>13.00</td>
<td>8663.713</td>
<td>1097.153</td>
<td>252.205</td>
</tr>
<tr>
<td>14.00</td>
<td>10215.474</td>
<td>1299.557</td>
<td>252.205</td>
</tr>
<tr>
<td>15.00</td>
<td>12000.000</td>
<td>1532.321</td>
<td>252.205</td>
</tr>
</tbody>
</table>

The present value of a value series to evaluate the viability in a project. `val` is a list of varying cash flows.

Example:
```
(%i1) load(finance)$
(%i2) npv(0.25,[100,500,323,124,300]);
```

```
(%o2) 714.4703999999999
```

IRR (Internal Rate of Return) is the value of rate which makes Net Present Value zero. `flowValues` is a list of varying cash flows, `i0` is the initial investment.

```
(%i1) load(finance)$
(%i2) irr([11000,7500,1000,2000,1500],0.25);
```

```
(%o2) 7.427030177300862
```
Example:

```lisp
(%i1) load(finance)$
(%i2) res:irr([-5000,0,800,1300,1500,2000],0)$
(%i3) rhs(res[1][1]);
(%o3) 0.03009250374237132
```

**benefit_cost (rate,input,output)**

[Function]
Calculates the ratio Benefit/Cost. Benefit is the Net Present Value (NPV) of the inputs, and Cost is the Net Present Value (NPV) of the outputs. Notice that if there is not an input or output value in a specific period, the input/output would be a zero for that period. *rate* is the interest rate, *input* is a list of input values, and *output* is a list of output values.

Example:

```lisp
(%i1) load(finance)$
(%i2) benefit_cost(0.24,[0,300,500,150],[100,320,0,180]);
(%o2) 1.427249324905784
```
59 fractals

59.1 Introduction to fractals

This package defines some well known fractals:

- with random IFS (Iterated Function System): the Sierpinsky triangle, a Tree and a Fern
- Complex Fractals: the Mandelbrot and Julia Sets
- the Koch snowflake sets
- Peano maps: the Sierpinski and Hilbert maps

Author: José Ramírez Labrador.
For questions, suggestions and bugs, please feel free to contact me at
pepe DOT ramirez AAATTT uca DOT es

59.2 Definitions for IFS fractals

Some fractals can be generated by iterative applications of contractive affine transformations in a random way; see


We define a list with several contractive affine transformations, and we randomly select the transformation in a recursive way. The probability of the choice of a transformation must be related with the contraction ratio.

You can change the transformations and find another fractal

\textbf{sierpinskiiale} \ (n) \quad \text{[Function]}

Sierpinski Triangle: 3 contractive maps; .5 contraction constant and translations; all maps have the same contraction ratio. Argument \( n \) must be great enough, 10000 or greater.

Example:

\begin{verbatim}
(%i1) load(fractals)$
(%i2) n: 10000$
(%i3) plot2d([discrete,sierpinskiiale(n)], [style,dots])$
\end{verbatim}

\textbf{treefale} \ (n) \quad \text{[Function]}

3 contractive maps all with the same contraction ratio. Argument \( n \) must be great enough, 10000 or greater.

Example:

\begin{verbatim}
(%i1) load(fractals)$
(%i2) n: 10000$
(%i3) plot2d([discrete,treefale(n)], [style,dots])$
\end{verbatim}

\textbf{fernfale} \ (n) \quad \text{[Function]}

4 contractive maps, the probability to choice a transformation must be related with the contraction ratio. Argument \( n \) must be great enough, 10000 or greater.
Example:

```maxima
(\%i1) load(fractals)$
(\%i2) n: 10000$
(\%i3) plot2d([discrete, fernfale(n)], [style, dots])$
```

59.3 Definitions for complex fractals

**mandelbrot_set (x, y)**

Mandelbrot set.

Example:

This program is time consuming because it must make a lot of operations; the computing time is also related with the number of grid points.

```maxima
(\%i1) load(fractals)$
(\%i2) plot3d (mandelbrot_set, [x, -2.5, 1], [y, -1.5, 1.5],
    [gnuplot_preamble, "set view map"],
    [gnuplot_pm3d, true],
    [grid, 150, 150])$
```

**julia_set (x, y)**

Julia sets.

This program is time consuming because it must make a lot of operations; the computing time is also related with the number of grid points.

Example:

```maxima
(\%i1) load(fractals)$
(\%i2) plot3d (julia_set, [x, -2, 1], [y, -1.5, 1.5],
    [gnuplot_preamble, "set view map"],
    [gnuplot_pm3d, true],
    [grid, 150, 150])$
```

See also **julia_parameter**.

**julia_parameter**

Default value: %i

Complex parameter for Julia fractals. Its default value is %i; we suggest the values -.745+%i*1.1303, -.39054-%i*.58679, -.15652+%i*1.03225, -.194+%i*.6557 and .011031-%i*.67037.

**julia_sin (x, y)**

While function julia_set implements the transformation julia_parameter+z^2, function julia_sin implements julia_parameter*sin(z). See source code for more details.

This program runs slowly because it calculates a lot of sines.

Example:

This program is time consuming because it must make a lot of operations; the computing time is also related with the number of grid points.

```maxima
(\%i1) load(fractals)$
```
Chapter 59: fractals

59.4 Definitions for Koch snowflakes

snowmap (ent, nn) [Function]
Koch snowflake sets. Function snowmap plots the snow Koch map over the vertex of an initial closed polygonal, in the complex plane. Here the orientation of the polygon is important. Argument nn is the number of recursive applications of Koch transformation; nn must be small (5 or 6).

Examples:

(\%i1) load(fractals)$
(\%i2) plot2d([discrete, snowmap([1,exp(\%i*\%pi*2/3),exp(-\%i*\%pi*2/3),1],4)])$
(\%i3) plot2d([discrete, snowmap([1,exp(-\%i*\%pi*2/3),exp(\%i*\%pi*2/3),1],4)])$
(\%i4) plot2d([discrete, snowmap([0,1,1+%i,%i,0],4)])$
(\%i5) plot2d([discrete, snowmap([0,%i,1+%i,1,0],4)])$

59.5 Definitions for Peano maps

Continuous curves that cover an area. Warning: the number of points exponentially grows with n.

hilbertmap (nn) [Function]
Hilbert map. Argument nn must be small (5, for example). Maxima can crash if nn is 7 or greater.

Example:

(\%i1) load(fractals)$
(\%i2) plot2d([discrete,hilbertmap(6)])$

sierpinskimap (nn) [Function]
Sierpinski map. Argument nn must be small (5, for example). Maxima can crash if nn is 7 or greater.

Example:

(\%i1) load(fractals)$
(\%i2) plot2d([discrete,sierpinskimap(6)])$
60  ggf

60.1 Functions and Variables for ggf

GGFINFINITY
[Option variable]
Default value: 3
This is an option variable for function ggf.
When computing the continued fraction of the generating function, a partial quotient
having a degree (strictly) greater than GGFINFINITY will be discarded and the
current convergent will be considered as the exact value of the generating function;
most often the degree of all partial quotients will be 0 or 1; if you use a greater
value, then you should give enough terms in order to make the computation accurate
enough.
See also ggf.

GGFCFMAX
[Option variable]
Default value: 3
This is an option variable for function ggf.
When computing the continued fraction of the generating function, if no good result
has been found (see the GGFINFINITY flag) after having computed GGFCFMAX
partial quotients, the generating function will be considered as not being a fraction
of two polynomials and the function will exit. Put freely a greater value for more
complicated generating functions.
See also ggf.

ggf  (1)
[Function]
Compute the generating function (if it is a fraction of two polynomials) of a sequence,
its first terms being given. I is a list of numbers.
The solution is returned as a fraction of two polynomials. If no solution has been
found, it returns with done.
This function is controlled by global variables GGFINFINITY and GGFCFMAX.
See also GGFINFINITY and GGFCFMAX.
To use this function write first load("ggf").
61 graphs

61.1 Introduction to graphs

The graphs package provides graph and digraph data structure for Maxima. Graphs and digraphs are simple (have no multiple edges nor loops), although digraphs can have a directed edge from \( u \) to \( v \) and a directed edge from \( v \) to \( u \).

Internally graphs are represented by adjacency lists and implemented as a lisp structures. Vertices are identified by their ids (an id is an integer). Edges/arcs are represented by lists of length 2. Labels can be assigned to vertices of graphs/digraphs and weights can be assigned to edges/arcs of graphs/digraphs.

There is a draw_graph function for drawing graphs. Graphs are drawn using a force based vertex positioning algorithm. draw_graph can also use graphviz programs available from http://www.graphviz.org. draw_graph is based on the maxima draw package.

To use the graphs package, first load it with load(graphs).

61.2 Functions and Variables for graphs

61.2.1 Building graphs

create_graph

\[\text{create}\_\text{graph}(v\_\text{list}, e\_\text{list})\]
\[\text{create}\_\text{graph}(n, e\_\text{list})\]
\[\text{create}\_\text{graph}(v\_\text{list}, e\_\text{list}, \text{directed})\]

Creates a new graph on the set of vertices \( v\_\text{list} \) and with edges \( e\_\text{list} \).

\( v\_\text{list} \) is a list of vertices ([v1, v2, ..., vn]) or a list of vertices together with vertex labels ([[v1,l1], [v2,l2],..., [vn,ln]])).

\( n \) is the number of vertices. Vertices will be identified by integers from 0 to n-1.

\( e\_\text{list} \) is a list of edges ([e1, e2, ..., em]) or a list of edges together with edge-weights ([[e1, w1], ..., [em, wm]])).

If \( \text{directed} \) is not false, a directed graph will be returned.

Example 1: create a cycle on 3 vertices:

\((\%i1) \text{ load } ("\text{graphs}\")\)
\((\%i2) \text{ g : create}\_\text{graph}([1,2,3], [[1,2], [2,3], [1,3]])\)
\((\%i3) \text{ print}\_\text{graph}(g)\)

Graph on 3 vertices with 3 edges.
Adjacencies:
\[
3 : 1 2 \\
2 : 3 1 \\
1 : 3 2
\]

Example 2: create a cycle on 3 vertices with edge weights:

\((\%i1) \text{ load } ("\text{graphs}\")\)
\((\%i2) \text{ g : create}\_\text{graph}([1,2,3], [[[1,2], 1.0], [[2,3], 2.0], [[1,3], 3.0]])\)
Example 3: create a directed graph:

```maxima
(%i1) load ("graphs")$
(%i2) d : create_graph(  
      [1,2,3,4],  
      [  
        [1,3], [1,4],  
        [2,3], [2,4]  
      ],  
      'directed = true)$
(%i3) print_graph(d)$
```

Digraph on 4 vertices with 4 arcs.
Adjacencies:
4 :
3 :
2 : 4 3
1 : 4 3

`copy_graph (g)`
Returns a copy of the graph g.

`circulant_graph (n, d)`
Returns the circulant graph with parameters n and d.

Example:

```maxima
(%i1) load ("graphs")$
(%i2) g : circulant_graph(10, [1,3])$
(%i3) print_graph(g)$
```

Graph on 10 vertices with 20 edges.
Adjacencies:
9 : 2 6 0 8
8 : 1 5 9 7
7 : 0 4 8 6
6 : 9 3 7 5
5 : 8 2 6 4
4 : 7 1 5 3
3 : 6 0 4 2
2 : 9 5 3 1
1 : 8 4 2 0
0 : 7 3 9 1

`clebsch_graph ()`
Returns the Clebsch graph.
complement_graph (g)  
Returns the complement of the graph g.

complete_bipartite_graph (n, m)  
Returns the complete bipartite graph on n+m vertices.

complete_graph (n)  
Returns the complete graph on n vertices.

cycle_digraph (n)  
Returns the directed cycle on n vertices.

cycle_graph (n)  
Returns the cycle on n vertices.

cuboctahedron_graph (n)  
Returns the cuboctahedron graph.

cube_graph (n)  
Returns the n-dimensional cube.

dodecahedron_graph ()  
Returns the dodecahedron graph.

empty_graph (n)  
Returns the empty graph on n vertices.

flower_snark (n)  
Returns the flower graph on 4n vertices.

Example:
(%i1) load ("graphs")$
(%i2) f5 : flower_snark(5)$
(%i3) chromatic_index(f5);
(%o3) 4

from_adjacency_matrix (A)  
Returns the graph represented by its adjacency matrix A.

frucht_graph ()  
Returns the Frucht graph.

graph_product (g1, g1)  
Returns the direct product of graphs g1 and g2.

Example:
(%i1) load ("graphs")$
(%i2) grid : graph_product(path_graph(3), path_graph(4))$
(%i3) draw_graph(grid)$

graph_union (g1, g1)  
Returns the union (sum) of graphs g1 and g2.
grid_graph (n, m)
  Returns the n x m grid.

great_rhombicosidodecahedron_graph ()
  Returns the great rhombicosidodecahedron graph.

great_rhombicuboctahedron_graph ()
  Returns the great rhombicuboctahedron graph.

grotzch_graph ()
  Returns the Grotzch graph.

heawood_graph ()
  Returns the Heawood graph.

icosahedron_graph ()
  Returns the icosahedron graph.

icosidodecahedron_graph ()
  Returns the icosidodecahedron graph.

induced_subgraph (V, g)
  Returns the graph induced on the subset V of vertices of the graph g.
  Example:
  (%i1) load ("graphs")$
  (%i2) p : petersen_graph()$
  (%i3) V : [0,1,2,3,4]$
  (%i4) g : induced_subgraph(V, p)$
  (%i5) print_graph(g)$
  Graph on 5 vertices with 5 edges.
  Adjacencies:
  4 : 3 0
  3 : 2 4
  2 : 1 3
  1 : 0 2
  0 : 1 4

line_graph (g)
  Returns the line graph of the graph g.

make_graph
  make_graph (vrt, f)
  make_graph (vrt, f, oriented)
  Creates a graph using a predicate function f.
  vrt is a list/set of vertices or an integer. If vrt is an integer, then vertices of the graph
  will be integers from 1 to vrt.
  f is a predicate function. Two vertices a and b will be connected if f(a,b)=true.
  If directed is not false, then the graph will be directed.
Chapter 61: graphs

Example 1:

```lisp
(%i1) load(graphs)$
(%i2) g : make_graph(powerset({1,2,3,4,5}, 2), disjointp)$
(%i3) is_isomorphic(g, petersen_graph());
     1
(%o3) true
(%i4) get_vertex_label(1, g);
     1
(%o4) {1, 2}
```

Example 2:

```lisp
(%i1) load(graphs)$
(%i2) f(i, j) := is (mod(j, i)=0)$
(%i3) g : make_graph(20, f, directed=true)$
(%i4) out_neighbors(4, g);
[8, 12, 16, 20]
(%i5) in_neighbors(18, g);
   1
[1, 2, 3, 6, 9]
```

**mycielski_graph** *(g)*

Returns the mycielskian graph of the graph *g*.

**new_graph** *

Returns the graph with no vertices and no edges.

**path_digraph** *(n)*

Returns the directed path on *n* vertices.

**path_graph** *(n)*

Returns the path on *n* vertices.

**petersen_graph** *

Returns the petersen graph *Pₙ,d*. The default values for *n* and *d* are **n=5** and **d=2**.

**random_bipartite_graph** *(a, b, p)*

Returns a random bipartite graph on *a+b* vertices. Each edge is present with probability *p*.

**random_digraph** *(n, p)*

Returns a random directed graph on *n* vertices. Each arc is present with probability *p*.

**random_regular_graph** *

Returns a random *d*-regular graph on *n* vertices. The default value for *d* is **d=3**.

**random_graph** *(n, p)*

Returns a random graph on *n* vertices. Each edge is present with probability *p*.
random_graph1 \((n, m)\)

Returns a random graph on \(n\) vertices and random \(m\) edges.

random_network \((n, p, w)\)

Returns a random network on \(n\) vertices. Each arc is present with probability \(p\) and has a weight in the range \([0, w]\). The function returns a list \([\text{network}, \text{source}, \text{sink}]\).

Example:

\[
\begin{align*}
(\text{i1}) \quad \text{load ("graphs")}$
(\text{i2}) \quad \text{[net, s, t]} : \text{random_network}(50, 0.2, 10.0); \\
(\text{o2}) \quad \text{[DIGRAPH, 50, 51]} \\
(\text{i3}) \quad \text{max_flow} (\text{net, s, t}); \\
(\text{o4}) \quad 27.65981397932507 \\
\end{align*}
\]

random_tournament \((n)\)

Returns a random tournament on \(n\) vertices.

random_tree \((n)\)

Returns a random tree on \(n\) vertices.

small_rhombicosidodecahedron_graph ()

Returns the small rhombicosidodecahedron graph.

small_rhombicuboctahedron_graph ()

Returns the small rhombicuboctahedron graph.

snub_cube_graph ()

Returns the snub cube graph.

snub_dodecahedron_graph ()

Returns the snub dodecahedron graph.

truncated_cube_graph ()

Returns the truncated cube graph.

truncated_dodecahedron_graph ()

Returns the truncated dodecahedron graph.

truncated_icosahedron_graph ()

Returns the truncated icosahedron graph.

truncated_tetrahedron_graph ()

Returns the truncated tetrahedron graph.

tutte_graph ()

Returns the Tutte graph.

underlying_graph \((g)\)

Returns the underlying graph of the directed graph \(g\).

wheel_graph \((n)\)

Returns the wheel graph on \(n+1\) vertices.
61.2.2 Graph properties

adjacency_matrix (gr)  
Returns the adjacency matrix of the graph gr.

Example:
(%i1) load ("graphs")$
(%i2) c5 : cycle_graph(4)$
(%i3) adjacency_matrix(c5);


average_degree (gr)  
Returns the average degree of vertices in the graph gr.

Example:
(%i1) load ("graphs")$
(%i2) average_degree(grotzch_graph());

biconnected_components (gr)  
Returns the (vertex sets of) 2-connected components of the graph gr.

Example:
(%i1) load ("graphs")$
(%i2) g : create_graph( 
[1,2,3,4,5,6,7],
[
[1,2],[2,3],[2,4],[3,4],
[4,5],[5,6],[4,6],[6,7]
])$
(%i3) biconnected_components(g);

bipartition (gr)  
Returns a bipartition of the vertices of the graph gr or an empty list if gr is not bipartite.

Example:
(%i1) load ("graphs")$
(%i2) h : heawood_graph()$
(%i3) [A,B]:bipartition(h);
(%i4) draw_graph(h, show_vertices=A, program=circular)$
chromatic_index (gr)  Returns the chromatic index of the graph gr.
    Example:
    (%i1) load ("graphs")$
    (%i2) p : petersen_graph()$
    (%i3) chromatic_index(p);                4
chromatic_number (gr)  Returns the chromatic number of the graph gr.
    Example:
    (%i1) load ('graphs')$
    (%i2) chromatic_number(cycle_graph(5));  3
    (%i3) chromatic_number(cycle_graph(6));  2
clear_edge_weight (e, gr)  Removes the weight of the edge e in the graph gr.
    Example:
    (%i1) load ("graphs")$
    (%i2) g : create_graph(3, [[[0,1], 1.5], [[1,2], 1.3]])$
    (%i3) get_edge_weight([0,1], g);        1.5
    (%i4) clear_edge_weight([0,1], g)$
    (%i5) get_edge_weight([0,1], g);        1
clear_vertex_label (v, gr)  Removes the label of the vertex v in the graph gr.
    Example:
    (%i1) load ("graphs")$
    (%i2) g : create_graph([[0,"Zero"], [1, "One"]], [[0,1]])$
    (%i3) get_vertex_label(0, g);            Zero
    (%i4) clear_vertex_label(0, g);         done
    (%i5) get_vertex_label(0, g);              false
connected_components (gr)  Returns the (vertex sets of) connected components of the graph gr.
    Example:
    (%i1) load ("graphs")$
    (%i2) g: graph_union(cycle_graph(5), path_graph(4))$
    (%i3) connected_components(g);          [[1, 2, 3, 4, 0], [8, 7, 6, 5]]
diameter \( (gr) \)

Returns the diameter of the graph \( gr \).

Example:

\[
\text{(i1)} \quad \text{load ("graphs")}$
\text{(i2)} \quad \text{diameter(dodecahedron_graph());$
\text{(o2)} \quad 5$
\]

edge_coloring \( (gr) \)

Returns an optimal coloring of the edges of the graph \( gr \).

The function returns the chromatic index and a list representing the coloring of the edges of \( gr \).

Example:

\[
\text{(i1)} \quad \text{load ("graphs")}$
\text{(i2)} \quad \text{p : petersen_graph();$
\text{(i3)} \quad \text{[ch_index, col] : edge_coloring(p);$
\text{(o3)} \quad [4, [[[0, 5], 3], [[5, 7], 1], [[0, 1], 1], [[1, 6], 2],
\quad [[6, 8], 1], [[1, 2], 3], [[2, 7], 4], [[7, 9], 2], [[2, 3], 2],
\quad [[3, 8], 3], [[5, 8], 2], [[3, 4], 1], [[4, 9], 4], [[6, 9], 3],
\quad [[0, 4], 2]]]$
\text{(i4)} \quad \text{assoc([0,1], col);$
\text{(o4)} \quad 1$
\text{(i5)} \quad \text{assoc([0,5], col);$
\text{(o5)} \quad 3$
\]

degree_sequence \( (gr) \)

Returns the list of vertex degrees of the graph \( gr \).

Example:

\[
\text{(i1)} \quad \text{load ("graphs")}$
\text{(i2)} \quad \text{degree_sequence(random_graph(10, 0.4));$
\text{(o2)} \quad [2, 2, 2, 2, 2, 2, 3, 3, 3, 3]$
\]

edge_connectivity \( (gr) \)

Returns the edge-connectivity of the graph \( gr \).

See also \text{min_edge_cut}.\]

degree_sequence \( (gr) \)

Returns the list of vertex degrees of the graph \( gr \).

Example:

\[
\text{(i1)} \quad \text{load ("graphs")}$
\text{(i2)} \quad \text{degree_sequence(random_graph(10, 0.4));$
\text{(o2)} \quad [2, 2, 2, 2, 2, 2, 3, 3, 3, 3]$
\]

edge_connectivity \( (gr) \)

Returns the edge-connectivity of the graph \( gr \).

See also \text{min_edge_cut}.\]

degree_sequence \( (gr) \)

Returns the list of vertex degrees of the graph \( gr \).

Example:

\[
\text{(i1)} \quad \text{load ("graphs")}$
\text{(i2)} \quad \text{degree_sequence(random_graph(10, 0.4));$
\text{(o2)} \quad [2, 2, 2, 2, 2, 2, 3, 3, 3, 3]$
\]

edge_connectivity \( (gr) \)

Returns the edge-connectivity of the graph \( gr \).

See also \text{min_edge_cut}.\]

degree_sequence \( (gr) \)

Returns the list of vertex degrees of the graph \( gr \).

Example:

\[
\text{(i1)} \quad \text{load ("graphs")}$
\text{(i2)} \quad \text{degree_sequence(random_graph(10, 0.4));$
\text{(o2)} \quad [2, 2, 2, 2, 2, 2, 3, 3, 3, 3]$
\]

get_edge_weight

Returns the weight of the edge \( e \) in the graph \( gr \).
If there is no weight assigned to the edge, the function returns 1. If the edge is not present in the graph, the function signals an error or returns the optional argument \texttt{ifnot}.

Example:

```maxima
(%i1) load ("graphs")$
(%i2) c5 : cycle_graph(5)$
(%i3) get_edge_weight([1,2], c5);
(%o3) 1
(%i4) set_edge_weight([1,2], 2.0, c5);
(%o4) done
(%i5) get_edge_weight([1,2], c5);
(%o5) 2.0
```

\textbf{get_vertex_label (v, gr)}

Returns the label of the vertex \texttt{v} in the graph \texttt{gr}.

Example:

```maxima
(%i1) load ("graphs")$
(%i2) g : create_graph([[[0,"Zero"], [1, "One"], [[0,1]]])$
(%i3) get_vertex_label(0, g);
(%o3) Zero
```

\textbf{graph_charpoly (gr, x)}

Returns the characteristic polynomial (in variable \texttt{x}) of the graph \texttt{gr}.

Example:

```maxima
(%i1) load ("graphs")$
(%i2) p : petersen_graph()$
(%i3) graph_charpoly(p, x), factor;
(%o3) (x - 3) (x - 1) (x + 2)
```

\textbf{graph_center (gr)}

Returns the center of the graph \texttt{gr}.

Example:

```maxima
(%i1) load ("graphs")$
(%i2) g : grid_graph(5,5)$
(%i3) graph_center(g);
(%o3) [12]
```

\textbf{graph_eigenvalues (gr)}

Returns the eigenvalues of the graph \texttt{gr}. The function returns eigenvalues in the same format as \texttt{maxima eigenvalues} function.

Example:

```maxima
(%i1) load ("graphs")$
(%i2) p : petersen_graph()$
(%i3) graph_eigenvalues(p);
(%o3) [[3, - 2, 1], [1, 4, 5]]
```
graph_periphery (gr)

Returns the periphery of the graph gr.

Example:
(%i1) load ("graphs")$
(%i2) g : grid_graph(5,5)$
(%i3) graph_periphery(g);
(%o3) [24, 20, 4, 0]

graph_size (gr)

Returns the number of edges in the graph gr.

Example:
(%i1) load ("graphs")$
(%i2) p : petersen_graph()$
(%i3) graph_size(p);
(%o3) 15

girth (gr)

Returns the length of the shortest cycle in gr.

Example:
(%i1) load ("graphs")$
(%i2) g : heawood_graph()$
(%i3) girth(g);
(%o3) 6

hamilton_cycle (gr)

Returns the Hamilton cycle of the graph gr or an empty list if gr is not hamiltonian.

Example:
(%i1) load ("graphs")$
(%i2) c : cube_graph(3)$
(%i3) hc : hamilton_cycle(c);
(%o3) [7, 3, 2, 6, 4, 0, 1, 5, 7]
(%i4) draw_graph(c, show_edges=vertices_to_cycle(hc))$

hamilton_path (gr)

Returns the Hamilton path of the graph gr or an empty list if gr does not have a Hamilton path.

Example:
(%i1) load ("graphs")$
(%i2) p : petersen_graph();
(%i3) hp : hamilton_path(p);
(%o3) [0, 5, 7, 2, 1, 6, 8, 3, 4, 9]
(%i4) draw_graph(p, show_edges=vertices_to_path(hp));

isomorphism (gr1, gr2)  [Function]
Returns an isomorphism between graphs/digraphs gr1 and gr2. If gr1 and gr2 are not isomorphic, it returns an empty list.
Example:
(%i1) load ("graphs")$
(%i2) clk5:complement_graph(line_graph(complete_graph(5)))$
(%i3) isomorphism(clk5, petersen_graph());
(%o3) [9 -> 0, 2 -> 1, 6 -> 2, 5 -> 3, 0 -> 4, 1 -> 5, 3 -> 6, 4 -> 7, 7 -> 8, 8 -> 9]

in_neighbors (v, gr)  [Function]
Returns the list of in-neighbors of the vertex v in the directed graph gr.
Example:
(%i1) load ("graphs")$
(%i2) p : path_digraph(3)$
(%i3) in_neighbors(2, p);
(%o3) [1]
(%i4) out_neighbors(2, p);
(%o4) []

is_biconnected (gr)  [Function]
Returns true if gr is 2-connected and false otherwise.
Example:
(%i1) load ("graphs")$
(%i2) is_biconnected(cycle_graph(5));
(%o2) true
(%i3) is_biconnected(path_graph(5));
(%o3) false

is_bipartite (gr)  [Function]
Returns true if gr is bipartite (2-colorable) and false otherwise.
Example:
(%i1) load ("graphs")$
(%i2) is_bipartite(petersen_graph());
(%o2) false
(%i3) is_bipartite(heawood_graph());
(%o3) true

is_connected (gr)  [Function]
Returns true if the graph gr is connected and false otherwise.
Example:
(%i1) load ("graphs")$
is_connected(graph_union(cycle_graph(4), path_graph(3))); 
(%o2) false

is_digraph (gr)  
[Function]  
Returns true if gr is a directed graph and false otherwise.

Example:
(%i1) load ("graphs");
(%i2) is_digraph(path_graph(5));
(%o2) false
(%i3) is_digraph(path_digraph(5));
(%o3) true

is_edge_in_graph (e, gr)  
[Function]  
Returns true if e is an edge (arc) in the (directed) graph g and false otherwise.

Example:
(%i1) load ("graphs");
(%i2) c4 : cycle_graph(4)$
(%i3) is_edge_in_graph([2,3], c4);
(%o3) true
(%i4) is_edge_in_graph([3,2], c4);
(%o4) true
(%i5) is_edge_in_graph([2,4], c4);
(%o5) false
(%i6) is_edge_in_graph([3,2], cycle_digraph(4));
(%o6) false

is_graph (gr)  
[Function]  
Returns true if gr is a graph and false otherwise.

Example:
(%i1) load ("graphs");
(%i2) is_graph(path_graph(5));
(%o2) true
(%i3) is_graph(path_digraph(5));
(%o3) false

is_graph_or_digraph (gr)  
[Function]  
Returns true if gr is a graph or a directed graph and false otherwise.

Example:
(%i1) load ("graphs");
(%i2) is_graph_or_digraph(path_graph(5));
(%o2) true
(%i3) is_graph_or_digraph(path_digraph(5));
(%o3) true

is_isomorphic (gr1, gr2)  
[Function]  
Returns true if graphs/digraphs gr1 and gr2 are isomorphic and false otherwise.

See also isomorphism.
Example:

```
(%i1) load ("graphs")$
(%i2) clk5:complement_graph(line_graph(complete_graph(5)))$
(%i3) is_isomorphic(clk5, petersen_graph());
(%o3) true
```

**is_planar (gr)**

[Function]

Returns true if gr is a planar graph and false otherwise.

The algorithm used is the Demoucron’s algorithm, which is a quadratic time algorithm.

Example:

```
(%i1) load ("graphs")$
(%i2) is_planar(dodecahedron_graph());
(%o2) true
(%i3) is_planar(petersen_graph());
(%o3) false
(%i4) is_planar(petersen_graph(10,2));
(%o4) true
```

**is_sconnected (gr)**

[Function]

Returns true if the directed graph gr is strongly connected and false otherwise.

Example:

```
(%i1) load ("graphs")$
(%i2) is_sconnected(cycle_digraph(5));
(%o2) true
(%i3) is_sconnected(path_digraph(5));
(%o3) false
```

**is_vertex_in_graph (v, gr)**

[Function]

Returns true if v is a vertex in the graph g and false otherwise.

Example:

```
(%i1) load ("graphs")$
(%i2) c4 : cycle_graph(4)$
(%i3) is_vertex_in_graph(0, c4);
(%o3) true
(%i4) is_vertex_in_graph(6, c4);
(%o4) false
```

**is_tree (gr)**

[Function]

Returns true if gr is a tree and false otherwise.

Example:

```
(%i1) load ("graphs")$
(%i2) is_tree(random_tree(4));
(%o2) true
(%i3) is_tree(graph_union(random_tree(4), random_tree(5)));
(%o3) false
```
**laplacian_matrix (gr)**

Returns the laplacian matrix of the graph gr.

Example:

```
(%i1) load ("graphs")$
(%i2) laplacian_matrix(cycle_graph(5));

[ 2 - 1 0 0 - 1 ]
[              ]
[ - 1 2 - 1 0 0 ]
[              ]
[ 0 - 1 2 - 1 0 ]
[              ]
[ 0 0 - 1 2 - 1 ]
[              ]
[- 1 0 0 - 1 2 ]
(%o2)
```

**max_clique (gr)**

Returns a maximum clique of the graph gr.

Example:

```
(%i1) load ("graphs")$
(%i2) g : random_graph(100, 0.5)$
(%i3) max_clique(g);

[6, 12, 31, 36, 52, 59, 62, 63, 80]
(%o3)
```

**max_degree (gr)**

Returns the maximal degree of vertices of the graph gr and a vertex of maximal degree.

Example:

```
(%i1) load ("graphs")$
(%i2) g : random_graph(100, 0.02)$
(%i3) max_degree(g);

[6, 79]
(%o3)
(%i4) vertex_degree(95, g);

2
(%o4)
```

**max_flow (net, s, t)**

Returns a maximum flow through the network net with the source s and the sink t. The function returns the value of the maximal flow and a list representing the weights of the arcs in the optimal flow.

Example:

```
(%i1) load ("graphs")$
(%i2) net : create_graph(
        [1,2,3,4,5,6],
        [[1,2], 1.0],
        [[1,3], 0.3],
        [[2,4], 0.2],
        [[2,5], 0.3],
        [[3,5], 0.2],
        [[4,5], 0.2])$
(%i3) max_flow(net, 1, 6);

[2.0, [1, 3, 5, 6], [2, 4, 5]]
(%o3)
```
[[3, 4], 0.1],
[[3, 5], 0.1],
[[4, 6], 1.0],
[[5, 6], 1.0]],
directed=true)
(%i3) [flow_value, flow] : max_flow(net, 1, 6);
(%o3) [0.7, [[[1, 2], 0.5], [[1, 3], 0.2], [[2, 4], 0.2],
       [[2, 5], 0.3], [[3, 4], 0.1], [[3, 5], 0.1], [[4, 6], 0.3],
       [[5, 6], 0.4]]]
(%i4) fl : 0$
(%i5) for u in out_neighbors(1, net)
do fl : fl + assoc([1, u], flow)$
(%i6) fl;
(%o6) 0.7

max_independent_set (gr)
[Function]
Returns a maximum independent set of the graph gr.
Example:
(%i1) load ("graphs")$
(%i2) d : dodecahedron_graph();$
(%i3) mi : max_independent_set(d);
(%o3) [0, 3, 5, 9, 10, 11, 18, 19]
(%i4) draw_graph(d, show_vertices=mi)$

max_matching (gr)
[Function]
Returns a maximum matching of the graph gr.
Example:
(%i1) load ("graphs")$
(%i2) d : dodecahedron_graph();$
(%i3) m : max_matching(d);
(%o3) [[[5, 7], [8, 9], [6, 10], [14, 19], [13, 18], [12, 17],
       [11, 16], [0, 15], [3, 4], [1, 2]]
(%i4) draw_graph(d, show_edges=m)$

min_degree (gr)
[Function]
Returns the minimum degree of vertices of the graph gr and a vertex of minimum
degree.
Example:
(%i1) load ("graphs")$
(%i2) g : random_graph(100, 0.1)$
(%i3) min_degree(g); 
(%o3) [3, 49]
(%i4) vertex_degree(21, g); 
(%o4) 9

min_edge_cut (gr)
[Function]
Returns the minimum edge cut in the graph gr.
See also edge_connectivity.
min_vertex_cover (gr)
  Returns the minimum vertex cover of the graph gr.

min_vertex_cut (gr)
  Returns the minimum vertex cut in the graph gr.
  See also vertex_connectivity.

minimum_spanning_tree (gr)
  Returns the minimum spanning tree of the graph gr.
  Example:
    (%i1) load ("graphs")$
    (%i2) g : graph_product(path_graph(10), path_graph(10))$
    (%i3) t : minimum_spanning_tree(g)$
    (%i4) draw_graph(g, show_edges=edges(t))$

neighbors (v, gr)
  Returns the list of neighbors of the vertex v in the graph gr.
  Example:
    (%i1) load ("graphs")$
    (%i2) p : petersen_graph()$
    (%i3) neighbors(3, p);
    (%o3) [4, 8, 2]

odd_girth (gr)
  Returns the length of the shortest odd cycle in the graph gr.
  Example:
    (%i1) load ("graphs")$
    (%i2) g : graph_product(cycle_graph(4), cycle_graph(7))$
    (%i3) girth(g);
    (%o3) 4
    (%i4) odd_girth(g);
    (%o4) 7

out_neighbors (v, gr)
  Returns the list of out-neighbors of the vertex v in the directed graph gr.
  Example:
    (%i1) load ("graphs")$
    (%i2) p : path_digraph(3)$
    (%i3) in_neighbors(2, p);
    (%o3) [1]
    (%i4) out_neighbors(2, p);
    (%o4) []

planar_embedding (gr)
  Returns the list of facial walks in a planar embedding of gr and false if gr is not a planar graph.
  The graph gr must be biconnected.
The algorithm used is the Demoucron's algorithm, which is a quadratic time algorithm.

Example:

```maxima
(%i1) load ("graphs")$
(%i2) planar_embedding(grid_graph(3,3));
(%o2) [[3, 6, 7, 8, 5, 2, 1, 0], [4, 3, 0, 1], [3, 4, 7, 6],
[8, 7, 4, 5], [1, 2, 5, 4]]
```

**print_graph (gr)**

Prints some information about the graph `gr`.

Example:

```maxima
(%i1) load ("graphs")$
(%i2) c5 : cycle_graph(5)$
(%i3) print_graph(c5)$
Graph on 5 vertices with 5 edges.
Adjacencies:
4 : 0 3
3 : 4 2
2 : 3 1
1 : 2 0
0 : 4 1
(%i4) dc5 : cycle_digraph(5)$
(%i5) print_graph(dc5)$
Digraph on 5 vertices with 5 arcs.
Adjacencies:
4 : 0
3 : 4
2 : 3
1 : 2
0 : 1
(%i6) out_neighbors(0, dc5);
(%o6) [1]
```

**radius (gr)**

Returns the radius of the graph `gr`.

Example:

```maxima
(%i1) load ("graphs")$
(%i2) radius(dodecahedron_graph());
(%o2) 5
```

**set_edge_weight (e, w, gr)**

Assigns the weight `w` to the edge `e` in the graph `gr`.

Example:

```maxima
(%i1) load ("graphs")$
(%i2) g : create_graph([1, 2], [[[1,2], 1.2]])$
(%i3) get_edge_weight([1,2], g);
```
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1.2

(%i4) set_edge_weight([1,2], 2.1, g);
(%o4) done

(%i5) get_edge_weight([1,2], g);
(%o5) 2.1

\textbf{set_vertex_label} (v, l, gr)

Assigns the label \textit{l} to the vertex \textit{v} in the graph \textit{gr}.

Example:

(%i11) load ("graphs")$
(%i12) g : create_graph([[1, "One"], [2, "Two"], [[1,2]]]$%i13) get_vertex_label(1, g);
(%o13) One

(%i14) set_vertex_label(1, "oNE", g);
(%o14) done

(%i15) get_vertex_label(1, g);
(%o15) oNE

\textbf{shortest_path} (u, v, gr)

Returns the shortest path from \textit{u} to \textit{v} in the graph \textit{gr}.

Example:

(%i1) load ("graphs")$
(%i2) d : dodecahedron_graph()$
(%i3) path : shortest_path(0, 7, d);
(%o3) [0, 1, 19, 13, 7]

(%i4) draw_graph(d, show_edges=vertices_to_path(path))$

\textbf{shortest_weighted_path} (u, v, gr)

Returns the length of the shortest weighted path and the shortest weighted path from \textit{u} to \textit{v} in the graph \textit{gr}.

The length of a weighted path is the sum of edge weights of edges in the path. If an edge has no weight, then it has a default weight 1.

Example:

(%i1) load ("graphs")$
(%i2) g: petersen_graph(20, 2)$%i3) for e in edges(g) do set_edge_weight(e, random(1.0), g)$%i4) shortest_weighted_path(0, 10, g);
(%o4) [2.575143920268482, [0, 20, 38, 36, 34, 32, 30, 10]]

\textbf{strong_components} (gr)

Returns the strong components of a directed graph \textit{gr}.

Example:

(%i1) load ("graphs")$
(%i2) t : random_tournament(4)$%i3) strong_components(t);
(%o3) [[1], [0], [2], [3]]
(\%i4) vertex_out_degree(3, t);  
(\%o4) 3

\textbf{topological_sort (dag)}  
[Function]  
Returns a topological sorting of the vertices of a directed graph \textit{dag} or an empty list if \textit{dag} is not a directed acyclic graph.  

Example:  
(\%i1) load ("graphs")$  
(\%i2) g:create_graph(  
    [1,2,3,4,5],  
    [  
        [1,2], [2,5], [5,3],  
        [5,4], [3,4], [1,3]  
    ],  
    directed=true)$  
(\%i3) topological_sort(g);  
(\%o3) [1, 2, 5, 3, 4]

\textbf{vertex_connectivity (g)}  
[Function]  
Returns the vertex connectivity of the graph \textit{g}.  
See also \textit{min_vertex_cut}.  

\textbf{vertex_degree (v, gr)}  
[Function]  
Returns the degree of the vertex \textit{v} in the graph \textit{gr}.  

\textbf{vertex_distance (u, v, gr)}  
[Function]  
Returns the length of the shortest path between \textit{u} and \textit{v} in the (directed) graph \textit{gr}.  

Example:  
(\%i1) load ("graphs")$  
(\%i2) d : dodecahedron_graph();  
(\%i3) vertex_distance(0, 7, d);  
(\%o3) 4  
(\%i4) shortest_path(0, 7, d);  
(\%o4) [0, 1, 19, 13, 7]

\textbf{vertex_eccentricity (v, gr)}  
[Function]  
Returns the eccentricity of the vertex \textit{v} in the graph \textit{gr}.  

Example:  
(\%i1) load ("graphs")$  
(\%i2) g:cycle_graph(7)$  
(\%i3) vertex_eccentricity(0, g);  
(\%o3) 3

\textbf{vertex_in_degree (v, gr)}  
[Function]  
Returns the in-degree of the vertex \textit{v} in the directed graph \textit{gr}.  

Example:  
(\%i1) load ("graphs")$
(%i2) p5 : path_digraph(5)$
(%i3) print_graph(p5)$
Digraph on 5 vertices with 4 arcs.
Adjacencies:
   4 :
   3 : 4
   2 : 3
   1 : 2
   0 : 1
(%i4) vertex_in_degree(4, p5);
(%o4) 1
(%i5) in_neighbors(4, p5);
(%o5) [3]

vertex_out_degree (v, gr)  
Returns the out-degree of the vertex v in the directed graph gr.
Example:
   (%i1) load ("graphs")$
   (%i2) t : random_tournament(10)$
   (%i3) vertex_out_degree(0, t);
   (%o3) 2
   (%i4) out_neighbors(0, t);
   (%o4) [7, 1]

vertices (gr)  
Returns the list of vertices in the graph gr.
Example:
   (%i1) load ("graphs")$
   (%i2) vertices(complete_graph(4));
   (%o2) [3, 2, 1, 0]

vertex_coloring (gr)  
Returns an optimal coloring of the vertices of the graph gr.
The function returns the chromatic number and a list representing the coloring of the vertices of gr.
Example:
   (%i1) load ("graphs")$
   (%i2) p:petersen_graph();
   (%i3) vertex_coloring(p);
   (%o3) [3, [[0, 2], [1, 3], [2, 2], [3, 3], [4, 1], [5, 3],
       [6, 1], [7, 1], [8, 2], [9, 2]]]

wiener_index (gr)  
Returns the Wiener index of the graph gr.
Example:
   (%i2) wiener_index(dodecahedron_graph());
   (%o2) 500
61.2.3 Modifying graphs

add_edge (e, gr)   [Function]
 Adds the edge e to the graph gr.

Example:

(\%i1) load ("graphs")$
(\%i2) p : path_graph(4)$
(\%i3) neighbors(0, p);
(\%o3) [1]
(\%i4) add_edge([0,3], p);
(\%o4) done
(\%i5) neighbors(0, p);
(\%o5) [3, 1]

add_edges (e_list, gr)   [Function]
 Adds all edges in the list e_list to the graph gr.

Example:

(\%i1) load ("graphs")$
(\%i2) g : empty_graph(3)$
(\%i3) add_edges([[0,1],[1,2]], g)$
(\%i4) print_graph(g)$
 Graph on 3 vertices with 2 edges.
 Adjacencies:
 2 : 1
 1 : 2 0
 0 : 1

add_vertex (v, gr)   [Function]
 Adds the vertex v to the graph gr.

Example:

(\%i1) load ("graphs")$
(\%i2) g : path_graph(2)$
(\%i3) add_vertex(2, g)$
(\%i4) print_graph(g)$
 Graph on 3 vertices with 1 edges.
 Adjacencies:
 2 :
 1 : 0
 0 : 1

add_vertices (v_list, gr)   [Function]
 Adds all vertices in the list v_list to the graph gr.

connect_vertices (v_list, u_list, gr)   [Function]
 Connects all vertices from the list v_list with the vertices in the list u_list in the graph gr.

v_list and u_list can be single vertices or lists of vertices.
Example:

```
(%i1) load ("graphs")$
(%i2) g : empty_graph(4)$
(%i3) connect_vertices(0, [1,2,3], g)$
(%i4) print_graph(g)$
Graph on 4 vertices with 3 edges.
Adjacencies:
  3 : 0
  2 : 0
  1 : 0
  0 : 3 2 1
```

**contract_edge (e, gr)**

Contracts the edge $e$ in the graph $gr$.

Example:

```
(%i1) load ("graphs")$
(%i2) g: create_graph(
    8, [[0,3],[1,3],[2,3],[3,4],[4,5],[4,6],[4,7]])$
(%i3) print_graph(g)$
Graph on 8 vertices with 7 edges.
Adjacencies:
  7 : 4
  6 : 4
  5 : 4
  4 : 7 6 5 3
  3 : 4 2 1 0
  2 : 3
  1 : 3
  0 : 3
(%i4) contract_edge([3,4], g)$
(%i5) print_graph(g)$
Graph on 7 vertices with 6 edges.
Adjacencies:
  7 : 3
  6 : 3
  5 : 3
  3 : 5 6 7 2 1 0
  2 : 3
  1 : 3
  0 : 3
```

**remove_edge (e, gr)**

Removes the edge $e$ from the graph $gr$.

Example:

```
(%i1) load ("graphs")$
(%i2) c3 : cycle_graph(3)$
```
(%i3) remove_edge([0,1], c3)$
(%i4) print_graph(c3)$
Graph on 3 vertices with 2 edges.
Adjacencies:
 2 : 0 1
 1 : 2
 0 : 2

remove_vertex (v, gr) [Function]
Removes the vertex v from the graph gr.

61.2.4 Reading and writing to files

dimacs_export [Function]
  dimacs_export (gr, fl)
dimacs_export (gr, fl, comment1, ..., commentn)
Exports the graph into the file fl in the DIMACS format. Optional comments will be added to the top of the file.

dimacs_import (fl) [Function]
Returns the graph from file fl in the DIMACS format.

graph6_decode (str) [Function]
Returns the graph encoded in the graph6 format in the string str.

graph6_encode (gr) [Function]
Returns a string which encodes the graph gr in the graph6 format.

graph6_export (gr_list, fl) [Function]
Exports graphs in the list gr_list to the file fl in the graph6 format.

graph6_import (fl) [Function]
Returns a list of graphs from the file fl in the graph6 format.

sparse6_decode (str) [Function]
Returns the graph encoded in the sparse6 format in the string str.

sparse6_encode (gr) [Function]
Returns a string which encodes the graph gr in the sparse6 format.

sparse6_export (gr_list, fl) [Function]
Exports graphs in the list gr_list to the file fl in the sparse6 format.

sparse6_import (fl) [Function]
Returns a list of graphs from the file fl in the sparse6 format.
61.2.5 Visualization

\texttt{draw\_graph}\quad [\text{Function}]

\texttt{draw\_graph (graph)}
\texttt{draw\_graph (graph, option1, ..., optionk)}

Draws the graph using the Chapter 52 [draw-pkg], page 747, package.

The algorithm used to position vertices is specified by the optional argument \texttt{program}. The default value is \texttt{program=spring\_embedding}. \texttt{draw\_graph} can also use the graphviz programs for positioning vertices, but graphviz must be installed separately.

Example 1:
\begin{verbatim}
(%i1) load ("graphs")$
(%i2) g:grid_graph(10,10)$
(%i3) m:max\_matching(g)$
(%i4) draw\_graph(g, 
    spring\_embedding\_depth=100, 
    show\_edges=m, edge\_type=dots, 
    vertex\_size=0)$
\end{verbatim}

Example 2:
\begin{verbatim}
(%i1) load ("graphs")$
(%i2) g:create\_graph(16, 
    [ 
        [0,1],[1,3],[2,3],[0,2],[3,4],[2,4], 
        [5,6],[6,4],[4,7],[6,7],[7,8],[7,10],[7,11], 
        [8,10],[11,10],[8,9],[11,12],[9,15],[12,13], 
        [10,14],[15,14],[13,14] 
    ])$
(%i3) t:minimum\_spanning\_tree(g)$
(%i4) draw\_graph( 
    g, 
    show\_edges=edges(t), 
    show\_edge\_width=4, 
    show\_edge\_color=green, 
    vertex\_type=filled\_square, 
    vertex\_size=2 
)$(
\end{verbatim}

Example 3:
\begin{verbatim}
(%i1) load ("graphs")$
(%i2) g:create\_graph(16, 
    [ 
        [0,1],[1,3],[2,3],[0,2],[3,4],[2,4], 
        [5,6],[6,4],[4,7],[6,7],[7,8],[7,10],[7,11], 
        [8,10],[11,10],[8,9],[11,12],[9,15],[12,13], 
        [10,14],[15,14],[13,14] 
    ])$
(%i3) mi : max\_independent\_set(g)$
\end{verbatim}
(%i4) draw_graph(
  g,
  show_vertices=mi,
  show_vertex_type=filled_up_triangle,
  show_vertex_size=2,
  edge_color=cyan,
  edge_width=3,
  show_id=true,
  text_color=brown
 )$

Example 4:
(%i1) load ("graphs")$
(%i2) net : create_graph(
    [0,1,2,3,4,5],
    [ 
      [[0,1], 3], [[0,2], 2],
      [[1,3], 1], [[1,4], 3],
      [[2,3], 2], [[2,4], 2],
      [[4,5], 2], [[3,5], 2]
    ],
    directed=true
 )$
(%i3) draw_graph(
    net,
    show_weight=true,
    vertex_size=0,
    show_vertices=[0,5],
    show_vertex_type=filled_square,
    head_length=0.2,
    head_angle=10,
    edge_color="dark-green",
    text_color=blue
 )$

Example 5:
(%i1) load(graphs)$
(%i2) g: petersen_graph(20, 2);
(%o2) GRAPH
(%i3) draw_graph(g, redraw=true, program=planar_embedding);
(%o3) done

Example 6:
(%i1) load(graphs)$
(%i2) t: tutte_graph();
(%o2) GRAPH
(%i3) draw_graph(t, redraw=true,
    fixed_vertices=[1,2,3,4,5,6,7,8,9]);
(%o3) done
draw_graph_program
    Default value: {spring_embedding}
    The default value for the program used to position vertices in draw_graph program.

show_id
    Default value: false
    If true then ids of the vertices are displayed.

show_label
    Default value: false
    If true then labels of the vertices are displayed.

label_alignment
    Default value: center
    Determines how to align the labels/ids of the vertices. Can be left, center or right.

show_weight
    Default value: false
    If true then weights of the edges are displayed.

vertex_type
    Default value: circle
    Defines how vertices are displayed. See the point_type option for the draw package for possible values.

vertex_size
    The size of vertices.

vertex_color
    The color used for displaying vertices.

show_vertices
    Default value: []
    Display selected vertices in the using a different color.

show_vertex_type
    Defines how vertices specified in show_vertices are displayed. See the point_type option for the draw package for possible values.

show_vertex_size
    The size of vertices in show_vertices.

show_vertex_color
    The color used for displaying vertices in the show_vertices list.

vertex_partition
    Default value: []
    A partition [[v1,v2,...],...,[vk,...,vn]] of the vertices of the graph. The vertices of each list in the partition will be drawn in a different color.
vertex_coloring [draw_graph option]
   Specifies coloring of the vertices. The coloring col must be specified in the format as returned by vertex_coloring.

edge_color [draw_graph option]
   The color used for displaying edges.

eedge_width [draw_graph option]
   The width of edges.

eedge_type [draw_graph option]
   Defines how edges are displayed. See the line_type option for the draw package.

dshow_edges [draw_graph option]
   Display edges specified in the list e_list using a different color.

dshow_edge_color [draw_graph option]
   The color used for displaying edges in the show_edges list.

dshow_edge_width [draw_graph option]
   The width of edges in show_edges.

dshow_edge_type [draw_graph option]
   Defines how edges in show_edges are displayed. See the line_type option for the draw package.

dedge_partition [draw_graph option]
   A partition [[e1,e2,...],[ek,...,em]] of edges of the graph. The edges of each list in the partition will be drawn using a different color.

dedge_coloring [draw_graph option]
   The coloring of edges. The coloring must be specified in the format as returned by the function edge_coloring.

dredraw [draw_graph option]
   Default value: false
   If true, vertex positions are recomputed even if the positions have been saved from a previous drawing of the graph.

dhead_angle [draw_graph option]
   Default value: 15
   The angle for the arrows displayed on arcs (in directed graphs).

dhead_length [draw_graph option]
   Default value: 0.1
   The length for the arrows displayed on arcs (in directed graphs).

dspring_embedding_depth [draw_graph option]
   Default value: 50
   The number of iterations in the spring embedding graph drawing algorithm.
terminal
[draw_graph option]
The terminal used for drawing (see the terminal option in the draw package).

file_name
[draw_graph option]
The filename of the drawing if terminal is not screen.

program
[draw_graph option]
Defines the program used for positioning vertices of the graph. Can be one of the
graphviz programs (dot, neato, twopi, circ, fdp), circular, spring_embedding or planar_embedding. planar_embedding is only available for 2-connected planar graphs.
When program=spring_embedding, a set of vertices with fixed position can be specified with the fixed_vertices option.

fixed_vertices
[draw_graph option]
Specifies a list of vertices which will have positions fixed along a regular polygon. Can be used when program=spring_embedding.

vertices_to_path (v_list)
[Function]
Converts a list v_list of vertices to a list of edges of the path defined by v_list.

vertices_to_cycle (v_list)
[Function]
Converts a list v_list of vertices to a list of edges of the cycle defined by v_list.
62 grobner

62.1 Introduction to grobner

grobner is a package for working with Groebner bases in Maxima.
To use the following functions you must load the grobner.lisp package.

    load(grobner);

A demo can be started by

    demo("grobner.demo");

or

    batch("grobner.demo")

Some of the calculation in the demo will take a lot of time therefore the output
grobner-demo.output of the demo can be found in the same directory as the demo file.

62.1.1 Notes on the grobner package

The package was written by
Marek Rychlik

http://alamos.math.arizona.edu

and is released 2002-05-24 under the terms of the General Public License(GPL) (see file
grobner.lisp). This documentation was extracted from the files
README, grobner.lisp, grobner.demo, grobner-demo.output

by Günter Nowak. Suggestions for improvement of the documentation can be discussed
at the maxima-mailing-list maxima@math.utexas.edu. The code is a little bit out of date
now. Modern implementation use the fast \( F_4 \) algorithm described in

A new efficient algorithm for computing Gröbner bases (F4)
Jean-Charles Faugère
LIP6/CNRS Université Paris VI
January 20, 1999

62.1.2 Implementations of admissible monomial orders in grobner

- lex
  pure lexicographic, default order for monomial comparisons
- grlex
  total degree order, ties broken by lexicographic
- grevlex
  total degree, ties broken by reverse lexicographic
- invlex
  inverse lexicographic order
62.2 Functions and Variables for grobner

62.2.1 Global switches for grobner

**poly_monomial_order**  
[Option variable]  
Default value: lex  
This global switch controls which monomial order is used in polynomial and Groebner Bases calculations. If not set, lex will be used.

**poly_coefficient_ring**  
[Option variable]  
Default value: expression_ring  
This switch indicates the coefficient ring of the polynomials that will be used in grobner calculations. If not set, maxima’s general expression ring will be used. This variable may be set to ring_of_integers if desired.

**poly_primary_elimination_order**  
[Option variable]  
Default value: false  
Name of the default order for eliminated variables in elimination-based functions. If not set, lex will be used.

**poly_secondary_elimination_order**  
[Option variable]  
Default value: false  
Name of the default order for kept variables in elimination-based functions. If not set, lex will be used.

**poly_elimination_order**  
[Option variable]  
Default value: false  
Name of the default elimination order used in elimination calculations. If set, it overrides the settings in variables poly_primary_elimination_order and poly_secondary_elimination_order. The user must ensure that this is a true elimination order valid for the number of eliminated variables.

**poly_return_term_list**  
[Option variable]  
Default value: false  
If set to true, all functions in this package will return each polynomial as a list of terms in the current monomial order rather than a maxima general expression.

**poly_grobner_debug**  
[Option variable]  
Default value: false  
If set to true, produce debugging and tracing output.

**poly_grobner_algorithm**  
[Option variable]  
Default value: buchberger  
Possible values:
- buchberger
- parallel_buchberger
- gebauer_moeller

The name of the algorithm used to find the Groebner Bases.
poly_top_reduction_only

[Option variable]

Default value: false

If not false, use top reduction only whenever possible. Top reduction means that division algorithm stops after the first reduction.

62.2.2 Simple operators in grobner

poly_add, poly_subtract, poly_multiply and poly_expt are the arithmetical operations on polynomials. These are performed using the internal representation, but the results are converted back to the maxima general form.

poly_add (poly1, poly2, varlist)

[Function]

Adds two polynomials poly1 and poly2.

(%i1) poly_add(z+x^2*y,x-z,[x,y,z]);
2
(%o1) x y + x

poly_subtract (poly1, poly2, varlist)

[Function]

Subtracts a polynomial poly2 from poly1.

(%i1) poly_subtract(z+x^2*y,x-z,[x,y,z]);
2
(%o1) 2 z + x y - x

poly_multiply (poly1, poly2, varlist)

[Function]

Returns the product of polynomials poly1 and poly2.

(%i2) poly_multiply(z+x^2*y,x-z,[x,y,z])-(z+x^2*y)*(x-z),expand;
(%o1) 0

poly_s_polynomial (poly1, poly2, varlist)

[Function]

Returns the syzygy polynomial (S-polynomial) of two polynomials poly1 and poly2.

poly_primitive_part (poly1, varlist)

[Function]

Returns the polynomial poly divided by the GCD of its coefficients.

(%i1) poly_primitive_part(35*y+21*x,[x,y]);
(%o1) 5 y + 3 x

poly_normalize (poly, varlist)

[Function]

Returns the polynomial poly divided by the leading coefficient. It assumes that the division is possible, which may not always be the case in rings which are not fields.

62.2.3 Other functions in grobner

poly_expand (poly, varlist)

[Function]

This function parses polynomials to internal form and back. It is equivalent to expand(poly) if poly parses correctly to a polynomial. If the representation is not compatible with a polynomial in variables varlist, the result is an error. It can be
used to test whether an expression correctly parses to the internal representation. The following examples illustrate that indexed and transcendental function variables are allowed.

\[
\begin{align*}
\text{(%i1)} & \text{ poly-expand((x-y)*(y+x),[x,y]);} \\
\text{(%o1)} & x^2 - y^2 \\
\text{(%i2)} & \text{poly-expand((y+x)^2,[x,y]);} \\
\text{(%o2)} & y^2 + 2xy + x^2 \\
\text{(%i3)} & \text{poly-expand((y+x)^5,[x,y]);} \\
\text{(%o3)} & y^5 + 5xy^4 + 10x^2y^3 + 10x^3y^2 + 5x^4y + x^5 \\
\text{(%i4)} & \text{poly-expand(-1-x*exp(y)+x^2/sqrt(y),[x]);} \\
\text{(%o4)} & -x\ e^y + \frac{x^2}{\sqrt{y}} - 1 \\
\text{(%i5)} & \text{poly-expand(-1-sin(x)^2+sin(x),[sin(x)]);} \\
\text{(%o5)} & -\sin^2(x) + \sin(x) - 1
\end{align*}
\]

\text{poly_expt (poly, number, varlist)} \quad [\text{Function}]

exponentiates \text{poly} by a positive integer \text{number}. If \text{number} is not a positive integer number an error will be raised.

\[
\begin{align*}
\text{(%i1)} & \text{poly_expt(x-y,3,[x,y])-(x-y)^3,expand;} \\
\text{(%o1)} & 0
\end{align*}
\]

\text{poly_content (poly, varlist)} \quad [\text{Function}]

\text{poly_content} extracts the GCD of its coefficients

\[
\begin{align*}
\text{(%i1)} & \text{poly_content(35*y+21*x,[x,y]);} \\
\text{(%o1)} & 7
\end{align*}
\]

\text{poly_pseudo_divide (poly, polylist, varlist)} \quad [\text{Function}]

Pseudo-divide a polynomial \text{poly} by the list of \text{n} polynomials \text{polylist}. Return multiple values. The first value is a list of quotients \text{a}. The second value is the remainder \text{r}. The third argument is a scalar coefficient \text{c}, such that \text{c * poly} can be divided by \text{polylist} within the ring of coefficients, which is not necessarily a field. Finally, the fourth value is an integer count of the number of reductions performed. The resulting objects satisfy the equation:

\[
c \cdot poly = \sum_{i=1}^{n} (a_i \cdot polylist_i) + r
\]
poly_exact_divide (poly1, poly2, varlist)  [Function]
Divide a polynomial poly1 by another polynomial poly2. Assumes that exact division
with no remainder is possible. Returns the quotient.

poly_normal_form (poly, polylist, varlist)  [Function]
poly_normal_form finds the normal form of a polynomial poly with respect to a set
of polynomials polylist.

poly_buchberger_criterion (polylist, varlist)  [Function]
Returns true if polylist is a Groebner basis with respect to the current term order,
by using the Buchberger criterion: for every two polynomials h1 and h2 in polylist
the S-polynomial S(h1, h2) reduces to 0 modulo polylist.

poly_buchberger (polylist_f1 varlist)  [Function]
poly_buchberger performs the Buchberger algorithm on a list of polynomials and
returns the resulting Groebner basis.

62.2.4 Standard postprocessing of Groebner Bases
The k-th elimination ideal Ik of an ideal I over K[x1, ..., xk] is I ∩ K[xk+1, ..., xn].
The colon ideal I : J is the ideal \{h | ∀w ∈ J : wh ∈ I\}.
The ideal I : p∞ is the ideal \{h | ∃n ∈ N : pn h ∈ I\}.
The ideal I : J∞ is the ideal \{h | ∃n ∈ N, ∃p ∈ J : pn h ∈ I\}.
The radical ideal √I is the ideal \{h | ∃n ∈ N : hn ∈ I\}.

poly_reduction (polylist, varlist)  [Function]
poly_reduction reduces a list of polynomials polylist, so that each polynomial is
fully reduced with respect to the other polynomials.

poly_minimization (polylist, varlist)  [Function]
Returns a sublist of the polynomial list polylist spanning the same monomial ideal as
polylist but minimal, i.e. no leading monomial of a polynomial in the sublist divides
the leading monomial of another polynomial.

poly_normalize_list (polylist, varlist)  [Function]
poly_normalize_list applies poly_normalize to each polynomial in the list. That
means it divides every polynomial in a list polylist by its leading coefficient.

poly_grobner (polylist, varlist)  [Function]
Returns a Groebner basis of the ideal span by the polynomials polylist. Affected by
the global flags.

poly_reduced_grobner (polylist, varlist)  [Function]
Returns a reduced Groebner basis of the ideal span by the polynomials polylist.
Affected by the global flags.

poly_depends_p (poly, var, varlist)  [Function]
poly_depends tests whether a polynomial depends on a variable var.
poly_elimination_ideal (polylist, number, varlist)  [Function]
poly_elimination_ideal returns the grobner basis of the number-th elimination
ideal of an ideal specified as a list of generating polynomials (not necessarily Groebner
basis).

poly_colon_ideal (polylist1, polylist2, varlist)  [Function]
Returns the reduced Groebner basis of the colon ideal
I(polylist1) : I(polylist2)
where polylist1 and polylist2 are two lists of polynomials.

poly_ideal_intersection (polylist1, polylist2, varlist)  [Function]
poly_ideal_intersection returns the intersection of two ideals.

poly_lcm (poly1, poly2, varlist)  [Function]
Returns the lowest common multiple of poly1 and poly2.

poly_gcd (poly1, poly2, varlist)  [Function]
Returns the greatest common divisor of poly1 and poly2.
See also ezgcd, gcd, gcdex, and gcdivide.
Example:

(\%i1) p1:6*x^3+19*x^2+19*x+6;
3 2
(\%o1) 6 x + 19 x + 19 x + 6
(\%i2) p2:6*x^5+13*x^4+12*x^3+13*x^2+6*x;
5 4 3 2
(\%o2) 6 x + 13 x + 12 x + 13 x + 6 x
(\%i3) poly_gcd(p1, p2, [x]);
2
(\%o3) 6 x + 13 x + 6

poly_grobner_equal (polylist1, polylist2, varlist)  [Function]
poly_grobner_equal tests whether two Groebner Bases generate the same ideal. Re-
turns true if two lists of polynomials polylist1 and polylist2, assumed to be Groebner
Bases, generate the same ideal, and false otherwise. This is equivalent to checking
that every polynomial of the first basis reduces to 0 modulo the second basis and vice
versa. Note that in the example below the first list is not a Groebner basis, and thus
the result is false.

(\%i1) poly_grobner_equal([y+x,x-y],[x,y],[x,y]);
(\%o1) false

poly_grobner_subsetp (polylist1, polylist2, varlist)  [Function]
poly_grobner_subsetp tests whether an ideal generated by polylist1 is contained in
the ideal generated by polylist2. For this test to always succeed, polylist2 must be a
Groebner basis.

poly_grobner_member (poly, polylist, varlist)  [Function]
Returns true if a polynomial poly belongs to the ideal generated by the polynomial
list polylist, which is assumed to be a Groebner basis. Returns false otherwise.
poly_grobner_member tests whether a polynomial belongs to an ideal generated by a list of polynomials, which is assumed to be a Groebner basis. Equivalent to normal_form being 0.

poly_ideal_saturation1 (polylist, poly, varlist)  
Returns the reduced Groebner basis of the saturation of the ideal

\[ I(polylist) : \text{poly}^\infty \]

Geometrically, over an algebraically closed field, this is the set of polynomials in the ideal generated by polylist which do not identically vanish on the variety of poly.

poly_ideal_saturation (polylist1, polylist2, varlist)  
Returns the reduced Groebner basis of the saturation of the ideal

\[ I(polylist1) : I(polylist2)^\infty \]

Geometrically, over an algebraically closed field, this is the set of polynomials in the ideal generated by polylist1 which do not identically vanish on the variety of polylist2.

poly_ideal_polysaturation1 (polylist1, polylist2, varlist)  
\text{polylist2} is a list of n polynomials \([\text{poly}_1, \ldots, \text{poly}_n]\). Returns the reduced Groebner basis of the ideal

\[ I(polylist) : \text{poly}_1^\infty : \ldots : \text{poly}_n^\infty \]

obtained by a sequence of successive saturations in the polynomials of the polynomial list polylist2 of the ideal generated by the polynomial list polylist1.

poly_ideal_polysaturation (polylist, polylistlist, varlist)  
\text{polylistlist} is a list of n list of polynomials \([\text{polylist}_1, \ldots, \text{polylist}_n]\). Returns the reduced Groebner basis of the saturation of the ideal

\[ I(polylist) : I(polylist_1)^\infty : \ldots : I(polylist_n)^\infty \]

poly_saturation_extension (poly, polylist, varlist1, varlist2)  
poly_saturation_extension implements the famous Rabinowitz trick.

poly_polysaturation_extension (poly, polylist, varlist1, varlist2)
63 impdiff

63.1 Functions and Variables for impdiff

\texttt{implicit\_derivative (f,indvarlist,orderlist,depvar)} [Function]

This subroutine computes implicit derivatives of multivariable functions. \textit{f} is an array function, the indexes are the derivative degree in the \textit{indvarlist} order; \textit{indvarlist} is the independent variable list; \textit{orderlist} is the order desired; and \textit{depvar} is the dependent variable.

To use this function write first \texttt{load("impdiff")}. 
64 interpol

64.1 Introduction to interpol

Package **interpol** defines the Lagrangian, the linear and the cubic splines methods for polynomial interpolation.

For comments, bugs or suggestions, please contact me at 'mario AT edu DOT xunta DOT es'.

64.2 Functions and Variables for interpol

**lagrange**

[Function]

\[ \text{lagrange (points)} \]
\[ \text{lagrange (points, option)} \]

Computes the polynomial interpolation by the Lagrangian method. Argument **points** must be either:

- a two column matrix, \( p: \text{matrix}([2,4],[5,6],[9,3]) \);
- a list of pairs, \( p: [[2,4],[5,6],[9,3]] \);
- a list of numbers, \( p: [4,6,3] \), in which case the abscissas will be assigned automatically to 1, 2, 3, etc.

In the first two cases the pairs are ordered with respect to the first coordinate before making computations.

With the **option** argument it is possible to select the name for the independent variable, which is 'x by default; to define another one, write something like **varname**='z'.

Note that when working with high degree polynomials, floating point evaluations are unstable.

See also **linearinterp**, **cspline**, and **ratinterp**.

Examples:

```plaintext
(%i1) load(interpol)$
(%i2) p:[[7,2],[8,2],[1,5],[3,2],[6,7]]$
(%i3) lagrange(p);
(x - 7) (x - 6) (x - 3) (x - 1)
---
35
(x - 8) (x - 6) (x - 3) (x - 1)
---
12
7 (x - 8) (x - 7) (x - 3) (x - 1)
+---
30
(x - 8) (x - 7) (x - 6) (x - 1)
---
60
(x - 8) (x - 7) (x - 6) (x - 3)
```
(%i4) f(x) := (x - 7) (x - 6) (x - 3) (x - 1)
(%o4) f(x) := -------------------------------
             35
                    (x - 8) (x - 6) (x - 3) (x - 1)
                      - -------------------------------
                             12
                               7 (x - 8) (x - 7) (x - 3) (x - 1)
                             + -------------------------------
                                    30
                              (x - 8) (x - 7) (x - 6) (x - 1)
                             - -------------------------------
                                        60
                              (x - 8) (x - 7) (x - 6) (x - 3)
                             + -------------------------------
                                    84
(%i5) /* Evaluate the polynomial at some points */
    expand(map(f,[2.3,5/7,%pi]));
     4  3  2
 919062 73 %pi 701 %pi 8957 %pi
(%o5) [- 1.567535, ------, ------- - -------- + ---------
     84035 420 210 420
 5288 %pi 186
    - ------- + ---]
     105 5
(%i6) %,numer;
(%o6) [- 1.567535, 10.9366573451538, 2.89319655125692]
(%i7) load(draw)$ /* load draw package */
(%i8) /* Plot the polynomial together with points */
    draw2d(
      color = red,
      key = "Lagrange polynomial",
      explicit(f(x),x,0,10),
      point_size = 3,
      color = blue,
      key = "Sample points",
      points(p))$
(%i9) /* Change variable name */
    lagrange(p, varname=w);
     (w - 7) (w - 6) (w - 3) (w - 1)
(%o9) -------------------------------
             35
                    (w - 8) (w - 6) (w - 3) (w - 1)
                      - -------------------------------
                             12
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\[
\begin{align*}
7 (w - 8) (w - 7) (w - 3) (w - 1) \\
+ \frac{30}{(w - 8) (w - 7) (w - 6) (w - 1)} \\
- \frac{60}{(w - 8) (w - 7) (w - 6) (w - 3)} \\
+ \frac{84}{\cdots}
\end{align*}
\]

\texttt{charfun2 (x, a, b)}  \quad \text{[Function]}

Returns \texttt{true} if number \( x \) belongs to the interval \([a, b]\), and \texttt{false} otherwise.

\textbf{linearinterpol}  \quad \text{[Function]}

\texttt{linearinterpol (points)}

\texttt{linearinterpol (points, option)}

Computes the polynomial interpolation by the linear method. Argument \texttt{points} must be either:

- a two column matrix, \( p: \text{matrix}([2,4],[5,6],[9,3]) \),
- a list of pairs, \( p: [[2,4],[5,6],[9,3]] \),
- a list of numbers, \( p: [4,6,3] \), in which case the abscissas will be assigned automatically to 1, 2, 3, etc.

In the first two cases the pairs are ordered with respect to the first coordinate before making computations.

With the \texttt{option} argument it is possible to select the name for the independent variable, which is 'x by default; to define another one, write something like \texttt{varname='z}. See also \texttt{lagrange}, \texttt{cspline}, and \texttt{ratinterpol}.

Examples:

\begin{verbatim}
(%i1) load(interpol)$
(%i2) p: matrix([7,2],[8,3],[1,5],[3,2],[6,7])$
(%i3) linearinterpol(p); 13 3 x
   2 2
   + (x - 5) charfun2(x, 7, inf) + (37 - 5 x) charfun2(x, 6, 7)
   5 x
   + (--- - 3) charfun2(x, 3, 6)
   3

(%i4) f(x):=''%; 13 3 x
   2 2
   + (x - 5) charfun2(x, 7, inf) + (37 - 5 x) charfun2(x, 6, 7)
   5 x
\end{verbatim}
\[
+ (--- - 3) \text{charfun2}(x, 3, 6)
\]

(%i5) /* Evaluate the polynomial at some points */
map(f,[7.3,25/7,%pi]);
   62  5 \%pi
(%o5) [2.3, --, ------ - 3]
   21 3

(%i6) %,numer;
(%o6) [2.3, 2.952380952380953, 2.235987755982989]

(%i7) load(draw)$ /* load draw package */
(%i8) /* Plot the polynomial together with points */
draw2d(
   color = red,
   key = "Linear interpolator",
   explicit(f(x),x,-5,20),
   point_size = 3,
   color = blue,
   key = "Sample points",
   points(args(p)))$

(%i9) /* Change variable name */
linearinterpol(p, varname='s);

(%o9) 13 3 s
     (-- - ---) \text{charfun2}(s, \text{minf}, 3)
     2 2
  + (s - 5) \text{charfun2}(s, 7, \text{inf}) + (37 - 5 s) \text{charfun2}(s, 6, 7)
     5 s
+ (--- - 3) \text{charfun2}(s, 3, 6)
     3

\texttt{cspline}

\texttt{cspline(points)}

\texttt{cspline(points, option1, option2, ...)}

Computes the polynomial interpolation by the cubic splines method. Argument \texttt{points} must be either:

- a two column matrix, \texttt{p:matrix([2,4],[5,6],[9,3])},
- a list of pairs, \texttt{p: [[2,4],[5,6],[9,3]]},
- a list of numbers, \texttt{p: [4,6,3]}, in which case the abscissas will be assigned automatically to 1, 2, 3, etc.

In the first two cases the pairs are ordered with respect to the first coordinate before making computations.

There are three options to fit specific needs:

- \texttt{'d1}, default \texttt{'unknown}, is the first derivative at \(x_1\); if it is \texttt{'unknown}, the second derivative at \(x_1\) is made equal to 0 (natural cubic spline); if it is equal to a number, the second derivative is calculated based on this number.
• 'dn, default 'unknown, is the first derivative at \( x_n \); if it is 'unknown, the second derivative at \( x_n \) is made equal to 0 (natural cubic spline); if it is equal to a number, the second derivative is calculated based on this number.

• 'varname, default 'x, is the name of the independent variable.

See also lagrange, linearinterpol, and ratinterpol.

Examples:

```lisp
(%i1) load(interpol)$
(%i2) p:[[7,2],[8,2],[1,5],[3,2],[6,7]]$
(%i3) /* Unknown first derivatives at the extremes
is equivalent to natural cubic splines */
cspine(p);
3 2
1159 x 1159 x 6091 x 8283
3288 1096 3288 1096
3 2
2587 x 5174 x 494117 x 108928
1644 137 1644 137
3 2
4715 x 15209 x 579277 x 199575
1644 274 1644 274
3 2
3287 x 2223 x 48275 x 9609
1644 274 1644 274
(%o3) (%i4) f(x):='''$
(%i5) /* Some evaluations */
map(f,[2.3,5/7,%pi]), numer;
(%o5) [1.991460766423356, 5.823200187269903, 2.227405312429507]
(%i6) load(draw)$ /* load draw package */
(%i7) /* Plotting interpolating function */
draw2d(
    color = red,
    key = "Cubic splines",
    explicit(f(x),x,0,10),
    point_size = 3,
    color = blue,
    key = "Sample points",
    points(p))$
(%o7) new
(%i8) /* New call, but giving values at the derivatives */
cspine(p,d1=0,dn=0);
3 2
1949 x 11437 x 17027 x 1247
```
ratinterpol

ratinterpol (points, numdeg)
ratinterpol (points, numdeg, option1)

Generates a rational interpolator for data given by points and the degree of the numerator being equal to numdeg; the degree of the denominator is calculated automatically. Argument points must be either:

- a two column matrix, p: matrix([2,4],[5,6],[9,3]),
- a list of pairs, p: [[2,4],[5,6],[9,3]],
- a list of numbers, p: [4,6,3], in which case the abscissas will be assigned automatically to 1, 2, 3, etc.

In the first two cases the pairs are ordered with respect to the first coordinate before making computations.

There is one option to fit specific needs:

- 'varname, default 'x, is the name of the independent variable.

See also lagrange, linearinterpol, cspline, minpack_lsquares, and Chapter 66 [lbfgs-pkg], page 969,
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Examples:

(%i1) load(interpol)$
(%i2) load(draw)$
(%i3) p: [[7.2, 2.5], [8.5, 2.1], [1.6, 5.1], [3.4, 2.4], [6.7, 7.9]]$
(%i4) for k: 0 thru length(p)-1 do
    draw2d(
        explicit(ratinterpol(p,k),x,0,9),
        point_size = 3,
        points(p),
        title = concat("Degree of numerator = ",k),
        yrange=[0,10])$
65 lapack

65.1 Introduction to lapack

lapack is a Common Lisp translation (via the program f2cl) of the Fortran library LAPACK, as obtained from the SLATEC project.

65.2 Functions and Variables for lapack

dgeev

\texttt{dgeev (A)} \quad \texttt{dgeev (A, right\_p, left\_p)}

Computes the eigenvalues and, optionally, the eigenvectors of a matrix \( A \). All elements of \( A \) must be integer or floating point numbers. \( A \) must be square (same number of rows and columns). \( A \) might or might not be symmetric.

\( \texttt{dgeev(A)} \) computes only the eigenvalues of \( A \). \( \texttt{dgeev(A, right\_p, left\_p)} \) computes the eigenvalues of \( A \) and the right eigenvectors when \( right\_p = \text{true} \) and the left eigenvectors when \( left\_p = \text{true} \).

A list of three items is returned. The first item is a list of the eigenvalues. The second item is \texttt{false} or the matrix of right eigenvectors. The third item is \texttt{false} or the matrix of left eigenvectors.

The right eigenvector \( v(j) \) (the \( j \)-th column of the right eigenvector matrix) satisfies
\[
A.v(j) = \lambda(j).v(j)
\]
where \( \lambda(j) \) is the corresponding eigenvalue. The left eigenvector \( u(j) \) (the \( j \)-th column of the left eigenvector matrix) satisfies
\[
u(j)^*H.A = \lambda(j).u(j)^*H
\]
where \( u(j)^*H \) denotes the conjugate transpose of \( u(j) \). The Maxima function \texttt{ctranspose} computes the conjugate transpose.

The computed eigenvectors are normalized to have Euclidean norm equal to 1, and largest component has imaginary part equal to zero.

Example:

\begin{verbatim}
(%i1) load ("lapack")$
(%i2) fpprintprec : 6;
(%o2) 6
(%i3) M : matrix ([9.5, 1.75], [3.25, 10.45]);
(%o3) 
[ 9.5 1.75 ]
[ 3.25 10.45 ]
(%i4) dgeev (M);
(%o4) [[7.54331, 12.4067], false, false]
(%i5) [L, v, u] : dgeev (M, true, true);
(%o5) [[7.54331, 12.4067], [ - .666642 - .515792 ]
[ .745378 - .856714 ]
\end{verbatim}
(%i6) D : apply (diag_matrix, L);
       [ 7.54331  0 ]
[ ]
       [ 0 12.4067 ]
(%o6)

(%i7) M . v - v . D;
       [ 0.0 - 8.88178E-16 ]
[ ]
       [ - 8.88178E-16  0.0 ]
(%o7)

(%i8) transpose (u) . M - D . transpose (u);
       [ 0.0 - 4.44089E-16 ]
[ ]
       [ 0.0 0.0 ]
(%o8)

\textbf{dgeqrf} (A)

Computes the QR decomposition of the matrix A. All elements of A must be integer or floating point numbers. A may or may not have the same number of rows and columns.

A list of two items is returned. The first item is the matrix Q, which is a square, orthonormal matrix which has the same number of rows as A. The second item is the matrix R, which is the same size as A, and which has all elements equal to zero below the diagonal. The product \(Q \cdot R\), where "," is the noncommutative multiplication operator, is equal to A (ignoring floating point round-off errors).

(%i11) load ("lapack") $
(%i12) fpprintprec : 6 $
(%i13) M : matrix ([1, -3.2, 8], [-11, 2.7, 5.9]) $
(%i14) [q, r] : dgeqrf (M);
[ - .0905357 .995893 ]
[ ]
[ .995893 .0905357 ]
(%o14)

(%i15) q . r - M;
[ ]
[ 0.0 - 1.33227E-15  8.88178E-16 ]
(%o15)

\textbf{dgesv} (A, b)

Computes the solution x of the linear equation \(Ax = b\), where A is a square matrix, and b is a matrix of the same number of rows as A and any number of columns. The return value x is the same size as b.
The elements of \(A\) and \(b\) must evaluate to real floating point numbers via \texttt{float}; thus elements may be any numeric type, symbolic numerical constants, or expressions which evaluate to floats. The elements of \(x\) are always floating point numbers. All arithmetic is carried out as floating point operations.

dgesv computes the solution via the LU decomposition of \(A\).

Examples:

dgesv computes the solution of the linear equation \(Ax = b\).

\begin{verbatim}
(%i1) A : matrix ([[1, -2.5], [0.375, 5]]);
  [ 1 - 2.5 ]
  [          ]
  [ 0.375  5 ]
(%o1) [          ]

(%i2) b : matrix ([[1.75], [-0.625]]);
  [ 1.75 ]
  [      ]
  [ - 0.625 ]
(%o2) [      ]

(%i3) x : dgesv (A, b);
  [ 1.210526315789474 ]
  [                  ]
  [ - 0.215789473684211 ]
(%o3) [                  ]

(%i4) dlange (inf_norm, b - A.x);
(%o4) 0.0
\end{verbatim}

\(b\) is a matrix with the same number of rows as \(A\) and any number of columns. \(x\) is the same size as \(b\).

\begin{verbatim}
(%i1) A : matrix ([[1, -0.15], [1.82, 2]]);
  [ 1 - 0.15 ]
  [          ]
  [ 1.82  2 ]
(%o1) [          ]

(%i2) b : matrix ([[3.7, 1, 8], [-2.3, 5, -3.9]]);
  [ 3.7  1  8 ]
  [          ]
  [ - 2.3  5  - 3.9 ]
(%o2) [          ]

(%i3) x : dgesv (A, b);
  [ 1.210526315789474  1.20985481742191  6.781786185657722 ]
  [                  ]
  [ - 0.215789473684211  1.399032116146062 -8.121425428948527 ]
(%o3) [                  ]

(%i4) dlange (inf_norm, b - A.x);
(%o4) 1.1102230246251565E-15
\end{verbatim}

The elements of \(A\) and \(b\) must evaluate to real floating point numbers.

\begin{verbatim}
(%i1) A : matrix ([[5, -%pi], [1b0, 11/17]]);
  [ 5   - %pi ]
  [           ]
  [ 1.0b0     ]
(%o1) [           ]

(%i2) b : matrix ([[%e], [sin(1)]]);
\end{verbatim}
\( x : \text{dgesv} (A, b); \)
\[
\begin{bmatrix}
0.690375643155986 \\
0.233510982552952
\end{bmatrix}
\]
\( \text{dlange} (\text{inf\_norm}, b - A \cdot x); \)
\( 2.220446049250313\times10^{-16} \)

\( \text{dgesvd} (A) \)
\( \text{dgesvd} (A, \text{left\_p}, \text{right\_p}) \)

Computes the singular value decomposition (SVD) of a matrix \( A \), comprising the singular values and, optionally, the left and right singular vectors. All elements of \( A \) must be integer or floating point numbers. \( A \) might or might not be square (same number of rows and columns).

Let \( m \) be the number of rows, and \( n \) the number of columns of \( A \). The singular value decomposition of \( A \) comprises three matrices, \( U \), \( \Sigma \), and \( V^T \), such that \( A = U \Sigma V^T \)

where \( U \) is an \( m \)-by-\( m \) unitary matrix, \( \Sigma \) is an \( m \)-by-\( n \) diagonal matrix, and \( V^T \) is an \( n \)-by-\( n \) unitary matrix.

Let \( \sigma[i] \) be a diagonal element of \( \Sigma \), that is, \( \Sigma[i,i] = \sigma[i] \). The elements \( \sigma[i] \) are the so-called singular values of \( A \); these are real and nonnegative, and returned in descending order. The first \( \min(m, n) \) columns of \( U \) and \( V \) are the left and right singular vectors of \( A \). Note that \( \text{dgesvd} \) returns the transpose of \( V \), not \( V \) itself.

\( \text{dgesvd}(A) \) computes only the singular values of \( A \). \( \text{dgesvd}(A, \text{left\_p}, \text{right\_p}) \) computes the singular values of \( A \) and the left singular vectors when \( \text{left\_p} = \text{true} \) and the right singular vectors when \( \text{right\_p} = \text{true} \).

A list of three items is returned. The first item is a list of the singular values. The second item is \( \text{false} \) or the matrix of left singular vectors. The third item is \( \text{false} \) or the matrix of right singular vectors.

Example:

(\%2) \text{load} ("lapack");
(\%3) \text{fpprintprec : 6;}
(\%4) \text{M : matrix([1, 2, 3], [3.5, 0.5, 8], [-1, 2, -3], [4, 9, 7]);}
(\%5) \[
\begin{bmatrix}
1 & 2 & 3 \\
3.5 & 0.5 & 8 \\
-1 & 2 & -3 \\
4 & 9 & 7
\end{bmatrix}
\]
Chapter 65: lapack

(%i4) dgesvd (M);
(%o4) [[14.4744, 6.38637, .452547], false, false]

(%i5) [sigma, U, VT] : dgesvd (M, true, true);
(%o5) [[14.4744, 6.38637, .452547],
[- .256731 .00816168 .959029 -.119523 ]
[ - .526456 .672116 -.206236 -.478091 ]
[ .107997 -.532278 -.0708315 - .83666 ]
[ - .803287 -.514659 -.180867 .239046 ]
[ - .374486 -.538209 -.755044 ]
[ .130623 -.836799 .5317 ]
[ - .917986 .100488 .383672 ]
(%i6) m : length (U);
(%o6) 4
(%i7) n : length (VT);
(%o7) 3
(%i8) Sigma:
  genmatrix(lambda ([i, j], if i=j then sigma[i] else 0),
  m, n);
(%o8)
  [ 14.4744 0 0 ]
  [ 0 6.38637 0 ]
  [ 0 0 .452547 ]
  [ 0 0 0 ]

 (%i9) U . Sigma . VT - M;
 (%o9) 
  [ 1.11022E-15 0.0 1.77636E-15 ]
  [ 1.33227E-15 1.66533E-15 0.0 ]
  [ - 4.44089E-16 - 8.88178E-16 4.44089E-16 ]
  [ 8.88178E-16 1.77636E-15 8.88178E-16 ]

 (%i10) transpose (U) . U;
 (%o10) 
  [ 1.0 5.55112E-17 2.498E-16 2.77556E-17 ]
  [ 5.55112E-17 1.0 5.55112E-17 4.16334E-17 ]
  [ 2.498E-16 5.55112E-17 1.0 - 2.08167E-16 ]
  [ 2.77556E-17 4.16334E-17 - 2.08167E-16 1.0 ]

 (%i11) VT . transpose (VT);
[ 1.0 0.0 - 5.55112E-17 ]
[ ]
(%o11) [ 0.0 1.0 5.55112E-17 ]
[ ]
[ - 5.55112E-17 5.55112E-17 1.0 ]

dlange (norm, A)  [Function]

dlange computes a norm or norm-like function of the matrix A.

max
Compute \( \max(\text{abs}(A(i,j))) \) where \( i \) and \( j \) range over the rows and columns, respectively, of \( A \). Note that this function is not a proper matrix norm.

one_norm
Compute the \( L[1] \) norm of \( A \), that is, the maximum of the sum of the absolute value of elements in each column.

inf_norm
Compute the \( L[\infty] \) norm of \( A \), that is, the maximum of the sum of the absolute value of elements in each row.

frobenius
Compute the Frobenius norm of \( A \), that is, the square root of the sum of squares of the matrix elements.

dgemm
[Function]

dgemm (A, B)
dgemm (A, B, options)
Compute the product of two matrices and optionally add the product to a third matrix.

In the simplest form, \( \text{dgemm}(A, B) \) computes the product of the two real matrices, \( A \) and \( B \).

In the second form, \( \text{dgemm} \) computes the \( \alpha A + \beta B \) where \( A, B, C \) are real matrices of the appropriate sizes and \( \alpha \) and \( \beta \) are real numbers. Optionally, \( A \) and/or \( B \) can be transposed before computing the product. The extra parameters are specified by optional keyword arguments: The keyword arguments are optional and may be specified in any order. They all take the form \( \text{key}=\text{val} \). The keyword arguments are:

\( C \)  The matrix \( C \) that should be added. The default is \text{false}, which means no matrix is added.

\( \alpha \)  The product of \( A \) and \( B \) is multiplied by this value. The default is 1.

\( \beta \)  If a matrix \( C \) is given, this value multiplies \( C \) before it is added. The default value is 0, which implies that \( C \) is not added, even if \( C \) is given. Hence, be sure to specify a non-zero value for \( \beta \).

\( \text{transpose_a} \)  If \text{true}, the transpose of \( A \) is used instead of \( A \) for the product. The default is \text{false}. 
transpose_b

If true, the transpose of B is used instead of B for the product. The default is false.

(%i1) load ("lapack")$
(%i2) A : matrix([1,2,3],[4,5,6],[7,8,9]);
    [ 1  2  3 ]
    [     ]
(%o2)    [ 4  5  6 ]
    [     ]
    [ 7  8  9 ]
(%i3) B : matrix([-1,-2,-3],[-4,-5,-6],[-7,-8,-9]);
    [ -1  -2  -3 ]
    [        ]
(%o3)    [ -4  -5  -6 ]
    [        ]
    [ -7  -8  -9 ]
(%i4) C : matrix([3,2,1],[6,5,4],[9,8,7]);
    [ 3  2  1 ]
    [     ]
(%o4)    [ 6  5  4 ]
    [     ]
    [ 9  8  7 ]
(%i5) dgemm(A,B);     [ -30.0  -36.0  -42.0 ]
    [                   ]
(%o5)    [ -66.0  -81.0  -96.0 ]
    [                   ]
    [ -102.0 -126.0 -150.0 ]
(%i6) A . B;
    [ -30  -36  -42 ]
    [        ]
(%o6)    [ -66  -81  -96 ]
    [        ]
    [ -102 -126 -150 ]
(%i7) dgemm(A,B,transpose_a=true);
    [ -66.0  -78.0  -90.0 ]
    [                 ]
(%o7)    [ -78.0  -93.0  -108.0 ]
    [                 ]
    [ -90.0 -108.0 -126.0 ]
(%i8) transpose(A) . B;
    [ -66  -78  -90 ]
    [        ]
(%o8)    [ -78  -93  -108 ]
    [        ]
    [ -90 -108 -126 ]
(%i9) dgemm(A,B,c=C,beta=1);
[ - 27.0 - 34.0 - 41.0 ]
[ ]
(%o9) [ - 60.0 - 76.0 - 92.0 ]
[ ]
[ - 93.0 - 118.0 - 143.0 ]
(%i10) A . B + C;
[ - 27 - 34 - 41 ]
[ ]
(%o10) [ - 60 - 76 - 92 ]
[ ]
[ - 93 - 118 - 143 ]
(%i11) dgemm(A,B,c=C,beta=1, alpha=-1);
[ 33.0 38.0 43.0 ]
[ ]
(%o11) [ 72.0 86.0 100.0 ]
[ ]
[ 111.0 134.0 157.0 ]
(%i12) -A . B + C;
[ 33 38 43 ]
[ ]
(%o12) [ 72 86 100 ]
[ ]
[ 111 134 157 ]

zgeev

Function

zgeev (A)

zgeev (A, right_p, left_p)
Like dgeev, but the matrix A is complex.

zheev

Function

zheev (A)

zheev (A, eigvec_p)
Like zheev, but the matrix A is assumed to be a square complex Hermitian matrix.
If eigvec_p is true, then the eigenvectors of the matrix are also computed.
No check is made that the matrix A is, in fact, Hermitian.
A list of two items is returned, as in dgeev: a list of eigenvalues, and false or the matrix of the eigenvectors.
66 lbfgs

66.1 Introduction to lbfgs

lbfgs is an implementation of the L-BFGS algorithm [1] to solve unconstrained minimization problems via a limited-memory quasi-Newton (BFGS) algorithm. It is called a limited-memory method because a low-rank approximation of the Hessian matrix inverse is stored instead of the entire Hessian inverse. The program was originally written in Fortran [2] by Jorge Nocedal, incorporating some functions originally written by Jorge J. Moré and David J. Thuente, and translated into Lisp automatically via the program f2cl. The Maxima package lbfgs comprises the translated code plus an interface function which manages some details.

References:


66.2 Functions and Variables for lbfgs

lbfgs [Function]

\[ \text{lbfgs (FOM, X, X0, epsilon, iprint)} \]
\[ \text{lbfgs ([FOM, grad] X, X0, epsilon, iprint)} \]

Finds an approximate solution of the unconstrained minimization of the figure of merit FOM over the list of variables X, starting from initial estimates X0, such that \( \text{norm(grad(FOM)) < epsilon * max(1, norm(X))} \).

grad, if present, is the gradient of FOM with respect to the variables X. grad may be a list or a function that returns a list, with one element for each element of X. If not present, the gradient is computed automatically by symbolic differentiation. If FOM is a function, the gradient grad must be supplied by the user.

The algorithm applied is a limited-memory quasi-Newton (BFGS) algorithm [1]. It is called a limited-memory method because a low-rank approximation of the Hessian matrix inverse is stored instead of the entire Hessian inverse. Each iteration of the algorithm is a line search, that is, a search along a ray in the variables X, with the search direction computed from the approximate Hessian inverse. The FOM is always decreased by a successful line search. Usually (but not always) the norm of the gradient of FOM also decreases.

iprint controls progress messages printed by lbfgs.

iprint[1]

iprint[1] controls the frequency of progress messages.

iprint[1] < 0
No progress messages.

iprint[1] = 0
Messages at the first and last iterations.
iprint[1] > 0
Print a message every iprint[1] iterations.

iprint[2]

iprint[2] = 0
Print out iteration count, number of evaluations of FOM, value of FOM, norm of the gradient of FOM, and step length.

iprint[2] = 1
Same as iprint[2] = 0, plus X0 and the gradient of FOM evaluated at X0.

iprint[2] = 2
Same as iprint[2] = 1, plus values of X at each iteration.

iprint[2] = 3
Same as iprint[2] = 2, plus the gradient of FOM at each iteration.

The columns printed by lbfgs are the following.

I  Number of iterations. It is incremented for each line search.

NFN Number of evaluations of the figure of merit.

FUNC Value of the figure of merit at the end of the most recent line search.

GNORM Norm of the gradient of the figure of merit at the end of the most recent line search.

STEPLENGTH
An internal parameter of the search algorithm.

Additional information concerning details of the algorithm are found in the comments of the original Fortran code [2].

See also lbfgs_nfeval_max and lbfgs_ncorrections.

References:

Examples:
The same FOM as computed by FGCOMPUTE in the program sdrive.f in the LBFGS package from Netlib. Note that the variables in question are subscripted variables. The FOM has an exact minimum equal to zero at u[k] = 1 for k = 1, ..., 8.

(%i1) load ("lbfgs")$
(%i2) t1[j] := 1 - u[j];
(%o2)                      t1 := 1 - u
                                 j          j
(%i3) t2[j] := 10*(u[j + 1] - u[j]^2);
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\%(o3) \quad t2 := 10 (u_{j} - u_{j+1})

\%(i4) \quad n : 8;

\%(o4) \quad 8

\%(i5) \quad \text{FOM} := \sum (t1_{2j-1}^2 + t2_{2j-1}^2, j, 1, n/2);

\%(o5) \quad \begin{align*}
100 (u_7 - u_8)^2 + (1 - u_7)^2 + 100 (u_6 - u_7)^2 + (1 - u_6)^2 + \\
100 (u_5 - u_6)^2 + (1 - u_5)^2 + 100 (u_4 - u_5)^2 + (1 - u_4)^2 + \\
100 (u_3 - u_4)^2 + (1 - u_3)^2 + 100 (u_2 - u_3)^2 + (1 - u_2)^2
\end{align*}

\%(i6) \quad \text{lbfgs} (\text{FOM}, [u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8], \\
[-1.2, 1, -1.2, 1, -1.2, 1, -1.2, 1], 1e-3, [1, 0]);

*************

N= 8 NUMBER OF CORRECTIONS=25
INITIAL VALUES
F= 9.680000000000000D+01 GNORM= 4.65735755084533D+02

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<th>FUNC</th>
<th>GNORM</th>
<th>STEPLENGTH</th>
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</table>
A regression problem. The FOM is the mean square difference between the predicted value \( F(X[i]) \) and the observed value \( Y[i] \). The function \( F \) is a bounded monotone function (a so-called "sigmoidal" function). In this example, \texttt{lbfgs} computes approximate values for the parameters of \( F \) and \texttt{plot2d} displays a comparison of \( F \) with the observed data.

\[
\text{(\%i1) load ("lbfgs")$}
\]
\[
\text{(\%i2) FOM : '(@(1/length(X))*sum((F(X[i]) - Y[i])^2, i, 1, length(X))), i, 1, length(X))};
\]
\[
\text{(\%o2) \begin{align*}
&\frac{2}{\text{length(X)}} \\
&\sum_{i} ((F(X[i]) - Y[i])^2) \\
&\end{align*}}
\]
\[
\text{(\%i3) X : [1, 2, 3, 4, 5];}
\]
\[
\text{(\%o3) [1, 2, 3, 4, 5]}
\]
\[
\text{(\%i4) Y : [0, 0.5, 1, 1.25, 1.5];}
\]
\[
\text{(\%o4) [0, 0.5, 1, 1.25, 1.5]}
\]
\[
\text{(\%i5) F(x) := A/(1 + exp(-B*(x - C)))};
\]
\[
\text{(\%o5) F(x) := \frac{A}{1 + \exp(-B*(x - C))}}
\]
\[
\text{(\%i6) ''FOM};
\]
\[
\text{(\%o6) \begin{align*}
&\frac{A}{1 + \exp((-B)(x - C))} \\
&\end{align*}}
\]
Chapter 66: lbfgs

\[
\frac{A}{A + \left( \frac{2}{(3 - C)} - 1 \right) + \left( \frac{2}{(2 - C)} - \frac{1}{\varepsilon} \right) + 1} + \frac{B (3 - C) + 1}{\varepsilon} + 1
\]

\[
\frac{A}{5 - B (1 - C) + 1}
\]

\[
\%e + 1
\]

\[
\%e + 1
\]

\[
(\%i7) \text{estimates : lbfgs (FOM, '[A, B, C], [1, 1, 1], 1e-4, [1, 0]);}
\]

\[
\text{*************************************************}
\]

\[
N = 3 \quad \text{NUMBER OF CORRECTIONS}=25
\]

\[
\text{INITIAL VALUES}
\]

\[
F= 1.34873853426918D-01 \quad \text{GNORM}= 2.000215531936760D-01
\]

\[
\text{*************************************************}
\]

<table>
<thead>
<tr>
<th>I</th>
<th>NFN</th>
<th>FUNC</th>
<th>GNORM</th>
<th>STEPLENGTH</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3</td>
<td>1.177820636622582D-01</td>
<td>9.893138394953992D-01</td>
<td>8.55443596892371D-01</td>
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<tr>
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<td>6</td>
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<td>1.180098521565904D-02</td>
<td>2.10000000000000D+01</td>
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<tr>
<td>3</td>
<td>8</td>
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<td>5.25734056784071D-01</td>
</tr>
<tr>
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<td>9</td>
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<td>1.32504167391314D-02</td>
<td>1.00000000000000D+00</td>
</tr>
<tr>
<td>5</td>
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<td>1.510670810312226D-02</td>
<td>1.00000000000000D+00</td>
</tr>
<tr>
<td>6</td>
<td>11</td>
<td>6.750147275936668D-03</td>
<td>1.914964958023037D-02</td>
<td>1.00000000000000D+00</td>
</tr>
<tr>
<td>7</td>
<td>12</td>
<td>5.850716021108202D-03</td>
<td>1.028089145793828D-02</td>
<td>1.00000000000000D+00</td>
</tr>
<tr>
<td>8</td>
<td>13</td>
<td>5.778664230657800D-03</td>
<td>3.67866074532179D-04</td>
<td>1.00000000000000D+00</td>
</tr>
<tr>
<td>9</td>
<td>14</td>
<td>5.778188236507800D-03</td>
<td>3.01740179797108D-04</td>
<td>1.00000000000000D+00</td>
</tr>
</tbody>
</table>

THE MINIMIZATION TERMINATED WITHOUT DETECTING ERRORS.
IFLAG = 0

\[
(\%o7) [A = 1.461933911464101, B = 1.601593973254801, C = 2.528933072164855]
\]

\[
(\%i8) \text{plot2d ([F(x), \{discrete, X, Y\}], \{x, -1, 6\}), 'estimates;}
\]

Gradient of FOM is specified (instead of computing it automatically). Both the FOM and its gradient are passed as functions to lbfgs.

\[
(\%i11) \text{load ("lbfgs")$}
\]

\[
(\%i12) \text{F(a, b, c) := (a - 5)^2 + (b - 3)^4 + (c - 2)^6$}
\]

\[
(\%i13) \text{define(F_grad(a, b, c),}
\]

\[
\text{map (lambda ([x], diff (F(a, b, c), x)), [a, b, c]))$}
\]

\[
(\%i14) \text{estimates : lbfgs ([F, F_grad],}
\]

\[
[a, b, c], [0, 0, 0], 1e-4, [1, 0]);
\]

\[
\text{*************************************************}
\]

\[
N= 3 \quad \text{NUMBER OF CORRECTIONS}=25
\]

\[
\text{INITIAL VALUES}
\]

\[
F= 1.700000000000000D+02 \quad \text{GNORM}= 2.205175729958953D+02
\]
### INFN FUNC GNORM STEPLENGTH

<table>
<thead>
<tr>
<th>I</th>
<th>NFN</th>
<th>FUNC</th>
<th>GNORM</th>
<th>STEPLENGTH</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td>6.632967565917637D+01</td>
<td>6.498411132518770D+01</td>
<td>4.534785987412505D+03</td>
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<tr>
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<td>3.784147651974131D+01</td>
<td>1.000000000000000D+00</td>
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<td>1.000000000000000D+00</td>
</tr>
<tr>
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<td>9.733664001790506D+00</td>
<td>1.000000000000000D+00</td>
</tr>
<tr>
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<td>1.000000000000000D+00</td>
</tr>
<tr>
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<td>1.215263596054292D+00</td>
<td>2.204727876126877D+00</td>
<td>1.000000000000000D+00</td>
</tr>
<tr>
<td>7</td>
<td>8</td>
<td>1.080252896385329D+02</td>
<td>1.431637116951845D+01</td>
<td>1.000000000000000D+00</td>
</tr>
<tr>
<td>8</td>
<td>9</td>
<td>8.407195124830860D+03</td>
<td>1.126344579730008D+01</td>
<td>1.000000000000000D+00</td>
</tr>
<tr>
<td>9</td>
<td>10</td>
<td>5.02209168198525D+03</td>
<td>7.7507318225275D+02</td>
<td>1.000000000000000D+00</td>
</tr>
<tr>
<td>10</td>
<td>11</td>
<td>2.277152808939775D+03</td>
<td>5.032810859286796D+02</td>
<td>1.000000000000000D+00</td>
</tr>
<tr>
<td>11</td>
<td>12</td>
<td>6.489384688303218D+04</td>
<td>1.932007150210909D+02</td>
<td>1.000000000000000D+00</td>
</tr>
<tr>
<td>12</td>
<td>13</td>
<td>2.075791943844547D+04</td>
<td>6.964319310814365D+03</td>
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<tr>
<td>13</td>
<td>14</td>
<td>7.349472666162258D+05</td>
<td>4.017499067849554D+03</td>
<td>1.000000000000000D+00</td>
</tr>
<tr>
<td>14</td>
<td>15</td>
<td>2.293617477985238D+05</td>
<td>1.334590390856715D+03</td>
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<tr>
<td>15</td>
<td>16</td>
<td>7.683645404048675D+06</td>
<td>6.011057038099202D+04</td>
<td>1.000000000000000D+00</td>
</tr>
</tbody>
</table>

The minimization terminated without detecting errors.

\[
\text{IFLAG} = 0
\]

\[(%o4) \{ a = 5.000086823042934, b = 3.052395429705181, \\
\quad c = 1.927980629919583 \}\]

**lbfgs_nfeval_max**  
Default value: 100  

`lbfgs_nfeval_max` is the maximum number of evaluations of the figure of merit (FOM) in `lbfgs`. When `lbfgs_nfeval_max` is reached, `lbfgs` returns the result of the last successful line search.

**lbfgs_ncorrections**  
Default value: 25  

`lbfgs_ncorrections` is the number of corrections applied to the approximate inverse Hessian matrix which is maintained by `lbfgs`. 
67  lindstedt

67.1  Functions and Variables for lindstedt

Lindstedt (eq,pvar,torder,ic)  [Function]
This is a first pass at a Lindstedt code. It can solve problems with initial conditions entered, which can be arbitrary constants, (just not \%k1 and \%k2) where the initial conditions on the perturbation equations are \( z[i] = 0, z'[i] = 0 \) for \( i > 0 \). ic is the list of initial conditions.

Problems occur when initial conditions are not given, as the constants in the perturbation equations are the same as the zero order equation solution. Also, problems occur when the initial conditions for the perturbation equations are not \( z[i] = 0, z'[i] = 0 \) for \( i > 0 \), such as the Van der Pol equation.

Example:

\[
\begin{align*}
\text{%i1) load("makeOrders")$} \\
\text{%i2) load("lindstedt")$} \\
\text{%i3) Lindstedt('diff(x,t,2)+x-(e*x^3)/6,e,2,[1,0]);} \\
\quad 2 \\
\quad e \ (\cos(5 \ T) - 24 \ \cos(3 \ T) + 23 \ \cos(T)) \\
\text{%o3) \[\frac{36864}{192} \ e \ (\cos(3 \ T) - \cos(T)) } \\
\quad \frac{192}{2} \ \cos(T}\], \\
\quad 2 \\
\quad \frac{7 \ e \ e}{16} \\
\quad T = (- \ ---- - -- + 1) \ t] \\
\text{To use this function write first load("makeOrders") and load("lindstedt").}
\end{align*}
\]
68 linearalgebra

68.1 Introduction to linearalgebra

linearalgebra is a collection of functions for linear algebra.

Example:

(%i1) M : matrix ([1, 2], [1, 2]);
   [ 1 2 ]
   [ ]
   [ 1 2 ]

(%i2) nullspace (M);
   [ 1 ]
   [ ]
   [ ]

   span([ 1 ])
   [ - ]
   [ 2 ]

(%i3) columnspace (M);
   [ 1 ]
   [ ]
   [ ]

   span([ ])
   [ 1 ]

(%i4) ptriangularize (M - z*ident(2), z);
   [ 1 2 - z ]
   [ ]
   [ 2 ]

   [ 0 3 z - z ]

(%i5) M : matrix ([1, 2, 3], [4, 5, 6], [7, 8, 9]) - z*ident(3);
   [ 4 5 - z 6 ]
   [ ]
   [ 7 8 9 - z ]

(%i6) MM : ptriangularize (M, z);
   [ 4 5 - z 6 ]
   [ ]
   [ ]

   [ 2 ]
   [ 66 z 102 z 132 ]
   [ 0 -- -- + ----- + --- ]

(%o6)

(%i7) algebraic : true;
(%o7) true

(%i8) tellrat (MM [3, 3]);
(%o8)  
\[ z - 15 z - 18 z \]

(%i9) MM : ratsimp (MM);
\[
\begin{bmatrix}
4 & 5 - z & 6 \\
\end{bmatrix}
\]

(%o9) 
\[
\begin{bmatrix}
66 & 7 z - 102 z - 132 \\
0 & \frac{\text{-}12}{49} \\
49 & 49 \\
0 & 0 & 0
\end{bmatrix}
\]

(%i10) nullspace (MM);
\[
\begin{bmatrix}
1 \\
2 \\
z - 14 z - 16 \\
\text{-}\frac{z - 18 z - 12}{12}
\end{bmatrix}
\]

(%i11) M : matrix ([1, 2, 3, 4], [5, 6, 7, 8], [9, 10, 11, 12], [13, 14, 15, 16]);
\[
\begin{bmatrix}
1 & 2 & 3 & 4 \\
5 & 6 & 7 & 8 \\
9 & 10 & 11 & 12 \\
13 & 14 & 15 & 16
\end{bmatrix}
\]

(%i12) columnspace (M);
\[
\begin{bmatrix}
1 \\
2 \\
z - 18 z - 12 \\
\text{-}\frac{z - 18 z - 12}{12}
\end{bmatrix}
\]

(%i13) apply ('orthogonal_complement, args (nullspace (transpose (M))));
\[
\begin{bmatrix}
0 \\
1 \\
1 \\
0
\end{bmatrix}
\]

(%i14) span([ ], [ ])
\[
\begin{bmatrix}
2 \\
-1 \\
0 \\
0
\end{bmatrix}
\]
68.2 Functions and Variables for linearalgebra

addmatrices \((f, M_1, \ldots, M_n)\)  
Using the function \(f\) as the addition function, return the sum of the matrices \(M_1, \ldots, M_n\). The function \(f\) must accept any number of arguments (a Maxima nary function).
Examples:
\[
\begin{align*}
(\%i1) \quad & m1 : \text{matrix}([[1,2],[3,4]]) \\
(\%i2) \quad & m2 : \text{matrix}([[7,8],[9,10]]) \\
(\%i3) \quad & \text{addmatrices('max,m1,m2)}; \\
(\%o3) \quad & \text{matrix}([[7,8],[9,10]]) \\
(\%i4) \quad & \text{addmatrices('max,m1,m2,5*m1)}; \\
(\%o4) \quad & \text{matrix}([[7,10],[15,20]])
\end{align*}
\]

blockmatrixp \((M)\)  
Return true if and only if \(M\) is a matrix and every entry of \(M\) is a matrix.

columnop \((M, i, j, \text{theta})\)  
If \(M\) is a matrix, return the matrix that results from doing the column operation \(C_i \leftarrow C_i - \text{theta} \cdot C_j\). If \(M\) doesn’t have a row \(i\) or \(j\), signal an error.

columnswap \((M, i, j)\)  
If \(M\) is a matrix, swap columns \(i\) and \(j\). If \(M\) doesn’t have a column \(i\) or \(j\), signal an error.

columnspace \((M)\)  
If \(M\) is a matrix, return \(\text{span} (v_1, \ldots, v_n)\), where the set \(\{v_1, \ldots, v_n\}\) is a basis for the column space of \(M\). The span of the empty set is \(\{0\}\). Thus, when the column space has only one member, return \(\text{span} ()\).

cholesky  
\text{cholesky} \((M)\)  
\text{cholesky} \((M, \text{field})\)  
Return the Cholesky factorization of the matrix selfadjoint (or hermitian) matrix \(M\). The second argument defaults to ‘generalring.’ For a description of the possible values for \(\text{field}\), see lu_factor.

ctranspose \((M)\)  
Return the complex conjugate transpose of the matrix \(M\). The function \text{ctranspose} uses \text{matrix_element_transpose} to transpose each matrix element.

diag_matrix \((d_1, d_2, \ldots, d_n)\)  
Return a diagonal matrix with diagonal entries \(d_1, d_2, \ldots, d_n\). When the diagonal entries are matrices, the zero entries of the returned matrix are zero matrices of the appropriate size; for example:
\[
(\%i1) \quad \text{diag_matrix(diag_matrix(1,2),diag_matrix(3,4))};
\]
```lisp
(%i1) \([ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \])

(%i2) \([ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \])

(%i12) diag_matrix(p,q);

\([ \begin{bmatrix} p & 0 \\ 0 & q \end{bmatrix} \])

dotproduct (u, v)

Return the dotproduct of vectors u and v. This is the same as conjugate (transpose (u)) . v. The arguments u and v must be column vectors.

eigens_by_jacobi

eigens_by_jacobi (A)

eigens_by_jacobi (A, field_type)

Computes the eigenvalues and eigenvectors of A by the method of Jacobi rotations. A must be a symmetric matrix (but it need not be positive definite nor positive semidefinite). field_type indicates the computational field, either floatfield or bigfloatfield. If field_type is not specified, it defaults to floatfield.
The elements of A must be numbers or expressions which evaluate to numbers via float or bfloat (depending on field_type).

Examples:

(%i1) S: matrix([1/sqrt(2), 1/sqrt(2)],[1/sqrt(2), 1/sqrt(2)]);

(%o1) \([ \begin{bmatrix} 1 & 1 \\ \text{-------} & \text{-------} \end{bmatrix} \])

(%o1) \([ \begin{bmatrix} \text{sqrt(2)} & \text{sqrt(2)} \\ \text{-------} & \text{-------} \end{bmatrix} \])

(%i2) L : matrix ([sqrt(3), 0], [0, sqrt(5)]);

(%o2) \([ \begin{bmatrix} \text{sqrt(3)} & 0 \\ 0 & \text{sqrt(5)} \end{bmatrix} \])

(%i3) M : S . L . transpose (S);

(%o3) \([ \begin{bmatrix} \text{sqrt(5)} & \text{sqrt(3)} & \text{sqrt(5)} & \text{sqrt(3)} \\ \text{-------} + \text{-------} & \text{-------} + \text{-------} & \text{-------} + \text{-------} & \text{-------} + \text{-------} \end{bmatrix} \])
```

(%i4) eigens_by_jacobi (M);
The largest percent change was 0.1454972243679
The largest percent change was 0.0
number of sweeps: 2
number of rotations: 1
(%o4) \[[1.732050807568877, 2.23606797749979],
[ 0.70710678118655 0.70710678118655 ]
[ ]
[ -0.70710678118655 0.70710678118655 ]
(%i5) float ([[sqrt(3), sqrt(5)], S]);
(%o5) \[[1.732050807568877, 2.23606797749979],
[ 0.70710678118655 0.70710678118655 ]
[ ]
[ -0.70710678118655 0.70710678118655 ]
(%i6) eigens_by_jacobi (M, bigfloatfield);
The largest percent change was 1.454972243679028b-1
The largest percent change was 0.0b0
number of sweeps: 2
number of rotations: 1
(%o6) \[[1.732050807568877b0, 2.23606797749979b0],
[ 7.071067811865475b-1 7.071067811865475b-1 ]
[ ]
[ -7.071067811865475b-1 7.071067811865475b-1 ]

get_lu_factors (x)  
When x = lu_factor (A), then get_lu_factors returns a list of the form [P, L, U], where P is a permutation matrix, L is lower triangular with ones on the diagonal, and U is upper triangular, and A = P L U.

hankel  

hankel (col)  
hankel (col, row)  
Return a Hankel matrix H. The first column of H is col; except for the first entry, the last row of H is row. The default for row is the zero vector with the same length as col.

hessian (f, x)  
Returns the Hessian matrix of f with respect to the list of variables x. The (i, j)-th element of the Hessian matrix is diff(f, x[i], 1, x[j], 1).

Examples:

(%i1) hessian (x * sin (y), [x, y]);
     [ 0  cos(y) ]
     [ ]
     [cos(y) - x sin(y) ]
(%o1)
(%i2) depends (F, [a, b]);
(%o2) F(a, b)
(%i3) hessian (F, [a, b]);
hilbert_matrix (n)

Return the \( n \times n \) Hilbert matrix. When \( n \) isn’t a positive integer, signal an error.

identfor

identfor (M)
identfor (M, fld)

Return an identity matrix that has the same shape as the matrix \( M \). The diagonal entries of the identity matrix are the multiplicative identity of the field \( fld \); the default for \( fld \) is \texttt{generalring}.

The first argument \( M \) should be a square matrix or a non-matrix. When \( M \) is a matrix, each entry of \( M \) can be a square matrix – thus \( M \) can be a blocked Maxima matrix. The matrix can be blocked to any (finite) depth.

See also \texttt{zerofor}

invert_by_lu (M, (rng generalring))

Invert a matrix \( M \) by using the LU factorization. The LU factorization is done using the ring \( rng \).

jacobian (f, x)

Returns the Jacobian matrix of the list of functions \( f \) with respect to the list of variables \( x \). The \((i, j)\)-th element of the Jacobian matrix is \( \text{diff}(f_i, x_j) \).

Examples:

\begin{verbatim}
(%i1) jacobian ([sin (u - v), sin (u * v)], [u, v]);
(%o1)       [ cos(v - u)   - cos(v - u) ]
           [ v cos(u v)   u cos(u v) ]

(%i2) depends ([F, G], [y, z]);
(%o2)          [F(y, z), G(y, z)]

(%i3) jacobian ([F, G], [y, z]);
(%o3)       [ dF  dF ]
           [ --  -- ]
           [ dy  dz ]
\end{verbatim}
**kronecker_product** \((A, B)\)  
Return the Kronecker product of the matrices \(A\) and \(B\).

**listp**  
```
listp (e, p)
listp (e)
```
Given an optional argument \(p\), return \texttt{true} if \(e\) is a Maxima list and \(p\) evaluates to \texttt{true} for every list element. When \texttt{listp} is not given the optional argument, return \texttt{true} if \(e\) is a Maxima list. In all other cases, return \texttt{false}.

**locate_matrix_entry** \((M, r_1, c_1, r_2, c_2, f, \text{rel})\)  
The first argument must be a matrix; the arguments \(r_1\) through \(c_2\) determine a sub-matrix of \(M\) that consists of rows \(r_1\) through \(r_2\) and columns \(c_1\) through \(c_2\). Find a entry in the sub-matrix \(M\) that satisfies some property. Three cases:
1. \(\text{rel} = \texttt{bool}\) and \(f\) a predicate:  
Scan the sub-matrix from left to right then top to bottom, and return the index of the first entry that satisfies the predicate \(f\). If no matrix entry satisfies \(f\), return \texttt{false}.
2. \(\text{rel} = \texttt{max}\) and \(f\) real-valued:  
Scan the sub-matrix looking for an entry that maximizes \(f\). Return the index of a maximizing entry.
3. \(\text{rel} = \texttt{min}\) and \(f\) real-valued:  
Scan the sub-matrix looking for an entry that minimizes \(f\). Return the index of a minimizing entry.

**lu_backsub** \((M, b)\)  
When \(M = \text{lu_factor} (A, \text{field})\), then \text{lu_backsub} \((M, b)\) solves the linear system \(A x = b\).

**lu_factor** \((M, \text{field})\)  
Return a list of the form \([LU, \text{perm}, \text{fld}]\), or \([LU, \text{perm}, \text{fld}, \text{lower-cnd upper-cnd}]\), where

1. The matrix \(LU\) contains the factorization of \(M\) in a packed form. Packed form means three things: First, the rows of \(LU\) are permuted according to the list \(\text{perm}\). If, for example, \(\text{perm}\) is the list \([3, 2, 1]\), the actual first row of the \(LU\) factorization is the third row of the matrix \(LU\). Second, the lower triangular factor of \(M\) is the lower triangular part of \(LU\) with the diagonal entries replaced by all ones. Third, the upper triangular factor of \(M\) is the upper triangular part of \(LU\).
2. When the field is either \texttt{floatfield} or \texttt{complexfield}, the numbers \(\text{lower-cnd}\) and \(\text{upper-cnd}\) are lower and upper bounds for the infinity norm condition number of \(M\). For all fields, the condition number might not be estimated; for such fields, \text{lu_factor} returns a two item list. Both the lower and upper bounds can differ from their true values by arbitrarily large factors. (See also \texttt{mat_cond}.)

The argument \(M\) must be a square matrix.

The optional argument \(\text{fld}\) must be a symbol that determines a ring or field. The pre-defined fields and rings are:

- \texttt{generalring} – the ring of Maxima expressions,
(b) **floatfield** – the field of floating point numbers of the type double,
(c) **complexfield** – the field of complex floating point numbers of the type double,
(d) **crering** – the ring of Maxima CRE expressions,
(e) **rationalfield** – the field of rational numbers,
(f) **runningerror** – track the all floating point rounding errors,
(g) **noncommutingring** – the ring of Maxima expressions where multiplication is the non-commutative dot operator.

When the field is **floatfield**, **complexfield**, or **runningerror**, the algorithm uses partial pivoting; for all other fields, rows are switched only when needed to avoid a zero pivot.

Floating point addition arithmetic isn’t associative, so the meaning of ‘field’ differs from the mathematical definition.

A member of the field **runningerror** is a two member Maxima list of the form \([x, n]\), where \(x\) is a floating point number and \(n\) is an integer. The relative difference between the ‘true’ value of \(x\) and \(x\) is approximately bounded by the machine epsilon times \(n\). The running error bound drops some terms that of the order the square of the machine epsilon.

There is no user-interface for defining a new field. A user that is familiar with Common Lisp should be able to define a new field. To do this, a user must define functions for the arithmetic operations and functions for converting from the field representation to Maxima and back. Additionally, for ordered fields (where partial pivoting will be used), a user must define functions for the magnitude and for comparing field members. After that all that remains is to define a Common Lisp structure **mring**. The file **mring** has many examples.

To compute the factorization, the first task is to convert each matrix entry to a member of the indicated field. When conversion isn’t possible, the factorization halts with an error message. Members of the field needn’t be Maxima expressions. Members of the **complexfield**, for example, are Common Lisp complex numbers. Thus after computing the factorization, the matrix entries must be converted to Maxima expressions.

See also **get_lu_factors**.

Examples:

```lisp
(\%i1) w[i,j] := random (1.0) + %i * random (1.0);
(\%o1)  
(\%i2) showtime : true$
Evaluation took 0.00 seconds (0.00 elapsed)
(\%i3) M : genmatrix (w, 100, 100)$
Evaluation took 7.40 seconds (8.23 elapsed)
(\%i4) lu_factor (M, complexfield)$
Evaluation took 28.71 seconds (35.00 elapsed)
(\%i5) lu_factor (M, generalring)$
Evaluation took 109.24 seconds (152.10 elapsed)
(\%i6) showtime : false$
```
(%i7) M : matrix ([1 - z, 3], [3, 8 - z]);

(%o7) 
[ 1 - z  3 ]
[          ]
[ 3     8 - z ]

(%i8) lu_factor (M, generalring);

(%o8) 
[ [ 1 - z  3 ] ]
[            ]
[ [ 3     9   ] ]
[ [ ----- - z - ----- + 8 ] ]
[ [ 1 - z  1 - z ] ]

(%i9) get_lu_factors (%);

(%o9) 
[ [ 1 0 ] [ 1 - z  3 ]]
[ [ 1 0 ] [ ] ]

(%i10) (%[1] . %[2] . %[3]);

(%o10) 
[ [ 1 - z  3 ]]
[ [ 3     8 - z ]]

mat_cond

mat_cond (M, 1)
mat_cond (M, inf)

Return the $p$-norm matrix condition number of the matrix $m$. The allowed values for $p$ are 1 and inf. This function uses the LU factorization to invert the matrix $m$. Thus the running time for mat_cond is proportional to the cube of the matrix size; lu_factor determines lower and upper bounds for the infinity norm condition number in time proportional to the square of the matrix size.

mat_norm

mat_norm (M, 1)
mat_norm (M, inf)
mat_norm (M, frobenius)

Return the matrix $p$-norm of the matrix $M$. The allowed values for $p$ are 1, inf, and frobenius (the Frobenius matrix norm). The matrix $M$ should be an unblocked matrix.

matrixp

matrixp (e, p)

matrixp (e)

Given an optional argument $p$, return true if $e$ is a matrix and $p$ evaluates to true for every matrix element. When matrixp is not given an optional argument, return true if $e$ is a matrix. In all other cases, return false.

See also blockmatrixp
matrix_size (M)  [Function]
Return a two member list that gives the number of rows and columns, respectively of
the matrix M.

mat_fullunblocker (M)  [Function]
If M is a block matrix, unblock the matrix to all levels. If M is a matrix, return M;
otherwise, signal an error.

mat_trace (M)  [Function]
Return the trace of the matrix M. If M isn’t a matrix, return a noun form. When
M is a block matrix, mat_trace(M) returns the same value as does mat_trace(mat_
unblocker(m)).

mat_unblocker (M)  [Function]
If M is a block matrix, unblock M one level. If M is a matrix, mat_unblocker (M)
returns M; otherwise, signal an error.

Thus if each entry of M is matrix, mat_unblocker (M) returns an unblocked matrix,
but if each entry of M is a block matrix, mat_unblocker (M) returns a block matrix
with one less level of blocking.

If you use block matrices, most likely you’ll want to set matrix_element_mult to "."
and matrix_element_transpose to 'transpose. See also mat_fullunblocker.

Example:

(%i1) A : matrix ([1, 2], [3, 4]);
    [ 1 2 ]
(%o1)    [    ]
    [ 3 4 ]

(%i2) B : matrix ([7, 8], [9, 10]);
    [ 7 8 ]
(%o2)    [    ]
    [ 9 10 ]

(%i3) matrix ([A, B]);
    [ [ 1 2 ] [ 7 8 ] ]
(%o3)    [    [    ]    [    ] ]
    [ [ 3 4 ] [ 9 10 ] ]

(%i4) mat_unblocker (%);
    [ 1 2 7 8 ]
(%o4)    [    ]
    [ 3 4 9 10 ]

nullspace (M)  [Function]
If M is a matrix, return span (v_1, ..., v_n), where the set {v_1, ..., v_n} is
a basis for the nullspace of M. The span of the empty set is \{0\}. Thus, when the
nullspace has only one member, return span ()..

nullity (M)  [Function]
If M is a matrix, return the dimension of the nullspace of M.
orthogonal_complement \((v_1, \ldots, v_n)\)

Return \(\text{span} \ (u_1, \ldots, u_m)\), where the set \(\{u_1, \ldots, u_m\}\) is a basis for the orthogonal complement of the set \(\{v_1, \ldots, v_n\}\).

Each vector \(v_l\) through \(v_n\) must be a column vector.

polynomialp

polynomialp \((p, L, \text{coeffp}, \text{exponp})\)

polynomialp \((p, L, \text{coeffp})\)

polynomialp \((p, L)\)

Return \(\text{true}\) if \(p\) is a polynomial in the variables in the list \(L\). The predicate \(\text{coeffp}\) must evaluate to \(\text{true}\) for each coefficient, and the predicate \(\text{exponp}\) must evaluate to \(\text{true}\) for all exponents of the variables in \(L\). If you want to use a non-default value for \(\text{exponp}\), you must supply \(\text{coeffp}\) with a value even if you want to use the default for \(\text{coeffp}\).

The command \(\text{polynomialp} \ (p, L, \text{coeffp}, \text{'nonnegintegerp})\) and the command \(\text{polynomialp} \ (p, L)\) is equivalent to \(\text{polynomialp} \ (p, L, \text{coeffp}, \text{'nonnegintegerp})\).

The polynomial needn’t be expanded:

\[
\begin{align*}
(\%i1) \quad \text{polynomialp} \ ((x + 1)*(x + 2), [x]); \\
(\%o1) & \quad \text{true} \\
(\%i2) \quad \text{polynomialp} \ ((x + 1)*(x + 2)^a, [x]); \\
(\%o2) & \quad \text{false}
\end{align*}
\]

An example using non-default values for \(\text{coeffp}\) and \(\text{exponp}\):

\[
\begin{align*}
(\%i1) \quad \text{polynomialp} \ ((x + 1)*(x + 2)^{(3/2)}, [x], \text{numberp}, \text{numberp}); \\
(\%o1) & \quad \text{true} \\
(\%i2) \quad \text{polynomialp} \ ((x^{(1/2)} + 1)*(x + 2)^{(3/2)}, [x], \text{numberp}, \\
& \quad \text{numberp}); \\
(\%o2) & \quad \text{true}
\end{align*}
\]

Polynomials with two variables:

\[
\begin{align*}
(\%i1) \quad \text{polynomialp} \ (x^2 + 5*x*y + y^2, [x]); \\
(\%o1) & \quad \text{false} \\
(\%i2) \quad \text{polynomialp} \ (x^2 + 5*x*y + y^2, [x, y]); \\
(\%o2) & \quad \text{true}
\end{align*}
\]

polytocompanion \((p, x)\)

If \(p\) is a polynomial in \(x\), return the companion matrix of \(p\). For a monic polynomial \(p\) of degree \(n\), we have \(p = (-1)^n \ \text{charpoly} \ (\text{polytocompanion} \ (p, x))\).

When \(p\) isn’t a polynomial in \(x\), signal an error.

ptriangularize \((M, v)\)

If \(M\) is a matrix with each entry a polynomial in \(v\), return a matrix \(M2\) such that

1. \(M2\) is upper triangular,
2. \(M2 = E_n \ldots E_1 M\), where \(E_1\) through \(E_n\) are elementary matrices whose entries are polynomials in \(v\),
3. \(\det \ (M) = \det \ (M2)\),

Note: This function doesn’t check that every entry is a polynomial in \(v\).
rowop (M, i, j, theta)

If M is a matrix, return the matrix that results from doing the row operation \( R_i \leftarrow R_i - \theta \cdot R_j \). If M doesn’t have a row \( i \) or \( j \), signal an error.

rank (M)

Return the rank of that matrix \( M \). The rank is the dimension of the column space.

Example:

\[
\text{(\%i1) rank(matrix([1,2],[2,4]));}
\]
\[
\text{(\%o1) 1}
\]
\[
\text{(\%i2) rank(matrix([1,b],[c,d]));}
\]
\[
\text{Proviso: \{d - b c \neq 0\}}
\]
\[
\text{(\%o2) 2}
\]

rowswap (M, i, j)

If \( M \) is a matrix, swap rows \( i \) and \( j \). If \( M \) doesn’t have a row \( i \) or \( j \), signal an error.

toeplitz

toeplitz (col)
toeplitz (col, row)

Return a Toeplitz matrix \( T \). The first first column of \( T \) is \( col \); except for the first entry, the first row of \( T \) is \( row \). The default for \( row \) is complex conjugate of \( col \).

Example:

\[
\text{(\%i1) toeplitz([1,2,3],[x,y,z]);}
\]
\[
\begin{bmatrix}
 1 & y & z \\
 2 & 1 & y \\
 3 & 2 & 1
\end{bmatrix}
\]
\[
\text{(\%o1)}
\]
\[
\text{(\%i2) toeplitz([1,1+%i]);}
\]
\[
\begin{bmatrix}
 1 & 1 - \%i \\
 \%i + 1 & 1
\end{bmatrix}
\]

vandermonde_matrix ([x_1, ..., x_n])

Return a \( n \) by \( n \) matrix whose \( i \)-th row is \( [1, x_i, x_i^2, \ldots x_i^{(n-1)}] \).

zerofor

zerofor (M)
zerofor (M, fld)

Return a zero matrix that has the same shape as the matrix \( M \). Every entry of the zero matrix is the additive identity of the field \( fld \); the default for \( fld \) is generalring.

The first argument \( M \) should be a square matrix or a non-matrix. When \( M \) is a matrix, each entry of \( M \) can be a square matrix – thus \( M \) can be a blocked Maxima matrix. The matrix can be blocked to any (finite) depth.

See also identfor
zeromatrixp (M)  
If \( M \) is not a block matrix, return true if \( \text{is} \ (\text{equal} \ (e, \ 0)) \) is true for each element \( e \) of the matrix \( M \). If \( M \) is a block matrix, return true if \( \text{zeromatrixp} \) evaluates to true for each element of \( e \).
69 lsquares

69.1 Introduction to lsquares

lsquares is a collection of functions to implement the method of least squares to estimate parameters for a model from numerical data.

69.2 Functions and Variables for lsquares

lsquares_estimates

\texttt{lsquares\_estimates} (\texttt{D}, \texttt{x}, \texttt{e}, \texttt{a})

\texttt{lsquares\_estimates} (\texttt{D}, \texttt{x}, \texttt{e}, \texttt{a}, initial = \texttt{L}, tol = \texttt{t})

Estimate parameters \texttt{a} to best fit the equation \texttt{e} in the variables \texttt{x} and \texttt{a} to the data \texttt{D}, as determined by the method of least squares. \texttt{lsquares\_estimates} first seeks an exact solution, and if that fails, then seeks an approximate solution.

The return value is a list of lists of equations of the form \([a = \ldots, b = \ldots, c = \ldots].\) Each element of the list is a distinct, equivalent minimum of the mean square error.

The data \texttt{D} must be a matrix. Each row is one datum (which may be called a ‘record’ or ‘case’ in some contexts), and each column contains the values of one variable across all data. The list of variables \texttt{x} gives a name for each column of \texttt{D}, even the columns which do not enter the analysis. The list of parameters \texttt{a} gives the names of the parameters for which estimates are sought. The equation \texttt{e} is an expression or equation in the variables \texttt{x} and \texttt{a}; if \texttt{e} is not an equation, it is treated the same as \texttt{e} = 0.

Additional arguments to \texttt{lsquares\_estimates} are specified as equations and passed on verbatim to the function \texttt{lbfgs} which is called to find estimates by a numerical method when an exact result is not found.

If some exact solution can be found (via \texttt{solve}), the data \texttt{D} may contain non-numeric values. However, if no exact solution is found, each element of \texttt{D} must have a numeric value. This includes numeric constants such as \texttt{\%pi} and \texttt{\%e} as well as literal numbers (integers, rationals, ordinary floats, and bigfloats). Numerical calculations are carried out with ordinary floating-point arithmetic, so all other kinds of numbers are converted to ordinary floats for calculations.

If \texttt{lsquares\_estimates} needs excessive amounts of time or runs out of memory \texttt{lsquares\_estimates\_approximate}, which skips the attempt to find an exact solution, might still succeed.

\texttt{load(lquares)} loads this function.

See also \texttt{lsquares\_estimates\_exact}, \texttt{lsquares\_estimates\_approximate}, \texttt{lsquares\_mse}, \texttt{lsquares\_residuals}, and \texttt{lsquares\_residual\_mse}.

Examples:

A problem for which an exact solution is found.

\begin{verbatim}
(%i1) load ("lsquares")$
(%i2) M : matrix (
[1,1,1], [3/2,1,2], [9/4,2,1], [3,2,2], [2,2,1]);
[ 1 1 1 ]
[  ]
[ 3 ]  
[ - 1 2 ]
[ 2 ]  
[ ]
(%o2)
[ 9 ]
[ - 2 1 ]
[ 4 ]  
[ ]
[ 3 2 2 ]
[ ]
[ 2 2 1 ]

(%i3) lsquares_estimates (
M, [z,x,y], (z+D)^2 = A*x+B*y+C, [A,B,C,D]);
59  27
[59  27  10921  107]
(%o3)  [[A = - --, B = - --, C = ------, D = - ---]]
16  16  1024  32

A problem for which no exact solution is found, so lsquares_estimates resorts to numerical approximation.

(%i1) load ("lsquares")$
(%i2) M : matrix ([1, 1], [2, 7/4], [3, 11/4], [4, 13/4]);
[ 1 1 ]
[ ]
[ 7 ]
[ 2 - ]
[ 4 ]  
[ ]
(%o2)
[ 11 ]
[ 3 -- ]
[ 4 ]
[ ]
[ 13 ]
[ 4 -- ]
[ 4 ]

(%i3) lsquares_estimates (M, [x,y], y=a*x^b+c, [a,b,c], initial=[3,3,3], iprint=[-1,0]);

Exponential functions aren't well-conditioned for least square fitting. In case that fitting to them fails it might be possible to get rid of the exponential function using an logarithm.

(%i1) load ("lsquares")$
(%i2) yvalues:[1,3,5,60,200,203,80]$
(%i3) time:[1,2,4,5,6,8,10]$
(%i4) \( f: y = a \exp(b \cdot t) \);

(%o4) \( y = a \exp(b \cdot t) \)

(%i5) yvalues_log: log(yvalues)$

(%i6) \( f \_log: \log(\text{subst}(y=\exp(y), f)) \);

(%o6) \( y = \log(a \exp(b \cdot t)) \)

(%i7) lsquares_estimates(
    \text{transpose(matrix(yvalues\_log, time))},
    \{y, t\},
    f\_log,
    \{a, b\})

*************************************************
N= 2 NUMBER OF CORRECTIONS=25
INITIAL VALUES
F= 6.802906290754687D+00 GNORM= 2.851243373781393D+01
*************************************************

<table>
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<tr>
<th>I</th>
<th>NFN</th>
<th>FUNC</th>
<th>GNORM</th>
<th>STEPLENGTH</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3</td>
<td>1.141838765593467D+00</td>
<td>1.067358003667488D-01</td>
<td>1.390943719972406D-02</td>
</tr>
<tr>
<td>2</td>
<td>5</td>
<td>1.141118195694385D+00</td>
<td>1.237977833033414D-01</td>
<td>5.000000000000000D+00</td>
</tr>
<tr>
<td>3</td>
<td>6</td>
<td>1.136945723147959D+00</td>
<td>3.806696991691383D-01</td>
<td>1.000000000000000D+00</td>
</tr>
<tr>
<td>4</td>
<td>7</td>
<td>1.133958243220262D+00</td>
<td>3.865103550379243D-01</td>
<td>1.000000000000000D+00</td>
</tr>
<tr>
<td>5</td>
<td>8</td>
<td>1.131725773805499D+00</td>
<td>2.292258231154026D-02</td>
<td>1.000000000000000D+00</td>
</tr>
<tr>
<td>6</td>
<td>9</td>
<td>1.131625585698168D+00</td>
<td>2.66440547017370D-03</td>
<td>1.000000000000000D+00</td>
</tr>
<tr>
<td>7</td>
<td>10</td>
<td>1.131620564856599D+00</td>
<td>2.519366958715444D-04</td>
<td>1.000000000000000D+00</td>
</tr>
</tbody>
</table>

THE MINIMIZATION TERMINATED WITHOUT DETECTING ERRORS.
IFLAG = 0
(%o7) \([a = 1.155904145765554, b = 0.5772666876959847]\)

lsquares_estimates_exact (MSE, a) [Function]
Estimate parameters \( a \) to minimize the mean square error \( \text{MSE} \), by constructing a system of equations and attempting to solve them symbolically via \text{solve}. The mean square error is an expression in the parameters \( a \), such as that returned by \text{lsquares\_mse}.

The return value is a list of lists of equations of the form \([a = \ldots, b = \ldots, c = \ldots]\). The return value may contain zero, one, or two or more elements. If two or more elements are returned, each represents a distinct, equivalent minimum of the mean square error.

See also \text{lsquares\_estimates}, \text{lsquares\_estimates\_approximate}, \text{lsquares\_mse}, \text{lsquares\_residuals}, and \text{lsquares\_residual\_mse}.

Example:

(%i11) load ("lsquares")$
\begin{verbatim}
(%i2) M : matrix (
   [1,1,1],
   [3/2,1,2],
   [9/4,2,1],
   [3,2,2],
   [2,2,1]);

(%o2)
\begin{pmatrix}
  1 & 1 & 1 \\
  3 & 1 & 2 \\
  9 & 2 & 1 \\
  3 & 2 & 2 \\
  2 & 2 & 1 \\
\end{pmatrix}

(%i3) mse : lsquares_mse (M, [z, x, y], (z + D)^2 = A*x + B*y + C);

(%o3) \frac{5}{5}

(%i4) lsquares_estimates_exact (mse, [A, B, C, D]);

(%o4) \{A = \frac{59}{16}, B = \frac{27}{16}, C = \frac{10921}{1024}, D = \frac{107}{32}\}
\end{verbatim}

\textbf{lsquares_estimates_approximate} \quad \textbf{(MSE, a, initial = L, tol = t)}

Estimate parameters \(a\) to minimize the mean square error \(MSE\), via the numerical minimization function \texttt{lbfgs}. The mean square error is an expression in the parameters \(a\), such as that returned by \texttt{lsquares_mse}.

The solution returned by \texttt{lsquares_estimates_approximate} is a local (perhaps global) minimum of the mean square error. For consistency with \texttt{lsquares_estimates_exact}, the return value is a nested list which contains one element, namely a list of equations of the form \([a = \ldots, b = \ldots, c = \ldots]\).

Additional arguments to \texttt{lsquares_estimates_approximate} are specified as equations and passed on verbatim to the function \texttt{lbfgs}.

\(MSE\) must evaluate to a number when the parameters are assigned numeric values. This requires that the data from which \(MSE\) was constructed comprise only numeric constants such as \(\%\pi\) and \(\%e\) and literal numbers (integers, rationals, ordinary floats, and bignums). Numerical calculations are carried out with ordinary floating-point arithmetic, so all other kinds of numbers are converted to ordinary floats for calculations.
load(lsquares) loads this function.

See also lsquares_estimates, lsquares_estimates_exact, lsquares_mse, lsquares_residuals, and lsquares_residual_mse.

Example:

(%i1) load ("lsquares")$

(%i2) M : matrix (  
    [1,1,1], [3/2,1,2], [9/4,2,1], [3,2,2], [2,2,1];  
    [1 1 1]  
    [ ]  
    [ 3 ]  
    [ - 1 2 ]  
    [2 ]  
    [ ]  

(%o2)

    [ 9 ]
    [ - 2 1 ]
    [ 4 ]
    [ ]
    [ 3 2 2 ]
    [ ]
    [ 2 2 1 ]

(%i3) mse : lsquares_mse (M, [z, x, y], (z + D)^2 = A*x + B*y + C);  
   5
===
\ 
> ((- B M ) - A M + (M + D) - C)
/ i, 3 i, 2 i, 1
===

(%o3) -----------------------------------------------------------------------------

5

(%i4) lsquares_estimates_approximate (  
    mse, [A, B, C, D], iprint = [-1, 0]);

(%o4) [[A = 3.678504947401971, B = -1.683070351177937,  
    C = 10.63469950148714, D = -3.340357993175297]]

lsquares_mse (D, x, e)  
[Function]  

Returns the mean square error (MSE), a summation expression, for the equation e in the variables x, with data D.

The MSE is defined as:

\[
\frac{1}{n} \sum_{i=1}^{n} [\text{lhs}(e_i) - \text{rhs}(e_i)]^2,
\]

where \( n \) is the number of data and \( e[i] \) is the equation \( e \) evaluated with the variables in \( x \) assigned values from the \( i \)-th datum, \( D[i] \).

load(lsquares) loads this function.
Example:

(%i1) load ("lsquares")$
(%i2) M : matrix ([1,1,1], [3/2,1,2], [9/4,2,1], [3,2,2], [2,2,1]);

(%o2)
[ 1 1 1 ]
[ 3 - 1 2 ]
[ 2 2 ]
[ 9 ]
[- 2 1 ]
[ 4 ]
[ 2 2 2 ]
[ 2 2 1 ]

(%o2) [ 9 ]

(%i3) mse : lsquares_mse (M, [z, x, y], (z + D)^2 = A*x + B*y + C); 5

====
\ 
2 2
> ((- B M ) - A M + (M + D) - C)
/ 
i, 3 i, 2 i, 1
====
i = 1

(%o3) -------------------------------------------------------------
5

(%i4) diff (mse, D);

(%o4) 5

====
\ 
4 > (M + D) ((- B M ) - A M + (M + D) - C)
/ 
i, 1 i, 3 i, 2 i, 1
====
i = 1

-------------------------------------------------------------

(%i5) 'mse, nouns;

(%o5) (((D + 3) - C - 2 B - 2 A) + ((D + -) - C - 2 B - A))

2 2
+ ((D + -) - C - B - 2 A) + ((D + -) - C - 2 B - A)

2 2
+ ((D + 1) - C - B - A) )/5
Chapter 69: lsquares

(%i3) mse : lsquares_mse (M, [z, x, y], (z + D)^2 = A*x + B*y + C);

====
\ 2 2
> ((D + M ) - C - M B - M A)
/ i, 1 i, 3 i, 2
====
i = 1

(%o3) ---------------------------------------------

(%i4) diff (mse, D);

====
\ 2
4 > (D + M ) ((D + M ) - C - M B - M A)
/ i, 1 i, 1 i, 3 i, 2
====
i = 1

(%o4) ----------------------------------------------------------

(%i5) ''mse, nouns;

2 2 9 2 2
((D + 3) - C - 2 B - 2 A) + ((D + -) - C - B - 2 A)
2 2 3 2 2
+ ((D + 2) - C - B - 2 A) + ((D + -) - C - 2 B - A)
2
+ ((D + 1) - C - B - A )/5

lsquares_residuals (D, x, e, a)

Returns the residuals for the equation e with specified parameters a and data D.
D is a matrix, x is a list of variables, e is an equation or general expression; if not an
equation, e is treated as if it were e = 0. a is a list of equations which specify values
for any free parameters in e aside from x.
The residuals are defined as:

\[ \text{lhs}(e_i) - \text{rhs}(e_i) \]

where \(e[i]\) is the equation \(e\) evaluated with the variables in \(x\) assigned values from
the i-th datum, \(D[i]\), and assigning any remaining free variables from \(a\).

load(lsquares) loads this function.

Example:

(%i1) load ("lsquares")$

(%i2) M : matrix (]
   [1,1,1], [3/2,1,2], [9/4,2,1], [3,2,2], [2,2,1]);
\[
\begin{bmatrix}
1 & 1 & 1 \\
3 &  & \\
-1 & 2 & \\
2 &  & \\
 &  & \\
9 &  & \\
-2 & 1 & \\
4 &  & \\
 &  & \\
3 & 2 & 2 \\
 &  & \\
2 & 2 & 1 \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
59 & 27 & 10921 & 107 \\
16 & 16 & 1024 & 32 \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
- \frac{1}{16}, - \frac{1}{16}, \frac{10921}{1024}, - \frac{107}{32} \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
13 & 13 & 13 & 13 & 13 \\
64 & 64 & 32 & 64 & 64 \\
\end{bmatrix}
\]

\[
\operatorname{lsquares\_residual\_mse}(D, x, e, a)
\]

Returns the residual mean square error (MSE) for the equation \(e\) with specified parameters \(a\) and data \(D\).

The residual MSE is defined as:

\[
\frac{1}{n} \sum_{i=1}^{n} \left[ \text{lhs}(e_i) - \text{rhs}(e_i) \right]^2,
\]

where \(e[i]\) is the equation \(e\) evaluated with the variables in \(x\) assigned values from the \(i\)-th datum, \(D[i]\), and assigning any remaining free variables from \(a\).

\texttt{load(lsquares)} loads this function.

Example:

\[
\begin{bmatrix}
1 & 1 & 1 \\
3 &  & \\
-1 & 2 & \\
2 &  & \\
 &  & \\
9 &  & \\
-2 & 1 & \\
4 &  & \\
 &  & \\
3 & 2 & 2 \\
 &  & \\
2 & 2 & 1 \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
59 & 27 & 10921 & 107 \\
16 & 16 & 1024 & 32 \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
- \frac{1}{16}, - \frac{1}{16}, \frac{10921}{1024}, - \frac{107}{32} \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
13 & 13 & 13 & 13 & 13 \\
64 & 64 & 32 & 64 & 64 \\
\end{bmatrix}
\]
Chapter 69: lsquares

\[
\begin{bmatrix}
-2 & 1 \\
4 & \\
& \\
3 & 2 & 2 \\
& \\
2 & 2 & 1 \\
\end{bmatrix}
\]

(%i3) a : lsquares_estimates (M, \{z,x,y\}, (z+D)^2 = A*x+B*y+C, \{A,B,C,D\});

\[
\begin{bmatrix}
59 & 27 & 10921 & 107 \\
16 & 16 & 1024 & 32 \\
\end{bmatrix}
\]

(%o3) \{A = - --, B = - --, C = -------, D = - ---\}

(%i4) lsquares_residual_mse (M, \{z,x,y\}, (z + D)^2 = A*x + B*y + C, first (a));

\[
\frac{169}{2560}
\]

plsq

plsq

**plsq**

plsq

plsq

plsq

Multivariable polynomial adjustment of a data table by the "least squares" method. *Mat* is a matrix containing the data, *VarList* is a list of variable names (one for each *Mat* column, but use ",-*" instead of varnames to ignore *Mat* columns), *depvars* is the name of a dependent variable or a list with one or more names of dependent variables (which names should be in *VarList*), *maxexpon* is the optional maximum exponent for each independent variable (1 by default), and *maxdegree* is the optional maximum polynomial degree (*maxexpon* by default); note that the sum of exponents of each term must be equal or smaller than *maxdegree*, and if *maxdegree* = 0 then no limit is applied.

If *depvars* is the name of a dependent variable (not in a list), *plsq* returns the adjusted polynomial. If *depvars* is a list of one or more dependent variables, *plsq* returns a list with the adjusted polynomial(s). The Coefficients of Determination are displayed in order to inform about the goodness of fit, which ranges from 0 (no correlation) to 1 (exact correlation). These values are also stored in the global variable *DETCOEF* (a list if *depvars* is a list).

A simple example of multivariable linear adjustment:

(%i1) load("plsq")$

(%i2) plsq(matrix([[1,2,0],[3,5,4],[4,7,9],[5,8,10]], [x,y,z],z);

Determination Coefficient for z = .9897039897039897

\[
11 y - 9 x - 14
\]

(%o2) \[
\frac{z = \text{-----------}}{3}
\]

The same example without degree restrictions:
How many diagonals does a N-sides polygon have? What polynomial degree should be used?

(%i4) plsquares(matrix([3,0],[4,2],[5,5],[6,9],[7,14],[8,20]),
[N,diagonals],diagonals,5);

Determination Coefficient for diagonals = 1.0

2
-3 N
\[ \text{diagonals} = \frac{N - 3 N}{2} \]

(%o4)

%ev(%o4, N=9); /* Testing for a 9 sides polygon */

%o5)

How many ways do we have to put two queens without they are threatened into a n x n chessboard?

(%i6) plsquares(matrix([0,0],[1,0],[2,0],[3,8],[4,44]),
[n,positions],[positions],4);

Determination Coefficient for \([\text{positions}] = 1.0\]

4
3 2
3 N - 10 N + 9 N - 2 N
\[ \text{positions} = \frac{3 n - 10 n + 9 n - 2 n}{6} \]

(%o6)

%ev(%o6, n=8); /* Testing for a (8 x 8) chessboard */

%o7)

An example with six dependent variables:

(%i8) mtrx:matrix([0,0,0,0,0,1,1,1],[0,1,0,1,1,1,0,0],
[1,0,0,1,1,1,0,0],[1,1,1,1,0,0,0,1])$

(%i8) plsquares(mtrx,[a,b,_And,_Or,_Xor,_Nand,_Nor,_Nxor],
[_And,_Or,_Xor,_Nand,_Nor,_Nxor],1,0);

Determination Coefficient for
\[ _\text{And} = a \ b, _\text{Or} = - a \ b + b + a, _\text{Xor} = - 2 a b + b + a, _\text{Nand} = 1 - a b, _\text{Nor} = a b - b - a + 1, _\text{Nxor} = 2 a b - b - a + 1 \]

To use this function write first load("lsquares").
70 minpack

70.1 Introduction to minpack

Minpack is a Common Lisp translation (via f2cl) of the Fortran library MINPACK, as obtained from Netlib.

70.2 Functions and Variables for minpack

minpack_lsquares (flist, varlist, guess [tolerance, jacobian]) [Function]
Compute the point that minimizes the sum of the squares of the functions in the list flist. The variables are in the list varlist. An initial guess of the optimum point must be provided in guess.

The optional keyword arguments, tolerance and jacobian provide some control over the algorithm. tolerance is the estimated relative error desired in the sum of squares. jacobian can be used to specify the Jacobian. If jacobian is not given or is true (the default), the Jacobian is computed from flist. If jacobian is false, a numerical approximation is used.

minpack_lsquares returns a list. The first item is the estimated solution; the second is the sum of squares, and the third indicates the success of the algorithm. The possible values are

0 improper input parameters.
1 algorithm estimates that the relative error in the sum of squares is at most tolerance.
2 algorithm estimates that the relative error between x and the solution is at most tolerance.
3 conditions for info = 1 and info = 2 both hold.
4 fvec is orthogonal to the columns of the jacobian to machine precision.
5 number of calls to fcn with iflag = 1 has reached 100*(n+1).
6 tol is too small. no further reduction in the sum of squares is possible.
7 tol is too small. no further improvement in the approximate solution x is possible.

/* Problem 6: Powell singular function */
(\%i1) powell(x1,x2,x3,x4) :=
[x1+10*x2, sqrt(5)*(x3-x4), (x2-2*x3)^2,
sqrt(10)*(x1-x4)^2]$
(\%i2) minpack_lsquares(powell(x1,x2,x3,x4), [x1,x2,x3,x4],
[3,-1,0,1]);
(\%o2) [[1.652117596168394e-17, - 1.652117596168393e-18,
2.643388153869468e-18, 2.643388153869468e-18],
6.109327859207777e-34, 4]

/* Same problem but use numerical approximation to Jacobian */
\%(i3)\ \texttt{minpack\_lsquares(powell(x1,x2,x3,x4), [x1,x2,x3,x4],}
\texttt{[3,-1,0,1], jacobian = false);}\\
\%(o3) \texttt{[[5.060282149485331e-11, - 5.060282149491206e-12,}}
\texttt{2.179447843547218e-11, 2.179447843547218e-11],}
\texttt{3.534491794847031e-21, 5]}\\

\textbf{minpack\_solve (flist, varlist, guess [, tolerance, jacobian])} \quad \textbf{[Function]}\\

Solve a system of \(n\) equations in \(n\) unknowns. The \(n\) equations are given in the list \texttt{flist}, and the unknowns are in \texttt{varlist}. An initial guess of the solution must be provided in \texttt{guess}.

The optional keyword arguments, \texttt{tolerance} and \texttt{jacobian} provide some control over the algorithm. \texttt{tolerance} is the estimated relative error desired in the sum of squares. \texttt{jacobian} can be used to specify the Jacobian. If \texttt{jacobian} is not given or is \texttt{true} (the default), the Jacobian is computed from \texttt{flist}. If \texttt{jacobian} is \texttt{false}, a numerical approximation is used.

\texttt{minpack\_solve} returns a list. The first item is the estimated solution; the second is the sum of squares, and the third indicates the success of the algorithm. The possible values are

0 \quad \text{improper input parameters.}\\
1 \quad \text{algorithm estimates that the relative error in the solution is at most} \texttt{tolerance.}\\
2 \quad \text{number of calls to fcn with iflag = 1 has reached 100*(n+1).}\\
3 \quad \text{tol is too small. no further reduction in the sum of squares is possible.}\\
4 \quad \text{Iteration is not making good progress.}
71 makeOrders

71.1 Functions and Variables for makeOrders

makeOrders (indvarlist, orderlist)  [Function]
    Returns a list of all powers for a polynomial up to and including the arguments.

(%i1) load("makeOrders")$

(%i2) makeOrders([a,b],[2,3]);
(%o2) [[0, 0], [0, 1], [0, 2], [0, 3], [1, 0], [1, 1],
    [1, 2], [1, 3], [2, 0], [2, 1], [2, 2], [2, 3]]
(%i3) expand((1+a+a^2)*(1+b+b^2+b^3));
   2 3 3 3 2 2 2 2 2
(%o3) a b + a b + b + a b + a b + b + a b + a b + b + a + 1
    2
where [0, 1] is associated with the term b and [2, 3] with $a^2b^3$.

To use this function write first load("makeOrders").
72 mnewton

72.1 Introduction to mnewton

mnewton is an implementation of Newton’s method for solving nonlinear equations in one or more variables.

72.2 Functions and Variables for mnewton

newtonepsilon  [Option variable]

Default value: \(10.0^{-\left(-fpprec/2\right)}\)

Precision to determine when the mnewton function has converged towards the solution. If newtonepsilon is a bigfloat, then mnewton computations are done with bigfloats. See also mnewton.

newtonmaxiter  [Option variable]

Default value: 50

Maximum number of iterations to stop the mnewton function if it does not converge or if it converges too slowly. See also mnewton.

mnewton (FuncList, VarList, GuessList)  [Function]

Multiple nonlinear functions solution using the Newton method. FuncList is the list of functions to solve, VarList is the list of variable names, and GuessList is the list of initial approximations.

The solution is returned in the same format that solve() returns. If the solution is not found, [] is returned.

This function is controlled by global variables newtonepsilon and newtonmaxiter.

(%i1) load("mnewton")$

(%i2) mnewton([x1+3*log(x1)-x2^2, 2*x1^2-x1*x2-5*x1+1], [x1, x2], [5, 5]);

(%o2) \[[x1 = 3.756834008012769, x2 = 2.779849592817897]\]

(%i3) mnewton([2*a^a-5], [a], [1]);

(%o3) \[[a = 1.70927556786144]\]

(%i4) mnewton([2*3^u-v/u-5, u+2^v-4], [u, v], [2, 2]);

(%o4) \[[u = 1.066618389595407, v = 1.552564766841786]\]

The variable newtonepsilon controls the precision of the approximations. It also controls if computations are performed with floats or bigfloats.

(%i1) load(mnewton)$

(%i2) (fpprec : 25, newtonepsilon : bfloat(10^(-fpprec+5)))$

(%i3) mnewton([2*3^u-v/u-5, u+2^v-4], [u, v], [2, 2]);

(%o3) \[[u = 1.066618389595407, v = 1.552564766841786]\]
v = 1.552564766841786450100418b0]

To use this function write first load("mnewton"). See also newtonepsilon and newtonmaxiter.
73 numericalio

73.1 Introduction to numericalio

numericalio is a collection of functions to read and write files and streams. Functions for
plain-text input and output can read and write numbers (integer, float, or bigfloat), symbols,
and strings. Functions for binary input and output can read and write only floating-point
numbers.

If there already exists a list, matrix, or array object to store input data, numericalio
input functions can write data into that object. Otherwise, numericalio can guess, to
some degree, the structure of an object to store the data, and return that object.

73.1.1 Plain-text input and output

In plain-text input and output, it is assumed that each item to read or write is an atom: an
integer, float, bigfloat, string, or symbol, and not a rational or complex number or any other
kind of nonatomic expression. The numericalio functions may attempt to do something
sensible faced with nonatomic expressions, but the results are not specified here and subject
to change.

Atoms in both input and output files have the same format as in Maxima batch files
or the interactive console. In particular, strings are enclosed in double quotes, backslash
\ prevents any special interpretation of the next character, and the question mark ? is
recognized at the beginning of a symbol to mean a Lisp symbol (as opposed to a Maxima
symbol). No continuation character (to join broken lines) is recognized.

73.1.2 Separator flag values for input

The functions for plain-text input and output take an optional argument, separator_flag,
that tells what character separates data.

For plain-text input, these values of separator_flag are recognized: comma for comma
separated values, pipe for values separated by the vertical bar character |, semicolon for
values separated by semicolon ;, and space for values separated by space or tab characters.
If the file name ends in .csv and separator_flag is not specified, comma is assumed. If the
file name ends in something other than .csv and separator_flag is not specified, space
is assumed.

In plain-text input, multiple successive space and tab characters count as a single sepa-
rator. However, multiple comma, pipe, or semicolon characters are significant. Successive
comma, pipe, or semicolon characters (with or without intervening spaces or tabs) are con-
sidered to have false between the separators. For example, 1234,,Foo is treated the same
as 1234,false,Foo.

73.1.3 Separator flag values for output

For plain-text output, tab, for values separated by the tab character, is recognized as a
value of separator_flag, as well as comma, pipe, semicolon, and space.

In plain-text output, false atoms are written as such; a list [1234, false, Foo] is
written 1234,false,Foo, and there is no attempt to collapse the output to 1234,,Foo.
73.1.4 Binary floating-point input and output

`numericalio` functions can read and write 8-byte IEEE 754 floating-point numbers. These numbers can be stored either least significant byte first or most significant byte first, according to the global flag set by `assume_external_byte_order`. If not specified, `numericalio` assumes the external byte order is most-significant byte first.

Other kinds of numbers are coerced to 8-byte floats; `numericalio` cannot read or write binary non-numeric data.

Some Lisp implementations do not recognize IEEE 754 special values (positive and negative infinity, not-a-number values, denormalized values). The effect of reading such values with `numericalio` is undefined.

`numericalio` includes functions to open a stream for reading or writing a stream of bytes.

73.2 Functions and Variables for plain-text input and output

`read_matrix` [Function]

```lisp
read_matrix (S)
read_matrix (S, M)
read_matrix (S, separator_flag)
read_matrix (S, M, separator_flag)
```

`read_matrix(S)` reads the source `S` and returns its entire content as a matrix. The size of the matrix is inferred from the input data; each line of the file becomes one row of the matrix. If some lines have different lengths, `read_matrix` complains.

`read_matrix(S, M)` read the source `S` into the matrix `M`, until `M` is full or the source is exhausted. Input data are read into the matrix in row-major order; the input need not have the same number of rows and columns as `M`.

The source `S` may be a file name or a stream which for example allows skipping the very first line of a file (that may be useful, if you read CSV data, where the first line often contains the description of the columns):

```lisp
s : openr("data.txt");
readline(s); /* skip the first line */
M : read_matrix(s, 'comma); /* read the following (comma-separated) lines into matrix M */
close(s);
```

The recognized values of `separator_flag` are `comma`, `pipe`, `semicolon`, and `space`. If `separator_flag` is not specified, the file is assumed space-delimited.

`read_array` [Function]

```lisp
read_array (S, A)
read_array (S, A, separator_flag)
```

Reads the source `S` into the array `A`, until `A` is full or the source is exhausted. Input data are read into the array in row-major order; the input need not conform to the dimensions of `A`.

The source `S` may be a file name or a stream.

The recognized values of `separator_flag` are `comma`, `pipe`, `semicolon`, and `space`. If `separator_flag` is not specified, the file is assumed space-delimited.
**read_hashed_array**

read_hashed_array (S, A)
read_hashed_array (S, A, separator_flag)

Reads the source S and returns its entire content as a hashed array. The source S may be a file name or a stream.

read_hashed_array treats the first item on each line as a hash key, and associates the remainder of the line (as a list) with the key. For example, the line 567 12 17 32 55 is equivalent to A[567]: [12, 17, 32, 55]. Lines need not have the same numbers of elements.

The recognized values of separator_flag are comma, pipe, semicolon, and space. If separator_flag is not specified, the file is assumed space-delimited.

**read_nested_list**

read_nested_list (S)
read_nested_list (S, separator_flag)

Reads the source S and returns its entire content as a nested list. The source S may be a file name or a stream.

read_nested_list returns a list which has a sublist for each line of input. Lines need not have the same numbers of elements. Empty lines are not ignored: an empty line yields an empty sublist.

The recognized values of separator_flag are comma, pipe, semicolon, and space. If separator_flag is not specified, the file is assumed space-delimited.

**read_list**

read_list (S)
read_list (S, L)
read_list (S, separator_flag)
read_list (S, L, separator_flag)

read_list(S) reads the source S and returns its entire content as a flat list.

read_list(S, L) reads the source S into the list L, until L is full or the source is exhausted.

The source S may be a file name or a stream.

The recognized values of separator_flag are comma, pipe, semicolon, and space. If separator_flag is not specified, the file is assumed space-delimited.

**write_data**

write_data (X, D)
write_data (X, D, separator_flag)

Writes the object X to the destination D.

write_data writes a matrix in row-major order, with one line per row.

write_data writes an array created by array or make_array in row-major order, with a new line at the end of every slab. Higher-dimensional slabs are separated by additional new lines.

write_data writes a hashed array with each key followed by its associated list on one line.
write_data writes a nested list with each sublist on one line.
write_data writes a flat list all on one line.

The destination $D$ may be a file name or a stream. When the destination is a file name, the global variable file_output_append governs whether the output file is appended or truncated. When the destination is a stream, no special action is taken by write_data after all the data are written; in particular, the stream remains open.

The recognized values of separator_flag are comma, pipe, semicolon, space, and tab. If separator_flag is not specified, the file is assumed space-delimited.

73.3 Functions and Variables for binary input and output

assume_external_byte_order (byte_order_flag)  
Tells numericalio the byte order for reading and writing binary data. Two values of byte_order_flag are recognized: lsb which indicates least-significant byte first, also called little-endian byte order; and msb which indicates most-significant byte first, also called big-endian byte order.

If not specified, numericalio assumes the external byte order is most-significant byte first.

openr_binary (file_name)  
Returns an input stream of 8-bit unsigned bytes to read the file named by file_name.

openw_binary (file_name)  
Returns an output stream of 8-bit unsigned bytes to write the file named by file_name.

opena_binary (file_name)  
Returns an output stream of 8-bit unsigned bytes to append the file named by file_name.

read_binary_matrix (S, M)  
Reads binary 8-byte floating point numbers from the source $S$ into the matrix $M$ until $M$ is full, or the source is exhausted. Elements of $M$ are read in row-major order.

The source $S$ may be a file name or a stream.

The byte order in elements of the source is specified by assume_external_byte_order.

read_binary_array (S, A)  
Reads binary 8-byte floating point numbers from the source $S$ into the array $A$ until $A$ is full, or the source is exhausted. $A$ must be an array created by array or make_array. Elements of $A$ are read in row-major order.

The source $S$ may be a file name or a stream.

The byte order in elements of the source is specified by assume_external_byte_order.
read_binary_list
  read_binary_list(S)
  read_binary_list(S, L)
read_binary_list(S) reads the entire content of the source S as a sequence of binary 8-byte floating point numbers, and returns it as a list. The source S may be a file name or a stream.
read_binary_list(S, L) reads 8-byte binary floating point numbers from the source S until the list L is full, or the source is exhausted.
The byte order in elements of the source is specified by assume_external_byte_order.

write_binary_data(X, D)  [Function]
Writes the object X, comprising binary 8-byte IEEE 754 floating-point numbers, to the destination D. Other kinds of numbers are coerced to 8-byte floats. write_binary_data cannot write non-numeric data.
The object X may be a list, a nested list, a matrix, or an array created by array or make_array; X cannot be an undeclared array or any other type of object. write_binary_data writes nested lists, matrices, and arrays in row-major order.
The destination D may be a file name or a stream. When the destination is a file name, the global variable file_output_append governs whether the output file is appended or truncated. When the destination is a stream, no special action is taken by write_binary_data after all the data are written; in particular, the stream remains open.
The byte order in elements of the destination is specified by assume_external_byte_order.
74 operatingsystem

74.1 Introduction to operatingsystem

Package operatingsystem contains functions for operatingsystem-tasks, like file system operations.

74.2 Directory operations

chdir (dir)  
Change to directory dir

mkdir (dir)  
Create directory dir

rmdir (dir)  
remove directory dir

getcode()  
returns the current working directory.
See also directory.

Examples:
(%i1) load("operatingsystem")$
(%i2) mkdir("testdirectory")$
(%i3) chdir("testdirectory")$
(%i4) chdir("..")$
(%i5) rmdir("testdirectory")$

74.3 File operations

copy_file (file1, file2)  
copies file file1 to file2

rename_file (file1, file2)  
renames file file1 to file2

delete_file (file1)  
deletes file file1

74.4 Environment operations

getenv (env)  
Get the value of the environment variable env

Example:
(%i1) load("operatingsystem")$
(%i2) getenv("PATH");
(%o2) /usr/local/sbin:/usr/local/bin:/usr/sbin:/usr/bin:/sbin:/bin
75 opsubst

75.1 Functions and Variables for opsubst

\texttt{opsubst} [Function]

\texttt{opsubst}(f, g, e)
\texttt{opsubst}(g=f, e)
\texttt{opsubst}([g1=f1, g2=f2, ..., gn=fn], e)

The function \texttt{opsubst} is similar to the function \texttt{subst}, except that \texttt{opsubst} only makes substitutions for the operators in an expression. In general, when \( f \) is an operator in the expression \( e \), substitute \( g \) for \( f \) in the expression \( e \).

To determine the operator, \texttt{opsubst} sets \texttt{inflag} to true. This means \texttt{opsubst} substitutes for the internal, not the displayed, operator in the expression.

Examples:

\begin{verbatim}
(%i1) load ("opsubst")$

(%i2) opsubst(f, g, g(g(x)));
(%o2) f(f(x))

(%i3) opsubst(f, g, g(g));
(%o3) f(g)

(%i4) opsubst(f, g[x], g[x](z));
(%o4) f(z)

(%i5) opsubst(g[x], f, f(z));
(%o5) g(z)

(%i6) opsubst(tan, sin, sin(sin));
(%o6) tan(sin)

(%i7) opsubst([f=g, g=h], f(x));
(%o7) h(x)

Internally, Maxima does not use the unary negation, division, or the subtraction operators; thus:

\begin{verbatim}
(%i8) opsubst("+", "-", a-b);
(%o8) a - b

(%i9) opsubst("f", "-", -a);
(%o9) - a

(%i10) opsubst("\-\-", "/", a/b);
(%o10) a

(%i11) opsubst("[", "*, -a*b);
(%o11) [-1, a, b]
\end{verbatim}

When either operator isn't a Maxima symbol, generally some other function will signal an error:

\begin{verbatim}
(%i12) opsubst(a+b, f, f(x));
\end{verbatim}

Improper name or value in functional position:
b + a
-- an error. Quitting. To debug this try debugmode(true);

However, subscripted operators are allowed:
   (%i13) opsubst(g[5], f, f(x));
   (%o13)                                
          g (x)
                 5

To use this function write first load("opsubst").
76 orthopoly

76.1 Introduction to orthogonal polynomials

orthopoly is a package for symbolic and numerical evaluation of several kinds of orthogonal polynomials, including Chebyshev, Laguerre, Hermite, Jacobi, Legendre, and ultraspherical (Gegenbauer) polynomials. Additionally, orthopoly includes support for the spherical Bessel, spherical Hankel, and spherical harmonic functions.

For the most part, orthopoly follows the conventions of Abramowitz and Stegun Handbook of Mathematical Functions, Chapter 22 (10th printing, December 1972); additionally, we use Gradshteyn and Ryzhik, Table of Integrals, Series, and Products (1980 corrected and enlarged edition), and Eugen Merzbacher Quantum Mechanics (2nd edition, 1970).

Barton Willis of the University of Nebraska at Kearney (UNK) wrote the orthopoly package and its documentation. The package is released under the GNU General Public License (GPL).

76.1.1 Getting Started with orthopoly

load ("orthopoly") loads the orthopoly package.

To find the third-order Legendre polynomial,

```maxima
(%i1) legendre_p (3, x);
(%o1) 3 2
      5 (1 - x) 15 (1 - x)
     - ------------ + ------------ - 6 (1 - x) + 1
       2       2
```

To express this as a sum of powers of x, apply ratsimp or rat to the result.

```maxima
(%i2) ratsimp (%), rat (%);
(%o2)/R/[3 3
      5 x - 3 x 5 x - 3 x
     [---------------, ------------]
       2       2
```

Alternatively, make the second argument to legendre_p (its "main" variable) a canonical rational expression (CRE).

```maxima
(%i1) legendre_p (3, rat (x));
(%o1)/R/ 3
      5 x - 3 x
     [--------------]
       2
```

For floating point evaluation, orthopoly uses a running error analysis to estimate an upper bound for the error. For example,

```maxima
(%i1) jacobi_p (150, 2, 3, 0.2);
(%o1) interval(- 0.062017037936715, 1.533267919277521E-11)
```

Intervals have the form interval (c, r), where c is the center and r is the radius of the interval. Since Maxima does not support arithmetic on intervals, in some situations, such
as graphics, you want to suppress the error and output only the center of the interval. To do this, set the option variable `orthopoly_returns_intervals` to `false`.

```
(%i1) orthopoly_returns_intervals : false;
(%o1) false
(%i2) jacobi_p (150, 2, 3, 0.2);
(%o2) - 0.062017037936715
```

Refer to the section see [Floating point Evaluation], page 1021, for more information.

Most functions in `orthopoly` have a `gradef` property; thus

```
(%i1) diff (hermite (n, x), x);
(%o1) 2 n H (x)
    n - 1

(%i2) diff (gen_laguerre (n, a, x), x);
   (a) (a)
   n L (x) - (n + a) L (x) unit_step(n) 
   n   n - 1
   -------------------------------
   x
```

The unit step function in the second example prevents an error that would otherwise arise by evaluating with \( n \) equal to 0.

```
(%i3) ev (% , n = 0);
(%o3) 0
```

The `gradef` property only applies to the “main” variable; derivatives with respect other arguments usually result in an error message; for example

```
(%i1) diff (hermite (n, x), x);
(%o1) 2 n H (x)
    n - 1

(%i2) diff (hermite (n, x), n);
```

Maxima doesn't know the derivative of hermite with respect the first argument
-- an error. Quitting. To debug this try debugmode(true);

Generally, functions in `orthopoly` map over lists and matrices. For the mapping to fully evaluate, the option variables `doallmxops` and `listarith` must both be `true` (the defaults). To illustrate the mapping over matrices, consider

```
(%i1) hermite (2, x);
2
   2
(%o1) - 2 (1 - 2 x )

(%i2) m : matrix ([0, x], [y, 0]);
    [ 0 x ]
    [ ]
    [ y 0 ]

(%o2) 

(%i3) hermite (2, m);

```

```
    [ 2 ]
    [ - 2 - 2 (1 - 2 x ) ]
(%o3) 
```
In the second example, the $i, j$ element of the value is $\text{hermite}(2, m[i,j])$; this is not the same as computing $-2 + 4 m \cdot m$, as seen in the next example.

```
(%i4) -2 * matrix([1, 0], [0, 1]) + 4 * m . m;
(%o4) [ 4 x y - 2 0 ]
        [ 0 4 x y - 2 ]
```

If you evaluate a function at a point outside its domain, generally `orthopoly` returns the function unevaluated. For example,

```
(%i1) legendre_p(2/3, x);
(%o1) P_{2/3}(x)
```

`orthopoly` supports translation into TeX; it also does two-dimensional output on a terminal.

```
(%i1) spherical_harmonic(1, m, theta, phi);
       m
(%o1) Y(theta, phi)
       l
(%i2) tex (%);
$$Y_{l}^{m}\left(\vartheta,\varphi\right)$$
(%o2) false
(%i3) jacobi_p(n, a, a - b, x/2);
       (a, a - b) x
(%o3) P_{(-)}^{n}\left({{x}\over{2}}\right)
       n 2
(%i4) tex (%);
$$P_{n}^{\left(a,a-b\right)}\left({{x}\over{2}}\right)$$
(%o4) false
```

### 76.1.2 Limitations

When an expression involves several orthogonal polynomials with symbolic orders, it’s possible that the expression actually vanishes, yet Maxima is unable to simplify it to zero. If you divide by such a quantity, you’ll be in trouble. For example, the following expression vanishes for integers $n$ greater than 1, yet Maxima is unable to simplify it to zero.

```
(%i1) (2*n - 1) * legendre_p(n - 1, x) * x - n * legendre_p(n, x) + (1 - n) * legendre_p(n - 2, x);
(%o1) (2 n - 1) P_{n-1}(x) x - n P_{n}(x) + (1 - n) P_{n-2}(x)
```

For a specific $n$, we can reduce the expression to zero.

```
(%i2) ev (% , n = 10, ratsimp);
(%o2) 0
```

Generally, the polynomial form of an orthogonal polynomial is ill-suited for floating point evaluation. Here’s an example.

```
(%i1) p : jacobi_p(100, 2, 3, x)$
```
(%i2) subst (0.2, x, p);
(%o2) 3.442767023833592E+35
(%i3) jacobi_p (100, 2, 3, 0.2);
(%o3) interval(0.18413609135169, 6.8990300925815987E-12)
(%i4) float(jacobi_p (100, 2, 3, 2/10));
(%o4) 0.18413609135169

The true value is about 0.184; this calculation suffers from extreme subtractive cancellation error. Expanding the polynomial and then evaluating, gives a better result.

(%i5) p : expand(p)$
(%i6) subst (0.2, x, p);
(%o6) 0.18413609766122982

This isn't a general rule; expanding the polynomial does not always result in an expression that is better suited for numerical evaluation. By far, the best way to do numerical evaluation is to make one or more of the function arguments floating point numbers. By doing that, specialized floating point algorithms are used for evaluation.

Maxima’s float function is somewhat indiscriminate; if you apply float to an expression involving an orthogonal polynomial with a symbolic degree or order parameter, these parameters may be converted into floats; after that, the expression will not evaluate fully. Consider

(%i1) assoc_legendre_p (n, 1, x);
   1
P (x)
   n
(%i2) float (%);
   1.0
P (x)
   n
(%i3) ev (% , n=2, x=0.9);
   1.0
P (0.9)
   2

The expression in (%o3) will not evaluate to a float; orthopoly doesn’t recognize floating point values where it requires an integer. Similarly, numerical evaluation of the pochhammer function for orders that exceed pochhammer_max_index can be troublesome; consider

(%i1) x : pochhammer (1, 10), pochhammer_max_index : 5;
(%o1) (1)

Applying float doesn’t evaluate x to a float

(%i2) float (x);
(%o2) (1.0)

To evaluate x to a float, you’ll need to bind pochhammer_max_index to 11 or greater and apply float to x.

(%i3) float (x), pochhammer_max_index : 11;
The default value of `pochhammer_max_index` is 100; change its value after loading `orthopoly`.

Finally, be aware that reference books vary on the definitions of the orthogonal polynomials; we’ve generally used the conventions of Abramowitz and Stegun.

Before you suspect a bug in `orthopoly`, check some special cases to determine if your definitions match those used by `orthopoly`. Definitions often differ by a normalization; occasionally, authors use “shifted” versions of the functions that makes the family orthogonal on an interval other than $(-1,1)$. To define, for example, a Legendre polynomial that is orthogonal on $(0,1)$, define

```
(%i1) shifted_legendre_p (n, x) := legendre_p (n, 2*x - 1)$
(%i2) shifted_legendre_p (2, rat (x));
2
6 x - 6 x + 1
(%o2)/R/ 6 x - 6 x + 1
(%i3) legendre_p (2, rat (x));
2
3 x - 1
(%o3)/R/ ---
2
```

### 76.1.3 Floating point Evaluation

Most functions in `orthopoly` use a running error analysis to estimate the error in floating point evaluation; the exceptions are the spherical Bessel functions and the associated Legendre polynomials of the second kind. For numerical evaluation, the spherical Bessel functions call SLATEC functions. No specialized method is used for numerical evaluation of the associated Legendre polynomials of the second kind.

The running error analysis ignores errors that are second or higher order in the machine epsilon (also known as unit roundoff). It also ignores a few other errors. It’s possible (although unlikely) that the actual error exceeds the estimate.

Intervals have the form `interval (c, r)`, where $c$ is the center of the interval and $r$ is its radius. The center of an interval can be a complex number, and the radius is always a positive real number.

Here is an example.

```
(%i1) fpprec : 50$
(%i2) y0 : jacobi_p (100, 2, 3, 0.2);
(%o2) interval(0.1841360913516871, 6.899030925815987E-12)
(%i3) y1 : bfloat (jacobi_p (100, 2, 3, 1/5));
(%o3) 1.8413609135168563091370224958913493690868904463668b-1
```

Let’s test that the actual error is smaller than the error estimate.

```
(%i4) is (abs (part (y0, 1) - y1) < part (y0, 2));
(%o4) true
```

Indeed, for this example the error estimate is an upper bound for the true error.
Maxima does not support arithmetic on intervals.

(%i1) legendre_p (7, 0.1) + legendre_p (8, 0.1);
(%o1) interval(0.18032072148437508, 3.1477135311021797E-15) + interval(- 0.19949294375000004, 3.3769353084291579E-15)

A user could define arithmetic operators that do interval math. To define interval addition, we can define

(%i1) infix (@+)$
(%i2) @+(x,y) := interval (part (x, 1) + part (y, 1), part (x, 2) + part (y, 2))$
(%i3) legendre_p (7, 0.1) @+ legendre_p (8, 0.1);
(%o3) interval(- 0.019172222265624955, 6.5246488395313372E-15)

The special floating point routines get called when the arguments are complex. For example,

(%i1) legendre_p (10, 2 + 3.0*%i);
(%o1) interval(- 3.876378825E+7 %i - 6.0787748E+7, 1.2089173052721777E-6)

Let’s compare this to the true value.

(%i1) float (expand (legendre_p (10, 2 + 3*%i)));
(%o1) - 3.876378825E+7 %i - 6.0787748E+7

Additionally, when the arguments are big floats, the special floating point routines get called; however, the big floats are converted into double floats and the final result is a double.

(%i1) ultraspherical (150, 0.5b0, 0.9b0);
(%o1) interval(- 0.043009481257265, 3.3750051301228864E-14)

### 76.1.4 Graphics and orthopoly

To plot expressions that involve the orthogonal polynomials, you must do two things:

1. Set the option variable `orthopoly_returns_intervals` to `false`.
2. Quote any calls to `orthopoly` functions.

If function calls aren’t quoted, Maxima evaluates them to polynomials before plotting; consequently, the specialized floating point code doesn’t get called. Here is an example of how to plot an expression that involves a Legendre polynomial.

(%i1) plot2d ('(legendre_p (5, x)), [x, 0, 1]),
    orthopoly_returns_intervals : false;
The *entire* expression `legendre_p (5, x)` is quoted; this is different than just quoting the function name using `'legendre_p (5, x)`.

### 76.1.5 Miscellaneous Functions

The *orthopoly* package defines the Pochhammer symbol and a unit step function. *orthopoly* uses the Kronecker delta function and the unit step function in `gradef` statements.

To convert Pochhammer symbols into quotients of gamma functions, use `makegamma`.

```
(%i1) makegamma (pochhammer (x, n));
   gamma(x + n)
(%o1) ------------
    gamma(x)
(%i2) makegamma (pochhammer (1/2, 1/2));
      1
(%o2) ----------
     sqrt(%pi)
```

Derivatives of the Pochhammer symbol are given in terms of the `psi` function.

```
(%i1) diff (pochhammer (x, n), x);
   (x) (psi (x + n) - psi (x))
(%o1) ---------------
      n  0  0
(%i2) diff (pochhammer (x, n), n);
   (x) psi (x + n)
(%o2) ------------
      n  0
```

You need to be careful with the expression in `%o1`; the difference of the `psi` functions has polynomials when `x = -1, -2, .., -n`. These polynomials cancel with factors in `pochhammer (x, n)` making the derivative a degree `n - 1` polynomial when `n` is a positive integer.

The Pochhammer symbol is defined for negative orders through its representation as a quotient of gamma functions. Consider

```
(%i1) q : makegamma (pochhammer (x, n));
```
\[ \frac{\Gamma(x+n)}{\Gamma(x)} \]

%i2) sublis ([x=11/3, n=-6], q);

\[
\frac{729}{2240}
\]

Alternatively, we can get this result directly.

%i1) pochhammer (11/3, -6);

\[
\frac{729}{2240}
\]

The unit step function is left-continuous; thus

%i1) [unit_step (-1/10), unit_step (0), unit_step (1/10)];

\[
[0, 0, 1]
\]

If you need a unit step function that is neither left or right continuous at zero, define your own using `signum`; for example,

%i1) xunit_step (x) := (1 + signum (x))/2$

%i2) [xunit_step (-1/10), xunit_step (0), xunit_step (1/10)];

\[
[0, -1, 1]
\]

Do not redefine `unit_step` itself; some code in `orthopoly` requires that the unit step function be left-continuous.

### 76.1.6 Algorithms

Generally, `orthopoly` does symbolic evaluation by using a hypergeometric representation of the orthogonal polynomials. The hypergeometric functions are evaluated using the (undocumented) functions `hypergeo11` and `hypergeo21`. The exceptions are the half-integer Bessel functions and the associated Legendre function of the second kind. The half-integer Bessel functions are evaluated using an explicit representation, and the associated Legendre function of the second kind is evaluated using recursion.

For floating point evaluation, we again convert most functions into a hypergeometric form; we evaluate the hypergeometric functions using forward recursion. Again, the exceptions are the half-integer Bessel functions and the associated Legendre function of the second kind. Numerically, the half-integer Bessel functions are evaluated using the SLATEC code.

### 76.2 Functions and Variables for orthogonal polynomials

`assoc_legendre_p (n, m, x)`

The associated Legendre function of the first kind of degree \( n \) and order \( m \).

Reference: Abramowitz and Stegun, equations 22.5.37, page 779, 8.6.6 (second equation), page 334, and 8.2.5, page 333.
assoc_legendre_q (n, m, x)

The associated Legendre function of the second kind of degree n and order m.
Reference: Abramowitz and Stegun, equation 8.5.3 and 8.1.8.

chebyshev_t (n, x)

The Chebyshev polynomial of the first kind of degree n.
Reference: Abramowitz and Stegun, equation 22.5.47, page 779.

chebyshev_u (n, x)

The Chebyshev polynomial of the second kind of degree n.
Reference: Abramowitz and Stegun, equation 22.5.48, page 779.

gen_laguerre (n, a, x)

The generalized Laguerre polynomial of degree n.
Reference: Abramowitz and Stegun, equation 22.5.54, page 780.

hermite (n, x)

The Hermite polynomial of degree n.
Reference: Abramowitz and Stegun, equations 22.5.55, page 780.

intervalp (e)

Return true if the input is an interval and return false if it isn’t.

jacobi_p (n, a, b, x)

The Jacobi polynomial.
The Jacobi polynomials are actually defined for all a and b; however, the Jacobi polynomial weight \((1 - x)^a (1 + x)^b\) isn’t integrable for \(a \leq -1\) or \(b \leq -1\).
Reference: Abramowitz and Stegun, equation 22.5.42, page 779.

laguerre (n, x)

The Laguerre polynomial of degree n.
Reference: Abramowitz and Stegun, equations 22.5.16 and 22.5.54, page 780.

legendre_p (n, x)

The Legendre polynomial of the first kind of degree n.
Reference: Abramowitz and Stegun, equations 22.5.50 and 22.5.51, page 779.

legendre_q (n, x)

The Legendre function of the second kind of degree n.
Reference: Abramowitz and Stegun, equations 8.5.3 and 8.1.8.

orthopoly_recur (f, args)

Returns a recursion relation for the orthogonal function family f with arguments args.
The recursion is with respect to the polynomial degree.

(%i1) orthopoly_recur (legendre_p, [n, x]);
(2 n + 1) P (x) x - n P (x)
(\text{n}) (\text{n} - 1)

(%o1) P (x) = \text{-----------------------------}
\[ n + 1 \quad n + 1 \]

The second argument to \texttt{orthopoly_recur} must be a list with the correct number of arguments for the function \( f \); if it isn't, Maxima signals an error.

\begin{verbatim}(%i1) orthopoly_recur (jacobi_p, [n, x]);
\end{verbatim}

Function \texttt{jacobi}} needs 4 arguments, instead it received 2
-- an error. Quitting. To debug this try debugmode(true);

Additionally, when \( f \) isn't the name of one of the families of orthogonal polynomials, an error is signalled.

\begin{verbatim}(%i1) orthopoly_recur (foo, [n, x]);
\end{verbatim}

A recursion relation for \texttt{foo} isn't known to Maxima
-- an error. Quitting. To debug this try debugmode(true);

\texttt{orthopoly_returns_intervals} \quad \texttt{[Variable]}

Default value: \texttt{true}

When \texttt{orthopoly_returns_intervals} is \texttt{true}, floating point results are returned in the form \texttt{interval \((c, r)\)}, where \( c \) is the center of an interval and \( r \) is its radius. The center can be a complex number; in that case, the interval is a disk in the complex plane.

\texttt{orthopoly_weight} \((f, \text{args})\) \quad \texttt{[Function]}

Returns a three element list; the first element is the formula of the weight for the orthogonal polynomial family \( f \) with arguments given by the list \texttt{args}; the second and third elements give the lower and upper endpoints of the interval of orthogonality. For example,

\begin{verbatim}(%i1) w : orthopoly_weight (hermite, [n, x]);
\end{verbatim}

\begin{verbatim}(%o1) \[\%e ^{- \frac{x}{2}}, -\text{inf, inf}\]
\end{verbatim}

\begin{verbatim}(%i2) integrate(w[1]*hermite(3, x)*hermite(2, x), x, w[2], w[3]);
\end{verbatim}

\begin{verbatim}(%o2) 0
\end{verbatim}

The main variable of \( f \) must be a symbol; if it isn't, Maxima signals an error.

\texttt{pochhammer} \((x, n)\) \quad \texttt{[Function]}

The Pochhammer symbol. For nonnegative integers \( n \) with \( n \leq \texttt{pochhammer}\_\texttt{max}\_\texttt{index} \), the expression \texttt{pochhammer} \((x, n)\) evaluates to the product \( x \ (x + 1) \ (x + 2) \ldots \ (x + n - 1) \) when \( n > 0 \) and to 1 when \( n = 0 \). For negative \( n \), \texttt{pochhammer} \((x, n)\) is defined as \((-1)^n / \texttt{pochhammer} (1 - x, -n)\). Thus

\begin{verbatim}(%i1) pochhammer (x, 3);
\end{verbatim}

\begin{verbatim}(%o1) x \ (x + 1) \ (x + 2)
\end{verbatim}

\begin{verbatim}(%i2) pochhammer (x, -3);
\end{verbatim}

\begin{verbatim}(%o2) \frac{1}{1 - x} \ (2 - x) \ (3 - x)
\end{verbatim}
To convert a Pochhammer symbol into a quotient of gamma functions, (see Abramowitz and Stegun, equation 6.1.22) use makegamma; for example

```
(%i1) makegamma (pochhammer (x, n));
```

\[
\frac{\Gamma(x + n)}{\Gamma(x)}
\]

When \( n \) exceeds \texttt{pochhammer\_max\_index} or when \( n \) is symbolic, \texttt{pochhammer} returns a noun form.

```
(%i1) pochhammer (x, n);
```

\[
(x)^n
\]

\texttt{pochhammer\_max\_index} [Variable]

Default value: 100

\texttt{pochhammer (n, x)} expands to a product if and only if \( n \leq \texttt{pochhammer\_max\_index} \). Examples:

```
(%i1) pochhammer (x, 3), pochhammer\_max\_index : 3;
```

\[
x(x + 1)(x + 2)
\]

```
(%i2) pochhammer (x, 4), pochhammer\_max\_index : 3;
```

\[
(x)^4
\]

Reference: Abramowitz and Stegun, equation 6.1.16, page 256.

\texttt{spherical\_bessel\_j (n, x)} [Function]
The spherical Bessel function of the first kind.


\texttt{spherical\_bessel\_y (n, x)} [Function]
The spherical Bessel function of the second kind.


\texttt{spherical\_hankel1} [Function]
The spherical Hankel function of the first kind.

Reference: Abramowitz and Stegun, equation 10.1.36, page 439.

\texttt{spherical\_hankel2} [Function]
The spherical Hankel function of the second kind.


\texttt{spherical\_harmonic (n, m, x, y)} [Function]
The spherical harmonic function.

Reference: Merzbacher 9.64.

\texttt{unit\_step (x)} [Function]
The left-continuous unit step function; thus \texttt{unit\_step (x)} vanishes for \( x \leq 0 \) and equals 1 for \( x > 0 \).

If you want a unit step function that takes on the value 1/2 at zero, use \((1 + \texttt{signum (x)})/2\).
ultraspherical (n, a, x) [Function]
The ultraspherical polynomial (also known as the Gegenbauer polynomial).
Reference: Abramowitz and Stegun, equation 22.5.46, page 779.
The package `ratpow` provides functions that find the exponents of the denominator in a CRE polynomial. If the exponents in the denominator are needed instead `ratdenom` can be used to extract this denominator first. Returned coefficients are in CRE form except for numbers.

In order to get a list of vars in a CRE polynomial `showratvars` can be used.

For information about CREs see also `rat`, `ratdisrep` and `showratvars`.

### 77.1 Functions and Variables for ratpow

#### `ratp_hipow (expr, x)` [Function]
Finds the highest power of the main variable in `ratnumer(expr)`

```
(%i1) load("ratpow")$
(%i2) ratp_hipow( x^(5/2) + x^2 , x);
   2
(%o2) 2
(%i3) ratp_hipow( x^(5/2) + x^2 , sqrt(x));
   5
(%o3)
```

#### `ratp_lopow (expr, x)` [Function]
Finds the lowest power of the main variable in `ratnumer(expr)`

```
(%i1) load("ratpow")$
(%i2) ratp_lopow( x^5 + x^2 , x);
   2
(%o2) 2
The following example will return 0 since 1 equals x^0:
(%i1) load("ratpow")$
(%i2) ratp_lopow( x^5 + x^2 + 1, x);
   0
(%i3) ratp_lopow( g, x);
   0
(%i4) ratp_lopow( g, sqrt(x));
   0
```

#### `ratp_coeffs (expr, x)` [Function]
Generates a list of powers and coefficients of the main variable `ratnumer(expr)`

```
(%i1) load("ratpow")$
(%i2) ratp_coeffs( 4*x^3 + x + sqrt(x), x);
   [3, 4, 1, 1, 0, sqrt(x)]
```
ratp_dense_coeffs (expr, x)

Generates a list of coefficients in ratnumer(expr); returned coefficients are in CRE form except for numbers.

(%i1) load("ratpow")$
(%i2) ratp_dense_coeffs( 4*x^3 + x + sqrt(x), x);
(%o2)/R/ [4, 0, 1, sqrt(x)]
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78.1 Functions and Variables for romberg

romberg

romberg (expr, x, a, b)
romberg (F, a, b)

Computes a numerical integration by Romberg’s method.

romberg(expr, x, a, b) returns an estimate of the integral integrate(expr, x, a, b). expr must be an expression which evaluates to a floating point value when x is bound to a floating point value.

romberg(F, a, b) returns an estimate of the integral integrate(F(x), x, a, b) where x represents the unnamed, sole argument of F; the actual argument is not named x. F must be a Maxima or Lisp function which returns a floating point value when the argument is a floating point value. F may name a translated or compiled Maxima function.

The accuracy of romberg is governed by the global variables rombergabs and rombergtol. romberg terminates successfully when the absolute difference between successive approximations is less than rombergabs, or the relative difference in successive approximations is less than rombergtol. Thus when rombergabs is 0.0 (the default) only the relative error test has any effect on romberg.

romberg halves the stepsize at most rombergit times before it gives up; the maximum number of function evaluations is therefore 2^rombergit. If the error criterion established by rombergabs and rombergtol is not satisfied, romberg prints an error message. romberg always makes at least rombergmin iterations; this is a heuristic intended to prevent spurious termination when the integrand is oscillatory.

romberg repeatedly evaluates the integrand after binding the variable of integration to a specific value (and not before). This evaluation policy makes it possible to nest calls to romberg, to compute multidimensional integrals. However, the error calculations do not take the errors of nested integrations into account, so errors may be underestimated. Also, methods devised especially for multidimensional problems may yield the same accuracy with fewer function evaluations.

load(romberg) loads this function.

See also QUADPACK, a collection of numerical integration functions.

Examples:

A 1-dimensional integration.

(%i1) load ("romberg");
(%o1) /usr/share/maxima/5.11.0/share/numeric/romberg.lisp
(%i2) f(x) := 1/((x - 1)^2 + 1/100) + 1/((x - 2)^2 + 1/1000) + 1/((x - 3)^2 + 1/200); f(x) :=
1 1 1
2 1 2 1 2 1
--- (x - 1) + --- (x - 2) + ---- (x - 3) + ---
\begin{verbatim}
(%i13) rombergtol : 1e-6;
(%o13) 9.99999999999995E-7
(%i14) rombergit : 15;
(%o14) 15
(%i15) estimate : romberg (f(x), x, -5, 5);
(%o15) 173.6730736617464
(%i16) exact : integrate (f(x), x, -5, 5);
(%o16) 10 sqrt(10) atan(70 sqrt(10))
     + 10 sqrt(10) atan(30 sqrt(10)) + 10 sqrt(2) atan(80 sqrt(2))
     + 10 sqrt(2) atan(20 sqrt(2)) + 10 atan(60) + 10 atan(40)
(%i17) abs (estimate - exact) / exact, numer;
(%o17) 7.5527060865060088E-11
A 2-dimensional integration, implemented by nested calls to romberg.
(%i1) load ("romberg");
(%o1) /usr/share/maxima/5.11.0/share/numeric/romberg.lisp
(%i2) g(x, y) := x*y / (x + y);
     x y
(%o2) g(x, y) := -------
    x + y
(%i3) rombergtol : 1e-6;
(%o3) 9.99999999999995E-7
(%i4) estimate : romberg (romberg (g(x, y), y, 0, x/2), x, 1, 3);
(%o4) 0.81930239628356
(%i5) assume (x > 0);
(%o5) [x > 0]
(%i6) integrate (integrate (g(x, y), y, 0, x/2), x, 1, 3);
            3
     2 log(-) - 1
(%o6) - 9 log(-) + 9 log(3) + ---------------- + -
      2

     9
     2
     9
     2
     6
     2
(%i7) exact : radcan (%);
     26 log(3) - 26 log(2) - 13
     --------------------------
          3
(%i8) abs (estimate - exact) / exact, numer;
(%o8) 1.3711979871851024E-10
\end{verbatim}

**rombergabs**

[Option variable]

Default value: 0.0

The accuracy of `romberg` is governed by the global variables `rombergabs` and `rombergtol`. `romberg` terminates successfully when the absolute difference between successive approximations is less than `rombergabs`, or the relative difference in successive approximations is less than `rombergtol`. Thus when `rombergabs` is 0.0 (the default) only the relative error test has any effect on `romberg`. 
See also `rombergit` and `rombergmin`.

**rombergit**

[Option variable]
Default value: 11

`romberg` halves the stepsize at most `rombergit` times before it gives up; the maximum number of function evaluations is therefore $2^{\text{rombergit}}$. `romberg` always makes at least `rombergmin` iterations; this is a heuristic intended to prevent spurious termination when the integrand is oscillatory.

See also `rombergabs` and `rombergtol`.

**rombergmin**

[Option variable]
Default value: 0

`romberg` always makes at least `rombergmin` iterations; this is a heuristic intended to prevent spurious termination when the integrand is oscillatory.

See also `rombergit`, `rombergabs`, and `rombergtol`.

**rombergtol**

[Option variable]
Default value: 1e-4

The accuracy of `romberg` is governed by the global variables `rombergabs` and `rombergtol`. `romberg` terminates successfully when the absolute difference between successive approximations is less than `rombergabs`, or the relative difference in successive approximations is less than `rombergtol`. Thus when `rombergabs` is 0.0 (the default) only the relative error test has any effect on `romberg`.

See also `rombergit` and `rombergmin`.

79 simplex

79.1 Introduction to simplex

Simplex is a package for linear optimization using the simplex algorithm.

Example:

\begin{verbatim}
(%i1) load("simplex")$
(%i2) minimize_lp(x+y, [3*x+2*y>2, x+4*y>3]);
\end{verbatim}

\begin{verbatim}
   9    7     1
[-- , [y = --, x = --]]
10   10     5
\end{verbatim}

79.1.1 Tests for simplex

There are some tests in the directory share/simplex/Tests.

79.1.1.1 klee_minty

The function klee_minty produces input for linear_program, for which exponential time
for solving is required without scaling.

Example:

\begin{verbatim}
load(klee_minty)$
apply(linear_program, klee_minty(6));
\end{verbatim}

A better approach:

\begin{verbatim}
epsilon_sx : 0$
scale_sx : true$
apply(linear_program, klee_minty(10));
\end{verbatim}

79.1.1.2 NETLIB

Some smaller problems from netlib (http://www.netlib.org/lp/data/) test suite are
converted to a format, readable by Maxima. Problems are adlittle, afiro, kb2 and sc50a.
Each problem has three input files in CSV format for matrix A and vectors b and c.

Example:

\begin{verbatim}
A : read_matrix("adlittle_A.csv", 'csv)$
b : read_list("adlittle_b.csv", 'csv)$
c : read_list("adlittle_c.csv", 'csv)$
linear_program(A, b, c)$
\end{verbatim}

\begin{verbatim}
%[2]
=> 225494.963126615
\end{verbatim}

Results:

\begin{verbatim}
<table>
<thead>
<tr>
<th>PROBLEM</th>
<th>MINIMUM</th>
<th>SCALING</th>
</tr>
</thead>
<tbody>
<tr>
<td>adlittle</td>
<td>225494.963126615</td>
<td>no</td>
</tr>
<tr>
<td>afiro</td>
<td>-464.7531428571429</td>
<td>no</td>
</tr>
<tr>
<td>kb2</td>
<td>-1749.900129055996</td>
<td>yes</td>
</tr>
<tr>
<td>sc50a</td>
<td>-64.5750770585645</td>
<td>no</td>
</tr>
</tbody>
</table>
\end{verbatim}
### 79.2 Functions and Variables for simplex

#### epsilon_lp

Default value: $10^{-8}$

Epsilon used for numerical computations in `linear_program`.

See also: `linear_program`.

#### linear_program (A, b, c)

`linear_program` is an implementation of the simplex algorithm. `linear_program(A, b, c)` computes a vector $x$ for which $c \cdot x$ is minimum possible among vectors for which $A \cdot x = b$ and $x \geq 0$. Argument $A$ is a matrix and arguments $b$ and $c$ are lists.

`linear_program` returns a list which contains the minimizing vector $x$ and the minimum value $c \cdot x$. If the problem is not bounded, it returns "Problem not bounded!" and if the problem is not feasible, it returns "Problem not feasible!".

To use this function first load the `simplex` package with `load(simplex);`.

Example:

```
(%i2) A: matrix([1,1,-1,0], [2,-3,0,-1], [4,-5,0,0])$  
(%i3) b: [1,1,6]$  
(%i4) c: [1,-2,0,0]$  
(%i5) linear_program(A, b, c);  
(%o5) \([---, 4, ---, 0], - -\)  
     2 2 2
```

See also: `minimize_lp`, `scale_lp`, and `epsilon_lp`.

#### maximize_lp (obj, cond, [pos])

Maximizes linear objective function $obj$ subject to some linear constraints $cond$. See `minimize_lp` for detailed description of arguments and return value.

See also: `minimize_lp`.

#### minimize_lp (obj, cond, [pos])

Minimizes a linear objective function $obj$ subject to some linear constraints $cond$. $cond$ a list of linear equations or inequalities. In strict inequalities $>$ is replaced by $\geq$ and $<$ by $\leq$. The optional argument $pos$ is a list of decision variables which are assumed to be positive.

If the minimum exists, `minimize_lp` returns a list which contains the minimum value of the objective function and a list of decision variable values for which the minimum is attained. If the problem is not bounded, `minimize_lp` returns "Problem not bounded!" and if the problem is not feasible, it returns "Problem not feasible!".

The decision variables are not assumed to be nonegative by default. If all decision variables are nonegative, set `nonegative_lp` to true. If only some of decision variables are positive, list them in the optional argument $pos$ (note that this is more efficient than adding constraints).

`minimize_lp` uses the simplex algorithm which is implemented in maxima `linear_program` function.
To use this function first load the `simplex` package with `load(simplex)`.

Examples:

```
(%i1) minimize_lp(x+y, [3*x+y=0, x+2*y>2]);
   4       6       2
(%o1) [-, [y = -, x = -]]
       5       5       5
(%i2) minimize_lp(x+y, [3*x+y>0, x+2*y>2]), nonegative_lp=true;
(%o2) [1, [y = 1, x = 0]]
(%i3) minimize_lp(x+y, [3*x+y=0, x+2*y>2]), nonegative_lp=true;
(%o3) Problem not feasible!
(%i4) minimize_lp(x+y, [3*x+y>0]);
(%o4) Problem not bounded!
```

See also: `maximize_lp`, `nonegative_lp`, `epsilon_lp`.

`nonegative_lp` [Option variable]
Default value: `false`

If `nonegative_lp` is true all decision variables to `minimize_lp` and `maximize_lp` are assumed to be positive.

See also: `minimize_lp`.

`scale_lp` [Option variable]
Default value: `false`

When `scale_lp` is true, `linear_program` scales its input so that the maximum absolute value in each row or column is 1.

`pivot_count_sx` [Variable]
After `linear_program` returns, `pivot_count_sx` is the number of pivots in last computation.

`pivot_max_sx` [Variable]
`pivot_max_sx` is the maximum number of pivots allowed by `linear_program`.
80 simplification

80.1 Introduction to simplification

The directory maxima/share/simplification contains several scripts which implement simplification rules and functions, and also some functions not related to simplification.

80.2 Package absimp

The absimp package contains pattern-matching rules that extend the built-in simplification rules for the abs and signum functions. absimp respects relations established with the built-in assume function and by declarations such as modedeclare (m, even, n, odd) for even or odd integers.

absimp defines unitramp and unitstep functions in terms of abs and signum.

load ("absimp") loads this package. demo (absimp) shows a demonstration of this package.

Examples:

(\%i1) load ("absimp")$
(\%i2) (abs (x))^2;
(\%o2) x
(\%i3) diff (abs (x), x);
(\%o3) -----
(\%i4) cosh (abs (x));
(\%o4) cosh(x)

80.3 Package facexp

The facexp package contains several related functions that provide the user with the ability to structure expressions by controlled expansion. This capability is especially useful when the expression contains variables that have physical meaning, because it is often true that the most economical form of such an expression can be obtained by fully expanding the expression with respect to those variables, and then factoring their coefficients. While it is true that this procedure is not difficult to carry out using standard Maxima functions, additional fine-tuning may also be desirable, and these finishing touches can be more difficult to apply.

The function facsum and its related forms provide a convenient means for controlling the structure of expressions in this way. Another function, collectterms, can be used to add two or more expressions that have already been simplified to this form, without resimplifying the whole expression again. This function may be useful when the expressions are very large.

load ("facexp") loads this package. demo (facexp) shows a demonstration of this package.
facsum (expr, arg_1, ..., arg_n)  [Function]
Returns a form of expr which depends on the arguments arg_1, ..., arg_n. The arguments can be any form suitable for ratvars, or they can be lists of such forms.
If the arguments are not lists, then the form returned is fully expanded with respect to the arguments, and the coefficients of the arguments are factored. These coefficients are free of the arguments, except perhaps in a non-rational sense.
If any of the arguments are lists, then all such lists are combined into a single list, and instead of calling factor on the coefficients of the arguments, facsum calls itself on these coefficients, using this newly constructed single list as the new argument list for this recursive call. This process can be repeated to arbitrary depth by nesting the desired elements in lists.
It is possible that one may wish to facsum with respect to more complicated subexpressions, such as log (x + y). Such arguments are also permissible.
Occasionally the user may wish to obtain any of the above forms for expressions which are specified only by their leading operators. For example, one may wish to facsum with respect to all log’s. In this situation, one may include among the arguments either the specific log’s which are to be treated in this way, or alternatively, either the expression operator (log) or 'operator (log). If one wished to facsum the expression expr with respect to the operators op_1, ..., op_n, one would evaluate facsum (expr, operator (op_1, ..., op_n)). The operator form may also appear inside list arguments.
In addition, the setting of the switches facsum_combine and nextlayerfactor may affect the result of facsum.

nextlayerfactor  [Global variable]
Default value: false
When nextlayerfactor is true, recursive calls of facsum are applied to the factors of the factored form of the coefficients of the arguments.
When false, facsum is applied to each coefficient as a whole whenever recursive calls to facsum occur.
Inclusion of the atom nextlayerfactor in the argument list of facsum has the effect of nextlayerfactor: true, but for the next level of the expression only. Since nextlayerfactor is always bound to either true or false, it must be presented single-quoted whenever it appears in the argument list of facsum.

facsum_combine  [Global variable]
Default value: true
facsum_combine controls the form of the final result returned by facsum when its argument is a quotient of polynomials. If facsum_combine is false then the form will be returned as a fully expanded sum as described above, but if true, then the expression returned is a ratio of polynomials, with each polynomial in the form described above.
The true setting of this switch is useful when one wants to facsum both the numerator and denominator of a rational expression, but does not want the denominator to be multiplied through the terms of the numerator.
factorfacsum (expr, arg_1, ... arg_n)  
Returns a form of expr which is obtained by calling facsum on the factors of expr with arg_1, ... arg_n as arguments. If any of the factors of expr is raised to a power, both the factor and the exponent will be processed in this way.

collectterms (expr, arg_1, ..., arg_n)  
Collects all terms that contain arg_1 ... arg_n. If several expressions have been simplified with the following functions facsum, factorfacsum, factenexpand, facepten or factorfacepten, and they are to be added together, it may be desirable to combine them using the function collectterms. collectterms can take as arguments all of the arguments that can be given to these other associated functions with the exception of nextlayerfactor, which has no effect on collectterms. The advantage of collectterms is that it returns a form similar to facsum, but since it is adding forms that have already been processed by facsum, it does not need to repeat that effort. This capability is especially useful when the expressions to be summed are very large.

See also factor.

Example:
```
(%i1) (exp(x)+2)*x+exp(x);
    x    x
(%o1) x (%e + 2) + %e
(%i2) collectterms(expand(%),exp(x));
         x
(%o2) (x + 1) %e + 2 x
```

80.4 Package functs

rempart (expr, n)  
Removes part n from the expression expr.

If n is a list of the form [l, m] then parts l thru m are removed.

To use this function write first load(functs).

wronskian ([f_1, ..., f_n], x)  
Returns the Wronskian matrix of the list of expressions [f_1, ..., f_n] in the variable x. The determinant of the Wronskian matrix is the Wronskian determinant of the list of expressions.

To use wronskian, first load(functs). Example:
```
(%i1) load ("functs")$
(%i2) wronskian([f(x), g(x)],x);
    [   f(x)    g(x) ]
    [                ]
[ d        d     ]
[ -- (f(x)) -- (g(x)) ]
[ dx        dx    ]
(%o2)
```

tracematrix (M)  
Returns the trace (sum of the diagonal elements) of matrix M.
To use this function write first `load(functs)`.

`rational (z)` [Function]
Multiplies numerator and denominator of z by the complex conjugate of denominator, thus rationalizing the denominator. Returns canonical rational expression (CRE) form if given one, else returns general form.

To use this function write first `load(functs)`.

`nonzeroandfreeof (x, expr)` [Function]
Returns `true` if `expr` is nonzero and `freeof (x, expr)` returns `true`. Returns `false` otherwise.

To use this function write first `load(functs)`.

`linear (expr, x)` [Function]
When `expr` is an expression of the form `a*x + b` where `a` is nonzero, and `a` and `b` are free of `x`, `linear` returns a list of three equations, one for each of the three formal variables `b`, `a`, and `x`. Otherwise, `linear` returns `false`.

`load(antid)` loads this function.

Example:
```
(%i1) load ("antid");
(%o1) /maxima/share/integration/antid.mac
(%i2) linear ((1 - w)*(1 - x)*z, z);
(%o2) [bargumentb = 0, aargumenta = (w - 1) x - w + 1,
xargumentx = z]
(%i3) linear (cos(u - v) + cos(u + v), u);
(%o3) false
```

`gcdivide (p, q)` [Function]
When the option variable `takegcd` is `true` which is the default, `gcdivide` divides the polynomials `p` and `q` by their greatest common divisor and returns the ratio of the results. `gcdivide` calls the function `ezgcd` to divide the polynomials by the greatest common divisor.

When `takegcd` is `false`, `gcdivide` returns the ratio `p/q`.

To use this function write first `load(functs)`.

See also `ezgcd`, `gcd`, `gcdex`, and `poly_gcd`.

Example:
```
(%i1) load(functs)$
(%i2) p1:6*x^3+19*x^2+19*x+6;
   3   2
(%o2) 6 x + 19 x + 19 x + 6
(%i3) p2:6*x^5+13*x^4+12*x^3+13*x^2+6*x;
   5   4 3 2
(%o3) 6 x + 13 x + 12 x + 13 x + 6 x
(%i4) gcdivide(p1, p2);
   x + 1
```
arithmetic (a, d, n)  [Function]
Returns the n-th term of the arithmetic series a, a + d, a + 2d, ..., a + (n - 1)d.
To use this function write first load(functs).

geometric (a, r, n)  [Function]
Returns the n-th term of the geometric series a, a*r, a*r^2, ..., a*r^(n - 1).
To use this function write first load(functs).

harmonic (a, b, c, n)  [Function]
Returns the n-th term of the harmonic series a/b, a/(b + c), a/(b + 2c), ..., a/(b + (n - 1)c).
To use this function write first load(functs).

arithsum (a, d, n)  [Function]
Returns the sum of the arithmetic series from 1 to n.
To use this function write first load(functs).

geosum (a, r, n)  [Function]
Returns the sum of the geometric series from 1 to n. If n is infinity (inf) then a sum is finite only if the absolute value of r is less than 1.
To use this function write first load(functs).

gaussprob (x)  [Function]
Returns the Gaussian probability function %e(-x^2/2) / sqrt(2*%pi).
To use this function write first load(functs).

gd (x)  [Function]
Returns the Gudermannian function 2.atan(%e^-x)-%pi/2.
To use this function write first load(functs).
agd (x)  
Returns the inverse Gudermannian function \( \log(\tan(\pi/4 + x/2)) \).
To use this function write first \texttt{load(funcs)}.

vers (x)  
Returns the versed sine \( 1 - \cos(x) \).
To use this function write first \texttt{load(funcs)}.

covers (x)  
Returns the covered sine \( 1 - \sin(x) \).
To use this function write first \texttt{load(funcs)}.

exsec (x)  
Returns the exsecant \( \sec(x) - 1 \).
To use this function write first \texttt{load(funcs)}.

hav (x)  
Returns the haversine \( (1 - \cos(x))/2 \).
To use this function write first \texttt{load(funcs)}.

combination (n, r)  
Returns the number of combinations of \( n \) objects taken \( r \) at a time.
To use this function write first \texttt{load(funcs)}.

permutation (n, r)  
Returns the number of permutations of \( r \) objects selected from a set of \( n \) objects.
To use this function write first \texttt{load(funcs)}.

80.5 Package \texttt{ineq}

The \texttt{ineq} package contains simplification rules for inequalities.

Example session:

\begin{verbatim}
(%i1) load(ineq)$

tellsimp: warning: rule will treat '+'
   ' as noncommutative and nonassociative.

tellsimp: warning: rule will treat '+'
   ' as noncommutative and nonassociative.

tellsimp: warning: rule will treat '+'
   ' as noncommutative and nonassociative.

tellsimp: warning: rule will treat '+'
   ' as noncommutative and nonassociative.

tellsimp: warning: rule will treat '+'
   ' as noncommutative and nonassociative.

tellsimp: warning: rule will treat '+'
   ' as noncommutative and nonassociative.

tellsimp: warning: rule will treat '+'
   ' as noncommutative and nonassociative.

tellsimp: warning: rule will treat '+'
   ' as noncommutative and nonassociative.

tellsimp: warning: rule will treat '+'
   ' as noncommutative and nonassociative.

tellsimp: warning: rule will treat '+'
   ' as noncommutative and nonassociative.

tellsimp: warning: rule will treat '+'
   ' as noncommutative and nonassociative.
\end{verbatim}
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tellsimp: warning: rule will treat '+' as noncommutative and nonassociative.
(%i2) a>=4; /* a sample inequality */
(%o2) a >= 4
(%o3) b + a > c + 4
(%o4) 7 x < 7 y
(%o5) - 2 x <= - 6 z
 2
(%o6) 1 <= a + 1
(%o8) 2 x < 3 x
(%o9) a >= b
(%o10) a + 3 >= b + 3
(%o11) a >= b
(%o12) a >= c - b
(%o13) b + a >= c
(%o14) (- c) + b + a >= 0
(%o15) c - b - a <= 0
 2
(%o16) (z - 1) > - 2 z
 2
(%o17) z + 1 > 0
(%o18) true
(%i19) (b>c)+%; /* add a second, strict inequality */

Be careful about using parentheses around the inequalities: when the user types in (A > B) + (C = 5) the result is A + C > B + 5, but A > B + C = 5 is a syntax error, and (A > B + C) = 5 is something else entirely.

Do disprule (all) to see a complete listing of the rule definitions.

The user will be queried if Maxima is unable to decide the sign of a quantity multiplying an inequality.

The most common mis-feature is illustrated by:

(%i1) eq: a > b;
(%o1)
(%i2) 2*eq;
(%o2) 2 (a > b)
(%i3) % - eq;
(%o3) a > b

Another problem is 0 times an inequality; the default to have this turn into 0 has been left alone. However, if you type X*some_inequality and Maxima asks about the sign of X and you respond zero (or z), the program returns X*some_inequality and not use the information that X is 0. You should do ev (%, x: 0) in such a case, as the database will only be used for comparison purposes in decisions, and not for the purpose of evaluating X.

The user may note a slower response when this package is loaded, as the simplifier is forced to examine more rules than without the package, so you might wish to remove the rules after making use of them. Do kill (rules) to eliminate all of the rules (including
any that you might have defined); or you may be more selective by killing only some of
them; or use \texttt{remrule} on a specific rule.

Note that if you load this package after defining your own rules you will clobber your
rules that have the same name. The rules in this package are: \texttt{*rule1}, ... , \texttt{*rule8}, \texttt{+rule1},
..., \texttt{+rule18}, and you must enclose the rulename in quotes to refer to it, as in \texttt{remrule}
("+", \texttt{"+rule1"}) to specifically remove the first rule on "+" or \texttt{disprule} (\texttt{"*rule2"}) to
display the definition of the second multiplicative rule.

\section{80.6 Package rducon}

\texttt{reduceconsts (expr)}

\texttt{reduceconsts} replaces constant subexpressions of \texttt{expr} with constructed constant atoms, saving
the definition of all these constructed constants in the list of equations \texttt{const_eqns}, and
returning the modified \texttt{expr}. Those parts of \texttt{expr} are constant which return \texttt{true}
when operated on by the function \texttt{constantp}. Hence, before invoking \texttt{reduceconsts}, one
should do
\begin{verbatim}
    declare ([objects to be given the constant property], constant)
\end{verbatim}
to set up a database of the constant quantities occurring in your expressions.

If you are planning to generate Fortran output after these symbolic calculations, one
of the first code sections should be the calculation of all constants. To generate this
code segment, do
\begin{verbatim}
    map ('fortran, const_eqns)
\end{verbatim}

Variables besides \texttt{const_eqns} which affect \texttt{reduceconsts} are:
\begin{enumerate}
\item \texttt{const_prefix} (default value: \texttt{xx}) is the string of characters used to prefix all symbols
      generated by \texttt{reduceconsts} to represent constant subexpressions.
\item \texttt{const_counter} (default value: 1) is the integer index used to generate unique symbols
      to represent each constant subexpression found by \texttt{reduceconsts}.
\end{enumerate}

\texttt{load ("rducon")} loads this function. \texttt{demo (rducon)} shows a demonstration of this
function.

\section{80.7 Package scifac}

\texttt{gcfac (expr)}

\texttt{gcfac} is a factoring function that attempts to apply the same heuristics which scient-
ists apply in trying to make expressions simpler. \texttt{gcfac} is limited to monomial-type
factoring. For a sum, \texttt{gcfac} does the following:
\begin{enumerate}
\item Factors over the integers.
\item Factors out the largest powers of terms occurring as coefficients, regardless of the
      complexity of the terms.
\item Uses (1) and (2) in factoring adjacent pairs of terms.
\item Repeatedly and recursively applies these techniques until the expression no longer
      changes.
\end{enumerate}
Item (3) does not necessarily do an optimal job of pairwise factoring because of the combinatorially-difficult nature of finding which of all possible rearrangements of the pairs yields the most compact pair-factored result.

(load "scifac") loads this function. demo (scifac) shows a demonstration of this function.

80.8 Package sqdnst

sqrtdenest (expr)  [Function]

Denests sqrt of simple, numerical, binomial surds, where possible. E.g.

(%i1) load ("sqdnst")$
(%i2) sqrt(sqrt(3)/2+1)/sqrt(11*sqrt(2)-12);
     sqrt(3)
     sqrt(------ + 1)
     2
     sqrt(11 sqrt(2) - 12)
(%o2) ---------------------

(%i3) sqrtdenest(%);
     sqrt(3)  1
     ------- + -
     2  2
     1/4  3/4
     3 2  - 2
(%o3) ------------

Sometimes it helps to apply sqrtdenest more than once, on such as (19601-13860 sqrt(2))^(7/4).

load ("sqdnst") loads this function.
81 solve_rec

81.1 Introduction to solve_rec

solve_rec is a package for solving linear recurrences with polynomial coefficients.

A demo is available with demo(solve_rec);

Example:

(%i1) load("solve_rec")
(%i2) solve_rec((n+4)*s[n+2] + s[n+1] - (n+1)*s[n], s[n]);

(%o2) s = \frac{(2n + 3)(-1)^k}{n(n + 1)(n + 2)(n + 1)(n + 2)}

81.2 Functions and Variables for solve_rec

reduce_order (rec, sol, var) [Function]

Reduces the order of linear recurrence rec when a particular solution sol is known. The reduced recurrence can be used to get other solutions.

Example:

(%i3) rec: x[n+2] = x[n+1] + x[n]/n;
(%i4) solve_rec(rec, x[n]);

WARNING: found some hypergeometrical solutions!

(%o4) x = \frac{n %k}{n + 1}

(%i5) reduce_order(rec, x[n]);

(%t5) x = n %z

(%o6) \frac{(- n - 2) %u - %u}{n + 1}
(%i6) solve_rec((n+2)*%u[n+1] + %u[n], %u[n]);
  n
  %k (- 1)
  1
(%)o6) %u = ----------
  n (n + 1)!

So the general solution is

\[ \sum_{j=0}^{n-1} \left( \frac{(-1)^j}{2^j (j+1)!} \right) + \frac{(-1)^n}{(n+1)!} \]

simplify_products
[Option variable]

Default value: true

If simplify_products is true, solve_rec will try to simplify products in result.
See also: solve_rec.

simplify_sum (expr)
[Function]

Tries to simplify all sums appearing in expr to a closed form.
To use this function first load the simplify_sum package with load(simplify_sum).
Example:

(%i1) load("simplify_sum")$
(%i2) sum(binomial(n+k,k)/2^k,k,1,n)+sum(binomial(2*n,2*k),k,1,n);
  n
  \sum_{k=1}^{n} \binomial{n+k}{k} \quad \sum_{k=1}^{2n} \binomial{2n}{2k}
\sum_{k=1}^{n} \binomial{n+k}{k} / \sum_{k=1}^{2n} \binomial{2n}{2k}

(%o2)
(%i3) simplify_sum(%);

\[ \frac{2n-1}{2} + \frac{n}{2} - 2 \]

solve_rec (eqn, var, [init])
[Function]

Solves for hypergeometrical solutions to linear recurrence eqn with polynomials coefficient in variable var. Optional arguments init are initial conditions.
solve_rec can solve linear recurrences with constant coefficients, finds hypergeometrical solutions to homogeneous linear recurrences with polynomial coefficients, rational solutions to linear recurrences with polynomial coefficients and can solve Ricatti type recurrences.
Note that the running time of the algorithm used to find hypergeometrical solutions is exponential in the degree of the leading and trailing coefficient.

To use this function first load the `solve_rec` package with `load(solve_rec);`.

Example of linear recurrence with constant coefficients:

```plaintext
(%i2) solve_rec(a[n]=a[n-1]+a[n-2]+n/2^n, a[n]);
```

```plaintext
(%o2) a = ------------------------- - ----
             n  n  n
           2  5  2
        (sqrt(5) - 1) %k (- 1)
        n
        1
        n
        (sqrt(5) + 1) %k
        2  2
        + ------------------ - ----
        n  n  n
        2  5  2
```

Example of linear recurrence with polynomial coefficients:

```plaintext
(%i7) 2*x*(x+1)*y[x] - (x^2+3*x-2)*y[x+1] + (x-1)*y[x+2];
```

```plaintext
(%o7) (x - 1) y - (x + 3 x - 2) y + 2 x (x + 1) y
```

```plaintext
(%i8) solve_rec(%i7, y[x], y[1]=1, y[3]=3);
```

```plaintext
(%o8) y = ---- - --
     x 4 2
```

Example of Ricatti type recurrence:

```plaintext
(%i2) x*y[x+1]*y[x] - y[x+1]/(x+2) + y[x]/(x-1) = 0;
```

```plaintext
(%o2) y y
       x + 1 x
       (x - 1) y
       x + 2 x + 1 x - 1
```

```plaintext
(%i3) solve_rec(%o2, y[x], y[3]=5)$
```

```plaintext
(%o4) y = - -------------------------
       x 6 5 4 3 2
       5 x - 3 x - 25 x + 15 x + 20 x - 12 x - 1584
```

See also: `solve_rec_rat`, `simplify_products` and `product_use_gamma`.

`solve_rec_rat (eqn, var, [init])`

Solves for rational solutions to linear recurrences. See `solve_rec` for description of arguments.
To use this function first load the solve_rec package with `load(solve_rec);`.

Example:

```maxima
(%i1) (x+4)*a[x+3] + (x+3)*a[x+2] - x*a[x+1] + (x^2-1)*a[x];
(%o1) (x + 4) a + (x + 3) a - x a
     x + 3     x + 2     x + 1
      2
     + (x - 1) a
     x

(%i2) solve_rec_rat(% = (x+2)/(x+1), a[x]);
(%o2) a = ---------------
        x (x - 1) (x + 1)
```

See also: solve_rec.

**product_use_gamma**

[Option variable]

Default value: true

When simplifying products, solve_rec introduces gamma function into the expression if `product_use_gamma` is `true`.

See also: simplify_products, solve_rec.

**summand_to_rec**

[Function]

```
summand_to_rec (summand, k, n)
summand_to_rec (summand, [k, lo, hi], n)
```

Returns the recurrence satisfied by the sum

```
hi
====
/ \> summand
/  
====
k = lo
```

where summand is hypergeometrical in \(k\) and \(n\). If \(lo\) and \(hi\) are omitted, they are assumed to be \(lo = -\infty\) and \(hi = \infty\).

To use this function first load the simplify_sum package with `load(simplify_sum)`.

Example:

```maxima
(%i1) load("simplify_sum")$
(%i2) summand: binom(n,k);
(%o2) binomial(n, k)
(%i3) summand_to_rec(summand,k,n);
(%o3) 2 sm - sm = 0
     n  n + 1
(%i7) summand: binom(n, k)/(k+1);
(%o7) ------------------
        k + 1
```
(%i8) summand_to_rec(summand, [k, 0, n], n);
(%o8) 2 \frac{(n + 1) s_m}{n} - \frac{(n + 2) s_m}{n + 1} = -1
82 stats

82.1 Introduction to stats

Package stats contains a set of classical statistical inference and hypothesis testing procedures.

All these functions return an inference_result Maxima object which contains the necessary results for population inferences and decision making.

Global variable stats_numer controls whether results are given in floating point or symbolic and rational format; its default value is true and results are returned in floating point format.

Package descriptive contains some utilities to manipulate data structures (lists and matrices); for example, to extract subsamples. It also contains some examples on how to use package numericalio to read data from plain text files. See descriptive and numericalio for more details.

Package stats loads packages descriptive, distrib and inference_result.

For comments, bugs or suggestions, please contact the author at 'mario AT edu DOT xunta DOT es'.

82.2 Functions and Variables for inference_result

inference_result (title, values, numbers)  [Function]

Constructs an inference_result object of the type returned by the stats functions. Argument title is a string with the name of the procedure; values is a list with elements of the form symbol = value and numbers is a list with positive integer numbers ranging from one to length(values), indicating which values will be shown by default.

Example:

This is a simple example showing results concerning a rectangle. The title of this object is the string "Rectangle", it stores five results, named 'base, 'height, 'diagonal, 'area, and 'perimeter, but only the first, second, fifth, and fourth will be displayed. The 'diagonal is stored in this object, but it is not displayed; to access its value, make use of function take_inference.

(%i1) load(inference_result)$
(%i2) b: 3$ h: 2$
(%i3) inference_result("Rectangle",
['base=b, 'height=h, 'diagonal=sqrt(b^2+h^2), 'area=b*h, 'perimeter=2*(b+h)],
[1,2,5,4]);
| Rectangle
|
See also `take_inference`.

\textbf{inferencep (obj)} \hfill [Function]

Returns true or false, depending on whether \textit{obj} is an \texttt{inference_result} object or not.

\textbf{items_inference (obj)} \hfill [Function]

Returns a list with the names of the items stored in \textit{obj}, which must be an \texttt{inference_result} object.

Example:

The \texttt{inference_result} object stores two values, named 'pi and 'e, but only the second is displayed. The \texttt{items_inference} function returns the names of all items, no matter they are displayed or not.

\begin{verbatim}
(\%i1) load(inference_result)$
(\%i2) inference_result("Hi", ['pi=%pi,'e=%e],[2]);
     | Hi
     | e = %e
(\%o2)
(\%i3) items_inference(%);
     [pi, e]
(\%o3)
\end{verbatim}

\textbf{take_inference} \hfill [Function]

\begin{verbatim}
take_inference (n, obj)
take_inference (name, obj)
take_inference (list, obj)
\end{verbatim}

Returns the \textit{n}-th value stored in \textit{obj} if \textit{n} is a positive integer, or the item named \textit{name} if this is the name of an item. If the first argument is a list of numbers and/or symbols, function \texttt{take_inference} returns a list with the corresponding results.

Example:

Given an \texttt{inference_result} object, function \texttt{take_inference} is called in order to extract some information stored in it.

\begin{verbatim}
(\%i1) load(inference_result)$
(\%i2) b: 3$ h: 2$
(\%i3) sol: inference_result("Rectangle",
     ['base=b, 'height=h, 'diagonal=sqrt(b^2+h^2),
     ...
\end{verbatim}
'area=b*h,
'perimeter=2*(b+h)],
[1,2,5,4] );

| Rectangle
| base = 3
| height = 2
| perimeter = 10
| area = 6

(%i4) take_inference('base,sol);
(%o4) 3
(%i5) take_inference(5,sol);
(%o5) 10
(%i6) take_inference([1,'diagonal],sol);
(%o6) [3, sqrt(13)]
(%i7) take_inference(items_inference(sol),sol);
(%o7) [3, 2, sqrt(13), 6, 10]

See also inference_result, and take_inference.

82.3 Functions and Variables for stats

\textbf{stats_numer}

[Option variable]

Default value: true

If \texttt{stats_numer} is true, inference statistical functions return their results in floating point numbers. If it is false, results are given in symbolic and rational format.

\textbf{test_mean}

[Function]

\begin{verbatim}
test_mean (x) test_mean (x, options ...)
\end{verbatim}

This is the mean t-test. Argument x is a list or a column matrix containing a one dimensional sample. It also performs an asymptotic test based on the Central Limit Theorem if option 'asymptotic is true.

Options:

- 'mean, default 0, is the mean value to be checked.
- 'alternative, default 'twosided, is the alternative hypothesis; valid values are: 'twosided, 'greater and 'less.
- 'dev, default 'unknown, this is the value of the standard deviation when it is known; valid values are: 'unknown or a positive expression.
- 'conflevel, default 95/100, confidence level for the confidence interval; it must be an expression which takes a value in (0,1).
- 'asymptotic, default false, indicates whether it performs an exact t-test or an asymptotic one based on the Central Limit Theorem; valid values are true and false.
The output of function \texttt{test\_mean} is an \texttt{inference\_result} Maxima object showing the following results:

1. \texttt{'mean\_estimate}: the sample mean.
2. \texttt{'conf\_level}: confidence level selected by the user.
3. \texttt{'conf\_interval}: confidence interval for the population mean.
4. \texttt{'method}: inference procedure.
5. \texttt{'hypotheses}: null and alternative hypotheses to be tested.
6. \texttt{'statistic}: value of the sample statistic used for testing the null hypothesis.
7. \texttt{'distribution}: distribution of the sample statistic, together with its parameter(s).
8. \texttt{'p\_value}: \textit{p}-value of the test.

Examples:

Performs an exact \textit{t}-test with unknown variance. The null hypothesis is \(H_0: \text{mean} = 50\) against the one sided alternative \(H_1: \text{mean} < 50\); according to the results, the \textit{p}-value is too great, there are no evidence for rejecting \(H_0\).

\begin{verbatim}
(%i1) load("stats")
(%i2) data: [78,64,35,45,45,75,43,74,42,42]
(%i3) test_mean(data,'conflevel=0.9,'alternative='less,'mean=50);
   MEAN TEST
   mean_estimate = 54.3
   conf_level = 0.9
   conf_interval = [minf, 61.51314273502712]
   method = Exact \textit{t}-test. Unknown variance.
   hypotheses = H0: mean = 50 , H1: mean < 50
   statistic = .8244705235071678
   distribution = [student\_t, 9]
   p_value = .7845100411786889
\end{verbatim}

This time Maxima performs an asymptotic test, based on the \textit{Central Limit Theorem}. The null hypothesis is \(H_0: \text{equal(mean, 50)}\) against the two sided alternative \(H_1: \text{notequal(mean, 50)}\); according to the results, the \textit{p}-value is very small, \(H_0\) should be rejected in favor of the alternative \(H_1\). Note that, as indicated by the \texttt{Method} component, this procedure should be applied to large samples.

\begin{verbatim}
(%i1) load("stats")
(%i2) test_mean([36,118,52,87,35,256,56,178,57,57,89,34,25,98,35,98,41,45,198,54,79,63,35,45,44,75,42,75,45,45,45,51,123,54,151],
\end{verbatim}
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'asymptotic=true,'mean=50);

MEAN TEST

mean_estimate = 74.88571428571429

conf_level = 0.95

conf_interval = [57.72848600856194, 92.04294256286663]

(%o2) method = Large sample z-test. Unknown variance.

hypotheses = H0: mean = 50 , H1: mean # 50

statistic = 2.842831192874313

distribution = [normal, 0, 1]

p_value = .004471474652002261

\textbf{test\_means\_difference}  
\texttt{test\_means\_difference (x1, x2) }  
\texttt{test\_means\_difference (x1, x2, options ...)}  
This is the difference of means $t$-test for two samples. Arguments $x1$ and $x2$ are lists or column matrices containing two independent samples. In case of different unknown variances (see options 'dev1, 'dev2 and 'varequal below), the degrees of freedom are computed by means of the Welch approximation. It also performs an asymptotic test based on the \textit{Central Limit Theorem} if option 'asymptotic is set to true.

Options:

- 'alternative, default 'twosided, is the alternative hypothesis; valid values are: 'twosided, 'greater and 'less.
- 'dev1, default 'unknown, this is the value of the standard deviation of the $x1$ sample when it is known; valid values are: 'unknown or a positive expression.
- 'dev2, default 'unknown, this is the value of the standard deviation of the $x2$ sample when it is known; valid values are: 'unknown or a positive expression.
- 'varequal, default false, whether variances should be considered to be equal or not; this option takes effect only when 'dev1 and/or 'dev2 are 'unknown.
- 'conflevel, default 95/100, confidence level for the confidence interval; it must be an expression which takes a value in (0,1).
- 'asymptotic, default false, indicates whether it performs an exact $t$-test or an asymptotic one based on the \textit{Central Limit Theorem}; valid values are true and false.

The output of function \texttt{test\_means\_difference} is an \texttt{inference\_result} Maxima object showing the following results:

1. 'diff\_estimate: the difference of means estimate.
2. 'conf_level: confidence level selected by the user.
3. 'conf_interval: confidence interval for the difference of means.
4. 'method: inference procedure.
5. 'hypotheses: null and alternative hypotheses to be tested.
6. 'statistic: value of the sample statistic used for testing the null hypothesis.
7. 'distribution: distribution of the sample statistic, together with its parameter(s).
8. 'p_value: p-value of the test.

Examples:
The equality of means is tested with two small samples \(x\) and \(y\), against the alternative \(H_1: m_1 > m_2\), being \(m_1\) and \(m_2\) the populations means; variances are unknown and supposed to be different.

```maxima
(%i1) load("stats")
(%i2) x: [20.4, 62.5, 61.3, 44.2, 11.1, 23.7]
(%i3) y: [1.2, 6.9, 38.7, 20.4, 17.2]
(%i4) test_means_difference(x,y,'alternative='greater);
DIFFERENCE OF MEANS TEST
| diff_estimate = 20.319999999999999
| conf_level = 0.95
| conf_interval = [ -0.045717812882298, inf ]
(%o4) | method = Exact t-test. Welch approx.
| hypotheses = H0: mean1 = mean2 , H1: mean1 > mean2
| statistic = 1.838004300728477
| distribution = [student_t, 8.62758740184604]
| p_value = 0.05032746527991905
```

The same test as before, but now variances are supposed to be equal.

```maxima
(%i1) load("stats")
(%i2) x: [20.4, 62.5, 61.3, 44.2, 11.1, 23.7]
(%i3) y: matrix([1.2],[6.9],[38.7],[20.4],[17.2])
(%i4) test_means_difference(x,y,'alternative='greater,
  'varequal=true);
DIFFERENCE OF MEANS TEST
| diff_estimate = 20.319999999999999
| conf_level = 0.95
```
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\[\text{conf_interval} = [-0.7722627696897568, \text{inf}]\]

\((\%o4)\)

\[\text{method} = \text{Exact t-test. Unknown equal variances}\]

\[\text{hypotheses} = \text{H0: mean1 = mean2, H1: mean1 > mean2}\]

\[\text{statistic} = 1.765996124515009\]

\[\text{distribution} = [\text{student_t, 9}]\]

\[\text{p_value} = 0.05560320992529344\]

test_variance

\[
\text{test_variance (x)}
\]

\[
\text{test_variance (x, options, ...)}
\]

This is the variance chi^2-test. Argument x is a list or a column matrix containing a one dimensional sample taken from a normal population.

Options:

- 'mean, default 'unknown, is the population’s mean, when it is known.
- 'alternative, default 'twosided, is the alternative hypothesis; valid values are: 'twosided, 'greater and 'less.
- 'variance, default 1, this is the variance value (positive) to be checked.
- 'conflevel, default 95/100, confidence level for the confidence interval; it must be an expression which takes a value in (0,1).

The output of function test_variance is an inference_result Maxima object showing the following results:

1. 'var_estimate: the sample variance.
2. 'conf_level: confidence level selected by the user.
3. 'conf_interval: confidence interval for the population variance.
4. 'method: inference procedure.
5. 'hypotheses: null and alternative hypotheses to be tested.
6. 'statistic: value of the sample statistic used for testing the null hypothesis.
7. 'distribution: distribution of the sample statistic, together with its parameter.
8. 'p_value: p-value of the test.

Examples:

It is tested whether the variance of a population with unknown mean is equal to or greater than 200.

\((\%i11)\) load("stats")$
(\%i12) x: [203, 229, 215, 220, 223, 233, 208, 228, 209]$ 
(\%i13) test_variance(x,'alternative='greater,'variance=200);
\[ \text{var}_{\text{estimate}} = 110.75 \]
\[ \text{conf}\_\text{level} = 0.95 \]
\[ \text{conf}\_\text{interval} = [57.1343376937479, \infty] \]

\((\%3)\) \text{method} = \text{Variance Chi-square test. Unknown mean.} \]
\hypotheses = H0: \text{var} = 200 , H1: \text{var} > 200
\[ \text{statistic} = 4.43 \]
\[ \text{distribution} = [\text{chi2}, 8] \]
\[ \text{p}\_\text{value} = .8163948512777689 \]

\text{test\_variance\_ratio} \quad [\text{Function}]
\text{test\_variance\_ratio (x1, x2)}
\text{test\_variance\_ratio (x1, x2, options ...)}

This is the variance ratio \(F\)-test for two normal populations. Arguments \(x1\) and \(x2\) are lists or column matrices containing two independent samples.

Options:
- 'alternative, default 'twosided, is the alternative hypothesis; valid values are: 'twosided, 'greater and 'less.
- 'mean1, default 'unknown, when it is known, this is the mean of the population from which \(x1\) was taken.
- 'mean2, default 'unknown, when it is known, this is the mean of the population from which \(x2\) was taken.
- 'conflevel, default 95/100, confidence level for the confidence interval of the ratio; it must be an expression which takes a value in \((0,1)\).

The output of function \text{test\_variance\_ratio} is an \text{inference\_result} Maxima object showing the following results:
1. 'ratio\_estimate: the sample variance ratio.
2. 'conf\_level: confidence level selected by the user.
3. 'conf\_interval: confidence interval for the variance ratio.
4. 'method: inference procedure.
5. 'hypotheses: null and alternative hypotheses to be tested.
6. 'statistic: value of the sample statistic used for testing the null hypothesis.
7. 'distribution: distribution of the sample statistic, together with its parameters.
8. 'p\_value: \(p\)-value of the test.

Examples:
The equality of the variances of two normal populations is checked against the alternative that the first is greater than the second.

(%i1) load("stats")$
(%i2) x: [20.4, 62.5, 61.3, 44.2, 11.1, 23.7]$
(%i3) y: [1.2, 6.9, 38.7, 20.4, 17.2]$
(%i4) test_variance_ratio(x,y,'alternative='greater);
   VARIANCE RATIO TEST
   ratio_estimate = 2.316933391522034
   conf_level = 0.95
   conf_interval = [.3703504689507268, inf]
   method = Variance ratio F-test. Unknown means.
   hypotheses = H0: var1 = var2, H1: var1 > var2
   statistic = 2.316933391522034
   distribution = [f, 5, 4]
   p_value = .2179269692254457

**test_proportion**

test_proportion (x, n)
test_proportion (x, n, options ...)

Inferences on a proportion. Argument x is the number of successes in n trials in a Bernoulli experiment with unknown probability.

Options:
- 'proportion, default 1/2, is the value of the proportion to be checked.
- 'alternative, default 'twosided, is the alternative hypothesis; valid values are: 'twosided, 'greater and 'less.
- 'conflevel, default 95/100, confidence level for the confidence interval; it must be an expression which takes a value in (0,1).
- 'asymptotic, default false, indicates whether it performs an exact test based on the binomial distribution, or an asymptotic one based on the Central Limit Theorem; valid values are true and false.
- 'correct, default true, indicates whether Yates correction is applied or not.

The output of function test_proportion is an inference_result Maxima object showing the following results:
1. 'sample_proportion: the sample proportion.
2. 'conf_level: confidence level selected by the user.
3. 'conf_interval: Wilson confidence interval for the proportion.
4. 'method: inference procedure.
5. 'hypotheses: null and alternative hypotheses to be tested.
6. 'statistic: value of the sample statistic used for testing the null hypothesis.
7. 'distribution: distribution of the sample statistic, together with its parameters.
8. 'p_value: p-value of the test.

Examples:

Performs an exact test. The null hypothesis is $H_0: p = 1/2$ against the one sided alternative $H_1: p < 1/2$.

```
(%i1) load("stats")
(%i2) test_proportion(45, 103, alternative = less);
```

```
PROPORTION TEST

| sample_proportion = .4368932038834951 |
| conf_level = 0.95 |
| conf_interval = [0, 0.522714149150231] |
```

```
(%o2) method = Exact binomial test.

| hypotheses = H0: p = 0.5 , H1: p < 0.5 |
| statistic = 45 |
| distribution = [binomial, 103, 0.5] |
| p_value = .1184509388901454 |
```

A two sided asymptotic test. Confidence level is 99/100.

```
(%i1) load("stats")
(%i2) fpprintprec:7$
(%i3) test_proportion(45, 103, conflevel = 99/100, asymptotic=true);
```

```
PROPORTION TEST

| sample_proportion = .43689 |
| conf_level = 0.99 |
| conf_interval = [.31422, .56749] |

(%o3) method = Asymptotic test with Yates correction.
```

| hypotheses = H0: p = 0.5 , H1: p # 0.5 |
test_proportions_difference

test_proportions_difference (x1, n1, x2, n2)
test_proportions_difference (x1, n1, x2, n2, options ...)

Inferences on the difference of two proportions. Argument x1 is the number of successes in n1 trials in a Bernoulli experiment in the first population, and x2 and n2 are the corresponding values in the second population. Samples are independent and the test is asymptotic.

Options:

- 'alternative, default 'twosided, is the alternative hypothesis; valid values are: 'twosided (p1 ≠ p2), 'greater (p1 > p2) and 'less (p1 < p2).
- 'conflevel, default 95/100, confidence level for the confidence interval; it must be an expression which takes a value in (0,1).
- 'correct, default true, indicates whether Yates correction is applied or not.

The output of function test_proportions_difference is an inference_result Maxima object showing the following results:

1. 'proportions: list with the two sample proportions.
2. 'conf_level: confidence level selected by the user.
3. 'conf_interval: Confidence interval for the difference of proportions p1 - p2.
4. 'method: inference procedure and warning message in case of any of the samples sizes is less than 10.
5. 'hypotheses: null and alternative hypotheses to be tested.
6. 'statistic: value of the sample statistic used for testing the null hypothesis.
7. 'distribution: distribution of the sample statistic, together with its parameters.
8. 'p_value: p-value of the test.

Examples:

A machine produced 10 defective articles in a batch of 250. After some maintenance work, it produces 4 defective in a batch of 150. In order to know if the machine has improved, we test the null hypothesis H0:p1=p2, against the alternative H0:p1>p2, where p1 and p2 are the probabilities for one produced article to be defective before and after maintenance. According to the p value, there is not enough evidence to accept the alternative.

(%i1) load("stats")$
(%i2) fpprintprec:7$
(%i3) test_proportions_difference(10, 250, 4, 150, alternative = greater);

<table>
<thead>
<tr>
<th></th>
<th>statistic = .43689</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>distribution = [normal, 0.5, .048872]</td>
</tr>
<tr>
<td></td>
<td>p_value = .19662</td>
</tr>
</tbody>
</table>
proportions = [0.04, 0.2666667]
conf_level = 0.95
conf_interval = [-0.02172761, 1]

(%o3) method = Asymptotic test. Yates correction.
hypotheses = H0: p1 = p2, H1: p1 > p2
statistic = 0.0133333
distribution = [normal, 0, 0.01898069]
p_value = 0.2411936

Exact standard deviation of the asymptotic normal distribution when the data are unknown.

(%i1) load("stats")$
(%i2) stats_numer: false$
(%i3) sol: test_proportions_difference(x1, n1, x2, n2)$
(%i4) last(take_inference('distribution,sol));

1 1 x2 + x1
--- + --- (x2 + x1) (1 - ----)
n2 n1 n2 + n1

(%o4) sqrt(---------------------------------)
n2 + n1

Function test_sign

This is the non parametric sign test for the median of a continuous population. Argument x is a list or a column matrix containing a one dimensional sample.

Options:
- 'alternative, default 'twosided, is the alternative hypothesis; valid values are: 'twosided, 'greater and 'less.
- 'median, default 0, is the median value to be checked.

The output of function test_sign is an inference_result Maxima object showing the following results:
1. 'med_estimate: the sample median.
2. 'method: inference procedure.
3. 'hypotheses: null and alternative hypotheses to be tested.
4. 'statistic: value of the sample statistic used for testing the null hypothesis.
5. 'distribution: distribution of the sample statistic, together with its parameter(s).
6. 'p_value: p-value of the test.

Examples:
Checks whether the population from which the sample was taken has median 6, against the alternative $H_1: \text{median} > 6$.

```
(%i1) load("stats")$
(%i2) x: [2,0.1,7,1.8,4,2.3,5.6,7.4,5.1,6.1,6]$
(%i3) test_sign(x,'median=6,'alternative='greater);
   SIGN TEST
   med_estimate = 5.1
   method = Non parametric sign test.
   hypotheses = H0: median = 6 , H1: median > 6
   statistic = 7
   distribution = [binomial, 10, 0.5]
   p_value = .05468749999999989
```

```
test_signed_rank
   test_signed_rank (x)
   test_signed_rank (x, options ...)
This is the Wilcoxon signed rank test to make inferences about the median of a continuous population. Argument x is a list or a column matrix containing a one dimensional sample. Performs normal approximation if the sample size is greater than 20, or if there are zeroes or ties.

See also pdf_rank_test and cdf_rank_test
```

Options:
- 'median, default 0, is the median value to be checked.
- 'alternative, default 'twosided, is the alternative hypothesis; valid values are: 'twosided, 'greater and 'less.

The output of function test_signed_rank is an inference_result Maxima object with the following results:
1. 'med_estimate: the sample median.
2. 'method: inference procedure.
3. 'hypotheses: null and alternative hypotheses to be tested.
4. 'statistic: value of the sample statistic used for testing the null hypothesis.
5. 'distribution: distribution of the sample statistic, together with its parameter(s).
6. 'p_value: p-value of the test.

Examples:
Checks the null hypothesis $H_0: \text{median} = 15$ against the alternative $H_1: \text{median} > 15$. This is an exact test, since there are no ties.

(%i1) load("stats")$
(%i2) x: [17.1,15.9,13.7,13.4,15.5,17.6]$
(%i3) test_signed_rank(x,median=15,alternative=greater);

| SIGNED RANK TEST |
| med_estimate = 15.7 |
| method = Exact test |

(%o3) hypotheses = H0: med = 15 , H1: med > 15 |
| statistic = 14 |
| distribution = [signed_rank, 6] |
| p_value = 0.28125 |

Checks the null hypothesis $H_0: \text{equal}(\text{median}, 2.5)$ against the alternative $H_1: \text{notequal}(\text{median}, 2.5)$. This is an approximated test, since there are ties.

(%i1) load("stats")$
(%i2) y: [1.9,2.3,2.6,1.9,1.6,3.3,4.2,4,2.4,2.9,1.5,3,2.9,4.2,3.1]$%
(%i3) test_signed_rank(y,median=2.5);

| SIGNED RANK TEST |
| med_estimate = 2.9 |
| method = Asymptotic test. Ties |

(%o3) hypotheses = H0: med = 2.5 , H1: med # 2.5 |
| statistic = 76.5 |
| distribution = [normal, 60.5, 17.58195097251724] |
| p_value = .3628097734643669 |

test_rank_sum

This is the Wilcoxon-Mann-Whitney test for comparing the medians of two continuous populations. The first two arguments x1 and x2 are lists or column matrices with the data of two independent samples. Performs normal approximation if any of the sample sizes is greater than 10, or if there are ties.

Option:
• 'alternative, default 'twosided, is the alternative hypothesis; valid values are:
  'twosided, 'greater and 'less.

The output of function test_rank_sum is an inference_result Maxima object with the following results:
1. 'method: inference procedure.
2. 'hypotheses: null and alternative hypotheses to be tested.
3. 'statistic: value of the sample statistic used for testing the null hypothesis.
4. 'distribution: distribution of the sample statistic, together with its parameters.
5. 'p_value: p-value of the test.

Examples:
Checks whether populations have similar medians. Samples sizes are small and an exact test is made.

```
(%i1) load("stats")$
(%i2) x: [12,15,17,38,42,10,23,35,28]$  
(%i3) y: [21,18,25,14,52,65,40,43]$  
(%i4) test_rank_sum(x,y);
```

```
| RANK SUM TEST |
| method = Exact test |
| hypotheses = H0: med1 = med2 , H1: med1 ≠ med2 |
| statistic = 22 |
| distribution = [rank_sum, 9, 8] |
| p_value = .199588646474702 |
```

Now, with greater samples and ties, the procedure makes normal approximation. The alternative hypothesis is \( H_1 : \text{median1} < \text{median2} \).

```
(%i1) load("stats")$
(%i2) x: [39,42,35,13,10,23,15,20,17,27]$  
(%i3) y: [20,52,66,19,41,32,44,25,14,39,43,35,19,56,27,15]$  
(%i4) test_rank_sum(x,y,'alternative='less);
```

```
| RANK SUM TEST |
| method = Asymptotic test. Ties |
| hypotheses = H0: med1 = med2 , H1: med1 < med2 |
| statistic = 48.5 |
| distribution = [normal, 79.5, 18.95419580097078] |
| p_value = .05096985666598441 |
```
test_normality (x)  [Function]
Shapiro-Wilk test for normality. Argument x is a list of numbers, and sample size must be greater than 2 and less or equal than 5000, otherwise, function test_normality signals an error message.

Reference:

The output of function test_normality is an inference_result Maxima object with the following results:
1. 'statistic: value of the W statistic.
2. 'p_value: p-value under normal assumption.

Examples:
Checks for the normality of a population, based on a sample of size 9.
(%i1) load("stats")$
(%i2) x:[12,15,17,38,42,10,23,35,28]$
(%i3) test_normality(x);
    | SHAPIRO - WILK TEST
    |
    | statistic = .9251055695162436
    |
    | p_value = .4361763918860381

linear_regression  [Function]
linear_regression (x)
linear_regression (x option)
Multivariate linear regression, \( y_i = b_0 + b_1*x_{1i} + b_2*x_{2i} + \ldots + b_k*x_{ki} + u_i \), where \( u_i \) are \( N(0, \sigma) \) independent random variables. Argument x must be a matrix with more than one column. The last column is considered as the responses \( (y_i) \).

Option:
- 'conflevel, default 95/100, confidence level for the confidence intervals; it must be an expression which takes a value in (0,1).

The output of function linear_regression is an inference_result Maxima object with the following results:
1. 'b_estimation: regression coefficients estimates.
2. 'b_covariances: covariance matrix of the regression coefficients estimates.
3. b_conf_int: confidence intervals of the regression coefficients.
4. b_statistics: statistics for testing coefficient.
5. b_p_values: p-values for coefficient tests.
6. b_distribution: probability distribution for coefficient tests.
7. v_estimation: unbiased variance estimator.
8. v_conf_int: variance confidence interval.
10. residuals: residuals.
11. **adc**: adjusted determination coefficient.

12. **aic**: Akaike’s information criterion.

13. **bic**: Bayes’s information criterion.

Only items 1, 4, 5, 6, 7, 8, 9 and 11 above, in this order, are shown by default. The rest remain hidden until the user makes use of functions `items_inference` and `take_inference`.

Example:

Fitting a linear model to a trivariate sample. The last column is considered as the responses ($y_i$).

```lisp
(%i2) load("stats")
(%i3) X:matrix(
   [58,111,64],
   [84,131,78],
   [78,158,83],
   [81,147,88],
   [82,121,89],
   [102,165,99],
   [85,174,101],
   [102,169,102])
(%i4) fpprintprec: 4$
(%i5) res: linear_regression(X);  
   LINEAR REGRESSION MODEL  
   | b_estimation = [9.054, .5203, .2397]  
   | b_statistics = [.6051, 2.246, 1.74]  
   | b_p_values = [.5715, .07466, .1423]  
   | b_distribution = [student_t, 5]  
   | v_estimation = 35.27  
   | v_conf_int = [13.74, 212.2]  
   | v_distribution = [chi2, 5]  
   | adc = .7922
(%i6) items_inference(res);
(%o6) [b_estimation, b_covariances, b_conf_int, b_statistics,  
   b_p_values, b_distribution, v_estimation, v_conf_int,  
   v_distribution, residuals, adc, aic, bic]
(%i7) take_inference('b_covariances, res);  
   [ 223.9  - 1.12  - .8532 ]  
   [                          ]  
   [ - 1.12  .05367  - .02305 ]  
   [                          ]  
   [ - .8532  - .02305  .01898 ]
(%i8) take_inference('bic, res);  
   30.98
```
82.4 Functions and Variables for special distributions

\textbf{pdf\_signed\_rank (x, n)} \quad \text{[Function]}

Probability density function of the exact distribution of the signed rank statistic. Argument \(x\) is a real number and \(n\) a positive integer.

See also \texttt{test\_signed\_rank}.

\textbf{cdf\_signed\_rank (x, n)} \quad \text{[Function]}

Cumulative density function of the exact distribution of the signed rank statistic. Argument \(x\) is a real number and \(n\) a positive integer.

See also \texttt{test\_signed\_rank}.

\textbf{pdf\_rank\_sum (x, n, m)} \quad \text{[Function]}

Probability density function of the exact distribution of the rank sum statistic. Argument \(x\) is a real number and \(n\) and \(m\) are both positive integers.

See also \texttt{test\_rank\_sum}.

\textbf{cdf\_rank\_sum (x, n, m)} \quad \text{[Function]}

Cumulative density function of the exact distribution of the rank sum statistic. Argument \(x\) is a real number and \(n\) and \(m\) are both positive integers.

See also \texttt{test\_rank\_sum}.
83 stirling

83.1 Functions and Variables for stirling

stirling [Function]

stirling (z,n)
stirling (z,n,pred)

Replace \( \gamma(x) \) with the \( O(1/x^{2n-1}) \) Stirling formula. When \( n \) isn't a nonnegative integer, signal an error. With the optional third argument \( \text{pred} \), the Stirling formula is applied only when \( \text{pred} \) is true.


Examples:

```
(%i1) load ("stirling")$

(%i2) stirling(gamma(%alpha+x)/gamma(x),1);
   1/2 - x       x + %alpha - 1/2
(%o2) x (x + %alpha)
         1  1
      ------------------ - ---- - %alpha
         12 (x + %alpha) 12 x
%e

(%i3) taylor(%,x,inf,1);
  %alpha 2 %alpha
(%o3) T/ x + ------------------------ + . . .
    2 x

(%i4) map('factor,%);
  %alpha - 1
(%o4) x + ---------------------
     2

The function \text{stirling} \ knows the difference between the variable 'gamma' and the function gamma:

```
(%i5) stirling(gamma + gamma(x),0);
   x - 1/2 - x
(%o5) gamma + sqrt(2) sqrt(%pi) x %e

(%i6) stirling(gamma(y) + gamma(x),0);
   y - 1/2 - y
(%o6) sqrt(2) sqrt(%pi) y %e
      x - 1/2 - x
      + sqrt(2) sqrt(%pi) x %e

To apply the Stirling formula only to terms that involve the variable \( k \), use an optional third argument; for example

```
(%i7) makegamma(pochhammer(a,k)/pochhammer(b,k));
```
\begin{verbatim}
(\%o7) \((\text{gamma}(b) \ast \text{gamma}(k+a))/(\text{gamma}(a) \ast \text{gamma}(k+b))\)
(\%i8) \text{stirling}(\%, 1, \lambda([s], \text{not(freeof}(k, s))));
(\%o8) \((\text{e}^{-b-a} \ast \text{gamma}(b) \ast (k+a)^{(k+a-1/2)} \ast (k+b)^{(-k-b+1/2)})/\text{gamma}(a)\)
\end{verbatim}

The terms \text{gamma}(a) and \text{gamma}(b) are free of k, so the Stirling formula was not
applied to these two terms.

To use this function write first \text{load("stirling")}. 
84 stringproc

84.1 Introduction to String Processing

The package stringproc contains functions for processing strings and characters including formatting, encoding and data streams. This package is completed by some tools for cryptography, e.g. base64 and hash functions.

It can be directly loaded via `load(stringproc)` or automatically by using one of its functions.

For questions and bug reports please contact the author. The following command prints his e-mail-address.

\[
\text{printf(true, } "\{-a\}@gmail.com", \text{split(sdowncase("Volker van Nek")))}$
\]

A string is constructed by typing e.g. "Text". When the option variable stringdisp is set to `false`, which is the default, the double quotes won’t be printed. [stringp], page 1092, is a test, if an object is a string.

\[
\begin{align*}
(\%i1) & \quad \text{str: } "\text{Text}"; \\
(\%o1) & \quad \text{Text} \\
(\%i2) & \quad \text{stringp(str)}; \\
(\%o2) & \quad \text{true}
\end{align*}
\]

Characters are represented by a string of length 1. [charp], page 1084, is the corresponding test.

\[
\begin{align*}
(\%i1) & \quad \text{char: } "\text{e}"; \\
(\%o1) & \quad \text{e} \\
(\%i2) & \quad \text{charp(char)}; \\
(\%o2) & \quad \text{true}
\end{align*}
\]

In Maxima position indices in strings are like in list 1-indexed which results to the following consistency.

\[
\begin{align*}
(\%i1) & \quad \text{is(charat("Lisp",1) = charlist("Lisp")}[1]); \\
(\%o1) & \quad \text{true}
\end{align*}
\]

A string may contain Maxima expressions. These can be parsed with [parse_string], page 1088.

\[
\begin{align*}
(\%i1) & \quad \text{map(parse_string, ["42", } "\text{sqrt(2)}", "\%pi"]); \\
(\%o1) & \quad \{42, \text{sqrt(2)}, \%pi\} \\
(\%i2) & \quad \text{map('float, %)}; \\
(\%o2) & \quad \{42.0, 1.414213562373095, 3.141592653589793\}
\end{align*}
\]

Strings can be processed as characters or in binary form as octets. Functions for conversions are [string_to_octets], page 1097, and [octets_to_string], page 1095. Usable encodings depend on the platform, the application and the underlying Lisp. (The following shows Maxima in GNU/Linux, compiled with SBCL.)

\[
\begin{align*}
(\%i1) & \quad \text{obase: 16.$} \\
(\%i2) & \quad \text{string_to_octets("$\text{£e}$", } "\text{cp1252}"); \\
(\%o2) & \quad \{24, 0A3, 80\} \\
(\%i3) & \quad \text{string_to_octets("$\text{£e}$", } "\text{utf-8}");
\end{align*}
\]
Strings may be written to character streams or as octets to binary streams. The following example demonstrates file in and output of characters.

[openw], page 1078, returns an output stream to a file, [printf], page 1079, writes formatted to that file and by e.g. [close], page 1076, all characters contained in the stream are written to the file.

```
(%i1) s: openw("file.txt");
(%o1) #<output stream file.txt>
(%i2) printf(s, "~%~d ~f ~a ~f ~e ~a~%", 42, 1.234, sqrt(2), %pi, 1.0e-2, 1.0e-2, 1.0b-2)$
(%i3) close(s)$
```

[openr], page 1078, then returns an input stream from the previously used file and [readline], page 1081, returns the line read as a string. The string may be tokenized by e.g. [split], page 1089, or [tokens], page 1092, and finally parsed by [parse_string], page 1088.

```
(%i4) s: openr("file.txt");
(%o4) #<input stream file.txt>
(%i5) readline(s);
(%o5) 42 1.234 sqrt(2) %pi 0.01 1.0E-2 1.0b-2
(%i6) map(parse_string, split(%));
(%o6) [42, 1.234, sqrt(2), %pi, 0.01, 0.01, 1.0b-2]
(%i7) close(s)$
```

84.2 Input and Output

Example: Formatted printing to a file.

```
(%i1) s: openw("file.txt");
(%o1) #<output stream file.txt>
(%i2) control: "~2tAn atom: ~20t~a~%~2t~and a list: ~20t~{~r ~}~%~2t
and an integer: ~20t~d~%$"$
(%i3) printf( s,control, 'true,[1,2,3],42 )$
(%o3) false
(%i4) close(s);
(%o4) true
(%i5) s: openr("file.txt");
(%o5) #<input stream file.txt>
(%i6) while stringp( tmp:readline(s) ) do print(tmp)$
   An atom: true
   and a list: one two three
   and an integer: 42
(%i7) close(s)$
```

`close (stream)`

Closes `stream` and returns `true` if `stream` had been open.
flength (stream) [Function]

stream has to be an open stream from or to a file. flength then returns the number
of bytes which are currently present in this file.
Example: See [writebyte], page 1082, .

flush_output (stream) [Function]

Flushes stream where stream has to be an output stream to a file.
Example: See [writebyte], page 1082, .

fposition [Function]

fposition (stream)
fposition (stream, pos)

Returns the current position in stream, if pos is not used. If pos is used, fposition
sets the position in stream. stream has to be a stream from or to a file and pos has
to be a positive number.
Positions in data streams are like in strings or lists 1-indexed, i.e. the first element
in stream is in position 1.

freshline [Function]

freshline ()
freshline (stream)

Writes a new line to the standard output stream if the position is not at the beginning
of a line und returns true. Using the optional argument stream the new line is written
to that stream. There are some cases, where freshline() does not work as expected.
See also [newline], page 1078.

get_output_stream_string (stream) [Function]

Returns a string containing all the characters currently present in stream which must
be an open string-output stream. The returned characters are removed from stream.
Example: See [make_string_output_stream], page 1077, .

make_string_input_stream [Function]

make_string_input_stream (string)
make_string_input_stream (string, start)
make_string_input_stream (string, start, end)

Returns an input stream which contains parts of string and an end of file. Without
optional arguments the stream contains the entire string and is positioned in front of
the first character. start and end define the substring contained in the stream. The
first character is available at position 1.

(%i1) istream : make_string_input_stream("text", 1, 4);
(%o1)  #<string-input stream from "text">
(%i2) (while (c : readchar(istream)) # false do sprint(c), newline())$
   t e x$
(%i3) close(istream)$

make_string_output_stream () [Function]

Returns an output stream that accepts characters. Characters currently present in
this stream can be retrieved by [get_output_stream_string], page 1077.

(%i11) ostream : make_string_output_stream();
(\%o1) #\<string-output stream 09622ea0>
(\%i2) printf(ostream, "foo")$

(\%i3) printf(ostream, "bar")$

(\%i4) string : get_output_stream_string(ostream);
(\%o4) foobar

(\%i5) printf(ostream, "baz")$

(\%i6) string : get_output_stream_string(ostream);
(\%o6) baz

(\%i7) close(ostream)$

newline [Function]

newline ()
newline (stream)

Writes a new line to the standard output stream. Using the optional argument stream the new line is written to that stream. There are some cases, where newline() does not work as expected.

See [sprint], page 1081, for an example of using newline().

opena (file) [Function]

Returns a character output stream to file. If an existing file is opened, opena appends elements at the end of file.

For binary output see Section 73.3 [opena_binary], page 1010,.

openr [Function]

openr (file)
openr (file, encoding)

Returns a character input stream to file. openr assumes that file already exists. If reading the file results in a lisp error about its encoding passing the correct string as the argument encoding might help. The available encodings and their names depend on the lisp being used. For sbcl a list of suitable strings can be found at http://www.sbcl.org/manual/#External-Formats.

For binary input see Section 73.3 [openr_binary], page 1010,. See also close and openw.

(\%i1) istream : openr("data.txt","EUC-JP");
(\%o1) #\<FD-STREAM for "file /home/gunter/data.txt" {10099A3AE3}>
(\%i2) close(istream);
(\%o2) true

openw (file) [Function]

Returns a character output stream to file. If file does not exist, it will be created. If an existing file is opened, openw destructively modifies file.

For binary output see Section 73.3 [openw_binary], page 1010,

See also close and openr.
printf  [Function]

printf (dest, string)
printf (dest, string, expr_1, ..., expr_n)

Produces formatted output by outputting the characters of control-string string and observing that a tilde introduces a directive. The character after the tilde, possibly preceded by prefix parameters and modifiers, specifies what kind of formatting is desired. Most directives use one or more elements of the arguments expr_1, ..., expr_n to create their output.

If dest is a stream or true, then printf returns false. Otherwise, printf returns a string containing the output. By default the streams stdin, stdout and stderr are defined. If maxima is running as a server (which is the normal case if maxima communicating with a graphical user interface) setup-client will define old_stdout and old_stderr, too.

printf provides the Common Lisp function format in Maxima. The following example illustrates the general relation between these two functions.

(%i1) printf(true, "R~dD~d~%", 2, 2);
R2D2
(%o1) false
(%i2) :lisp (format t "R~dD~d~%" 2 2)
R2D2
NIL

The following description is limited to a rough sketch of the possibilities of printf. The Lisp function format is described in detail in many reference books. Of good help is e.g. the free available online-manual "Common Lisp the Language" by Guy L. Steele. See chapter 22.3.3 there.

~%  new line
~&  fresh line
~t  tab
~$  monetary
~d  decimal integer
~b  binary integer
~o  octal integer
~x  hexadecimal integer
~br base-b integer
~r  spell an integer
~p  plural
~f  floating point
~e  scientific notation
~g  ~f or ~e, depending upon magnitude
~h  bigfloat
~a  uses Maxima function string
~s  like ~a, but output enclosed in "double quotes"
~~  ~
~<  justification, ~> terminates
~(  case conversion, ~) terminates
~[  selection, ~] terminates
"{ iteration, ~} terminates

The directive "h for bigfloat is no Lisp-standard and is therefore illustrated below. Note that the directive "* is not supported.

If dest is a stream or true, then printf returns false. Otherwise, printf returns a string containing the output.

```lisp
(%i1) printf( false, "~a ~a ~4f ~a ~@r", "String", sym, bound, sqrt(12), 144), bound = 1.234;
String sym 1.23 2*sqrt(3) CXLIV
(%i2) printf( false,"~-\{~a ~\}~",["one",2,"THREE"] );
one 2 THREE
(%i3) printf(true,"~-\{~9,1f ~}%~",mat ),
   mat = args(matrix([1.1,2,3.33],[4,5,6],[7,8.88,9]))$
   1.1 2.0 3.3
   4.0 5.0 6.0
   7.0 8.9 9.0
(%i4) control: "~:(~r~) bird~p ~[is~;are~] singing."
(%i5) printf( false,control, n,n,if n=1 then 1 else 2 ), n=2;
Two birds are singing.
```

The directive "h has been introduced to handle bigfloats.

```
~w,d,e,x,o,p@H
w : width
d : decimal digits behind floating point
e : minimal exponent digits
x : preferred exponent
o : overflow character
p : padding character
@ : display sign for positive numbers
```

```lisp
(%i1) fpprec : 1000$
(%i2) printf(true, "|~h|~", 2.0b^-64)$
|0.0000000000000000000542101086242752217003726400434970855712890625|
(%i3) fpprec : 26$
(%i4) printf(true, "|~h|~", sqrt(2))$
|1.4142135623730950488016887|
(%i5) fpprec : 24$
(%i6) printf(true, "|~h|~", sqrt(2))$
|1.41421356237309504880169|
(%i7) printf(true, "|~28h|~", sqrt(2))$
|1.41421356237309504880169|
(%i8) printf(true, "|~28,,,,,*h|~", sqrt(2))$
|***1.41421356237309504880169|
(%i9) printf(true, "|~,18h|~", sqrt(2))$
|1.414213562373095049|
(%i10) printf(true, "|~,3h|~", sqrt(2))$
|1.41421356237309504880169b-3|
(%i11) printf(true, "|~,2,3h|~", sqrt(2))$
```
readbyte (stream)  
Removes and returns the first byte in stream which must be a binary input stream.  
If the end of file is encountered readbyte returns false.

Example: Read the first 16 bytes from a file encrypted with AES in OpenSSL.

(%i1) ibase: obase: 16.$

(%i2) in: openr_binary("msg.bin");
(%o2) #<input stream msg.bin>
(%i3) (L: [], thru 16. do push(readbyte(in), L), L:reverse(L));
(%o3) [53, 61, 6C, 74, 65, 64, 5F, 5F, 88, 56, 0DE, 8A, 74, 0FD, 0AD, 0F0]
(%i4) close(in);
(%o4) true
(%i5) map(ascii, rest(L,-8));
(%o5) [S, a, l, t, e, d_, _, _]
(%i6) salt: octets_to_number(rest(L,8));
(%o6) 8856de8a74fdadf0

readchar (stream)  
Removes and returns the first character in stream. If the end of file is encountered readchar returns false.

Example: See [make_string_input_stream], page 1077.

readline (stream)  
Returns a string containing all characters starting at the current position in stream up to the end of the line or false if the end of the file is encountered.

sprint (expr_1, ..., expr_n)  
Evaluates and displays its arguments one after the other ‘on a line’ starting at the leftmost position. The expressions are printed with a space character right next to the number, and it disregards line length. newline() might be used for line breaking.

Example: Sequential printing with sprint. Creating a new line with newline().

(%i1) for n:0 thru 19 do sprint(fib(n))$
 0 1 1 2 3 5 8 13 21 34 55 89 144 233 377 610 987 1597 2584 4181
(%i2) for n:0 thru 22 do ($
  sprint(fib(n)),
    if mod(n,10) = 9 then newline() )$
 0 1 1 2 3 5 8 13 21 34 55 89 144 233 377 610 987 1597 2584 4181
  6765 10946 17711
writebyte (byte, stream)  [Function]

Writes byte to stream which must be a binary output stream. writebyte returns byte.

Example: Write some bytes to a binary file output stream. In this example all bytes correspond to printable characters and are printed by printfile. The bytes remain in the stream until flush_output or close have been called.

(%i1) ibase: obase: 16.$
(%i2) bytes: map(cint, charlist("GNU/Linux"));
(%o2) [47, 4E, 55, 2F, 4C, 69, 6E, 75, 78]
(%i3) out: openw_binary("test.bin");
(%o3) #<output stream test.bin>
(%i4) for i thru 3 do writebyte(bytes[i], out); done
(%i5) printfile("test.bin")$
(%i6) flength(out);
(%o6) 0
(%i7) flush_output(out);
(%o7) true
(%i8) flength(out);
(%o8) 3
(%i9) printfile("test.bin")$ GNU
(%i10) for b in rest(bytes,3) do writebyte(b, out); done
(%i10) close(out);
(%i10) true
(%i10) printfile("test.bin")$
GNU/Linux

84.3 Characters

Characters are strings of length 1.

adjust_external_format ()  [Function]

Prints information about the current external format of the Lisp reader and in case the external format encoding differs from the encoding of the application which runs Maxima adjust_external_format tries to adjust the encoding or prints some help or instruction. adjust_external_format returns true when the external format has been changed and false otherwise.

Functions like [cint], page 1084, [unicode], page 1085, [octets_to_string], page 1095, and [string_to_octets], page 1097, need UTF-8 as the external format of the Lisp reader to work properly over the full range of Unicode characters.

Examples (Maxima on Windows, March 2016): Using adjust_external_format when the default external format is not equal to the encoding provided by the application.
1. Command line Maxima

In case a terminal session is preferred it is recommended to use Maxima compiled with SBCL. Here Unicode support is provided by default and calls to `adjust_external_format` are unnecessary.

If Maxima is compiled with CLISP or GCL it is recommended to change the terminal encoding from CP850 to CP1252. `adjust_external_format` prints some help.

CCL reads UTF-8 while the terminal input is CP850 by default. CP1252 is not supported by CCL. `adjust_external_format` prints instructions for changing the terminal encoding and external format both to iso-8859-1.

2. wxMaxima

In wxMaxima SBCL reads CP1252 by default but the input from the application is UTF-8 encoded. Adjustment is needed.

Calling `adjust_external_format` and restarting Maxima permanently changes the default external format to UTF-8.

```lisp
(%i1) adjust_external_format();
```

The line
```lisp
(setf sb-impl::*default-external-format* :utf-8)
```
has been appended to the init file
```lisp
C:/Users/Username/.sbclrc
```
Please restart Maxima to set the external format to UTF-8.

```lisp
(%i1) false
```

Restarting Maxima.

```lisp
(%i1) adjust_external_format();
```

The external format is currently UTF-8 and has not been changed.

```lisp
(%i1) false
```

```
alphacharp (char)
```

Returns true if char is an alphabetic character.

To identify a non-US-ASCII character as an alphabetic character the underlying Lisp must provide full Unicode support. E.g. a German umlaut is detected as an alphabetic character with SBCL in GNU/Linux but not with GCL. (In Windows Maxima, when compiled with SBCL, must be set to UTF-8. See [adjust_external_format], page 1082, for more.)

Example: Examination of non-US-ASCII characters.

The underlying Lisp (SBCL, GNU/Linux) is able to convert the typed character into a Lisp character and to examine.

```lisp
(%i1) alphacharp("ü");
```

```lisp
(%o1) true
```

In GCL this is not possible. An error break occurs.

```lisp
(%i1) alphacharp("u");
```

```lisp
(%o1) true
```

```lisp
(%i2) alphacharp("ü");
```
package stringproc: ü cannot be converted into a Lisp character.
-- an error.

alphanumericp (char) [Function]
Returns true if char is an alphabetic character or a digit (only corresponding US-
ASCII characters are regarded as digits).

Note: See remarks on [alphacharp], page 1083.

ascii (int) [Function]
Returns the US-ASCII character corresponding to the integer int which has to be less
than 128.

See [unicode], page 1085, for converting code points larger than 127.

Examples:
(%i1) for n from 0 thru 127 do (
    ch: ascii(n),
    if alphacharp(ch) then sprint(ch),
    if n = 96 then newline() )$
A B C D E F G H I J K L M N O P Q R S T U V W X Y Z
a b c d e f g h i j k l m n o p q r s t u v w x y z

cequal (char_1, char_2) [Function]
Returns true if char_1 and char_2 are the same character.

cequalignore (char_1, char_2) [Function]
Like cequal but ignores case which is only possible for non-US-ASCII characters
when the underlying Lisp is able to recognize a character as an alphabetic character.

See remarks on [alphacharp], page 1083.

cgreaterp (char_1, char_2) [Function]
Returns true if the code point of char_1 is greater than the code point of char_2.

cgreaterpignore (char_1, char_2) [Function]
Like cgreaterp but ignores case which is only possible for non-US-ASCII characters
when the underlying Lisp is able to recognize a character as an alphabetic character.

See remarks on [alphacharp], page 1083.

charp (obj) [Function]
Returns true if obj is a Maxima-character. See introduction for example.

cint (char) [Function]
Returns the Unicode code point of char which must be a Maxima character, i.e. a
string of length 1.

Examples: The hexadecimal code point of some characters (Maxima with SBCL on
GNU/Linux).
(%i1) obase: 16.$
(%i2) map(cint, ["$","£","€"]);
(%o2) [24, 0A3, 20AC]
Warning: It is not possible to enter characters corresponding to code points larger than 16 bit in wxMaxima with SBCL on Windows when the external format has not been set to UTF-8. See [adjust_external_format], page 1082.

CMUCL doesn’t process these characters as one character. cint then returns false. Converting a character to a code point via UTF-8-octets may serve as a workaround:

```
utf8_to_unicode(string_to_octets(character));
```

See [utf8_to_unicode], page 1087, [string_to_octets], page 1097.

**Function clessp (char_1, char_2)**

Returns true if the code point of char_1 is less than the code point of char_2.

**Function clesspignore (char_1, char_2)**

Like clessp but ignores case which is only possible for non-US-ASCII characters when the underlying Lisp is able to recognize a character as an alphabetic character. See remarks on [alphacharp], page 1083.

**Function constituent (char)**

Returns true if char is a graphic character but not a space character. A graphic character is a character one can see, plus the space character. (constituent is defined by Paul Graham. See Paul Graham, ANSI Common Lisp, 1996, page 67.)

```
(%i1) for n from 0 thru 255 do (tmp: ascii(n), if constituent(tmp) then sprint(tmp))$
```

**Function digitcharp (char)**

Returns true if char is a digit where only the corresponding US-ASCII-character is regarded as a digit.

**Function lowercasep (char)**

Returns true if char is a lowercase character.

Note: See remarks on [alphacharp], page 1083.

**Variable newline**

The newline character (ASCII-character 10).

**Variable space**

The space character.

**Variable tab**

The tab character.

**Function unicode (arg)**

Returns the character defined by arg which might be a Unicode code point or a name string if the underlying Lisp provides full Unicode support.

Example: Characters defined by hexadecimal code points (Maxima with SBCL on GNU/Linux).

```
(%i1) ibase: 16.$
```
Warning: In wxMaxima with SBCL on Windows it is not possible to convert code points larger than 16 bit to characters when the external format has not been set to UTF-8. See [adjust_external_format], page 1082, for more information.

CMUCL doesn’t process code points larger than 16 bit. In these cases unicode returns false. Converting a code point to a character via UTF-8 octets may serve as a workaround:

```
octets_to_string(unicode_to_utf8(code_point));
```

See [octets_to_string], page 1095, [unicode_to_utf8], page 1086.

In case the underlying Lisp provides full Unicode support the character might be specified by its name. The following is possible in ECL, CLISP and SBCL, where in SBCL on Windows the external format has to be set to UTF-8. unicode(name) is supported by CMUCL too but again limited to 16 bit characters.

The string argument to unicode is basically the same string returned by printf using the "~@c" specifier. But as shown below the prefix "#" must be omitted. Underlines might be replaced by spaces and uppercase letters by lowercase ones.

Example (continued): Characters defined by names (Maxima with SBCL on GNU/Linux).

```
(%i3) printf(false, "~@c", unicode(0DF));
(%o3) #\LATIN_SMALL_LETTER_SHARP_S
(%i4) unicode("LATIN_SMALL_LETTER_SHARP_S");
(%o4) ß
(%i5) unicode("Latin small letter sharp s");
(%o5) ß
```

**unicode_to_utf8 (code_point)**

Returns a list containing the UTF-8 code corresponding to the Unicode code_point.

Examples: Converting Unicode code points to UTF-8 and vice versa.

```
(%i1) ibase: obase: 16.$
(%i2) map(cint, ["$","£","€"]);
(%o2) [24, 0A3, 20AC]
(%i3) map(unicode_to_utf8, %);
(%o3) [[24], [0C2, 0A3], [0E2, 82, 0AC]]
(%i4) map(utf8_to_unicode, %);
(%o4) [24, 0A3, 20AC]
```

**uppercasep (char)**

Returns true if char is an uppercase character.

Note: See remarks on [alphacharp], page 1083.

**us_ascii_only**

This option variable affects Maxima when the character encoding provided by the application which runs Maxima is UTF-8 but the external format of the Lisp reader is not equal to UTF-8.
On GNU/Linux this is true when Maxima is built with GCL and on Windows in wxMaxima with GCL- and SBCL-builds. With SBCL it is recommended to change the external format to UTF-8. Setting `us_ascii_only` is unnecessary then. See [adjust external format], page 1082, for details.

`us_ascii_only` is false by default. Maxima itself then (i.e. in the above described situation) parses the UTF-8 encoding.

When `us_ascii_only` is set to true it is assumed that all strings used as arguments to string processing functions do not contain Non-US-ASCII characters. Given that promise, Maxima avoids parsing UTF-8 and strings can be processed more efficiently.

### utf8_to_unicode (list)

[Function]

Returns a Unicode code point corresponding to the list which must contain the UTF-8 encoding of a single character.

Examples: See [unicode_to_utf8], page 1086.

### 84.4 String Processing

Position indices in strings are 1-indexed like in Maxima lists. See example in [charat], page 1087.

#### charat (string, n)

[Function]

Returns the n-th character of string. The first character in string is returned with n = 1.

```
(%i1) charat("Lisp",1);
(%o1) L
(%i2) charlist("Lisp")[1];
(%o2) L
```

#### charlist (string)

[Function]

Returns the list of all characters in string.

```
(%i1) charlist("Lisp");
(%o1) [L, i, s, p]
```

#### eval_string (str)

[Function]

Parse the string str as a Maxima expression and evaluate it. The string str may or may not have a terminator (dollar sign $ or semicolon ;). Only the first expression is parsed and evaluated, if there is more than one.

Complain if str is not a string.

Examples:

```
(%i1) eval_string ("foo: 42; bar: foo^2 + baz");
(%o1) 42
(%i2) eval_string ("(foo: 42, bar: foo^2 + baz)" );
(%o2) baz + 1764
```

See also [parse_string], page 1088.
parse_string (str)  [Function]
Parse the string str as a Maxima expression (do not evaluate it). The string str may or may not have a terminator (dollar sign $ or semicolon ;). Only the first expression is parsed, if there is more than one.
Complain if str is not a string.
Examples:
  (%i1) parse_string ("foo: 42; bar: foo^2 + baz");
  (%o1) foo : 42
  (%i2) parse_string ("(foo: 42, bar: foo^2 + baz)");
  (%o2) (foo : 42, bar : foo + baz)
See also [eval_string], page 1087.

scopy (string)  [Function]
Returns a copy of string as a new string.

sdowncase  [Function]
  sdowncase (string)
  sdowncase (string, start)
  sdowncase (string, start, end)
Like [supcase], page 1092, but uppercase characters are converted to lowercase.

sequal (string_1, string_2)  [Function]
Returns true if string_1 and string_2 contain the same sequence of characters.

sequalignore (string_1, string_2)  [Function]
Like sequal but ignores case which is only possible for non-US-ASCII characters when the underlying Lisp is able to recognize a character as an alphabetic character. See remarks on [alphacharp], page 1083.

sexplode (string)  [Function]
sexplode is an alias for function charlist.

simploade (list)  [Function]
simploade (list, delim)
simploade takes a list of expressions and concatenates them into a string. If no delimiter delim is specified, simploade uses no delimiter. delim can be any string.
Examples:
  (%i1) simploade(["xx",3,""],expand((x+y)^3));
  (%o1) xx[3]:y^3+3*x*y^2+3*x^2*y+x^3
  (%i2) simploade( sexplode("stars")," * ");
  (%o2) s * t * a * r * s
  (%i3) simploade( ["One","more","coffee."]," ");
  (%o3) One more coffee.
**sinsert (seq, string, pos)**

- **Function**
- Returns a string that is a concatenation of `substring(string, 1, pos-1)`, the string `seq` and `substring (string, pos)`. Note that the first character in `string` is in position 1.

Examples:

```
(%i1) s: "A submarine."$
(%i2) concat( substring(s,1,3),"yellow ",substring(s,3) );
     (%o2) A yellow submarine.
(%i3) sinsert("hollow ",s,3);
     (%o3) A hollow submarine.
```

**sinvertcase**

- **Function**
- `sinvertcase (string)`
- `sinvertcase (string, start)`
- `sinvertcase (string, start, end)`

Returns `string` except that each character from position `start` to `end` is inverted. If `end` is not given, all characters from `start` to the end of `string` are replaced.

Examples:

```
(%i1) sinvertcase("sInvertCase");
          (%o1) SiNVERTcASE
```

**slength (string)**

- **Function**
- Returns the number of characters in `string`.

**smake (num, char)**

- **Function**
- Returns a new string with a number of `num` characters `char`.

Example:

```
(%i1) smake(3,"w");
          (%o1) www
```

**smismatch (string_1, string_2)**

- **Function**
- `smismatch (string_1, string_2)`
- `smismatch (string_1, string_2, test)`

Returns the position of the first character of `string_1` at which `string_1` and `string_2` differ or `false`. Default test function for matching is `sequal`. If `smismatch` should ignore case, use `sequalignore` as test.

Example:

```
(%i1) smismatch("seven","seventh");
          (%o1) 6
```

**split (string)**

- **Function**
- `split (string)`
- `split (string, delim)`
- `split (string, delim, multiple)`

Returns the list of all tokens in `string`. Each token is an unparsed string. `split` uses `delim` as delimiter. If `delim` is not given, the space character is the default delimiter. `multiple` is a boolean variable with `true` by default. Multiple delimiters are read as
one. This is useful if tabs are saved as multiple space characters. If \textit{multiple} is set to \texttt{false}, each delimiter is noted.

Examples:

\begin{verbatim}
(%i1) split("1.2  2.3  3.4  4.5");
(%o1) [1.2, 2.3, 3.4, 4.5]
(%i2) split("first;;third;fourth",";",false);
(%o2) [first, , third, fourth]
\end{verbatim}

\texttt{sposition \char\ (char, string)}

Returns the position of the first character in \texttt{string} which matches \texttt{char}. The first character in \texttt{string} is in position 1. For matching characters ignoring case see \texttt{ssearch}, page 1090.

\texttt{sremove}

\begin{verbatim}
sremove (seq, string) 
sremove (seq, string, test) 
sremove (seq, string, test, start) 
sremove (seq, string, test, start, end)
\end{verbatim}

Returns a string like \texttt{string} but without all substrings matching \texttt{seq}. Default test function for matching is \texttt{sequal}. If \texttt{sremove} should ignore case while searching for \texttt{seq}, use \texttt{sequalignore} as test. Use \texttt{start} and \texttt{end} to limit searching. Note that the first character in \texttt{string} is in position 1.

Examples:

\begin{verbatim}
(%i1) sremove("n't","I don't like coffee.");
(%o1) I do like coffee.
(%i2) sremove ("DO ",",",'sequalignore);
(%o2) I like coffee.
\end{verbatim}

\texttt{sremovefirst}

\begin{verbatim}
sremovefirst (seq, string) 
sremovefirst (seq, string, test) 
sremovefirst (seq, string, test, start) 
sremovefirst (seq, string, test, start, end)
\end{verbatim}

Like \texttt{sremove} except that only the first substring that matches \texttt{seq} is removed.

\texttt{sreverse (string)}

Returns a string with all the characters of \texttt{string} in reverse order.

\texttt{ssearch}

\begin{verbatim}
ssearch (seq, string) 
ssearch (seq, string, test) 
ssearch (seq, string, test, start) 
ssearch (seq, string, test, start, end)
\end{verbatim}

Returns the position of the first substring of \texttt{string} that matches the string \texttt{seq}. Default test function for matching is \texttt{sequal}. If \texttt{ssearch} should ignore case, use \texttt{sequalignore} as test. Use \texttt{start} and \texttt{end} to limit searching. Note that the first character in \texttt{string} is in position 1.
Chapter 84: stringproc

Example:

```lisp
(%i1) ssearch("-s","-{S }-%","sequalignore");
(%o1) 4
```

**ssort**

```
ssort (string)
ssort (string, test)
```

Returns a string that contains all characters from `string` in an order such there are no two successive characters `c` and `d` such that `test (c, d)` is false and `test (d, c)` is true. Default test function for sorting is `clessp`. The set of test functions is `{clessp, clesspignore, cgreaterp, cgreaterpignore, cequal, cequalignore}`.

Examples:

```lisp
(%i1) ssort("I don't like Mondays.");
(%o1) '.IMaddeikInnoosty
(%i2) ssort("I don't like Mondays.",'cgreaterpignore);
(%o2) ytsoonnMlkIiedda.'
```

**ssubst**

```
ssubst (new, old, string)
ssubst (new, old, string, test)
ssubst (new, old, string, test, start)
ssubst (new, old, string, test, start, end)
```

Returns a string like `string` except that all substrings matching `old` are replaced by `new`. `old` and `new` need not to be of the same length. Default test function for matching is `sequal`. If `ssubst` should ignore case while searching for `old`, use `sequalignore` as test. Use `start` and `end` to limit searching. Note that the first character in `string` is in position 1.

Examples:

```lisp
(%i1) ssubst("like","hate","I hate Thai food. I hate green tea.";
(%o1) I like Thai food. I like green tea.
(%i2) ssubst("Indian","thai",%,'sequalignore,8,12);
(%o2) I like Indian food. I like green tea.
```

**ssubstfirst**

```
ssubstfirst (new, old, string)
ssubstfirst (new, old, string, test)
ssubstfirst (new, old, string, test, start)
ssubstfirst (new, old, string, test, start, end)
```

Like `ssubst` except that only the first substring that matches `old` is replaced.

**strim**

```
strim (seq,string)
```

Returns a string like `string`, but with all characters that appear in `seq` removed from both ends.

Examples:

```lisp
(%i1) "/* comment */"$
(%i2) strim(" /*",%);
(%o2) comment
```
(\%i3) slength(%);
(\%o3) 7

striml (seq, string)  [Function]
Like strim except that only the left end of string is trimmed.

strimr (seq, string)  [Function]
Like strim except that only the right end of string is trimmed.

stringp (obj)  [Function]
Returns true if obj is a string. See introduction for example.

substring  [Function]
substring (string, start)
substring (string, start, end)
Retuns the substring of string beginning at position start and ending at position end. The character at position end is not included. If end is not given, the substring contains the rest of the string. Note that the first character in string is in position 1.
Examples:
(\%i1) substring("substring",4);
(\%o1) string
(\%i2) substring(\%,4,6);
(\%o2) in

supcase  [Function]
supcase (string)
supcase (string, start)
supcase (string, start, end)
Returns string except that lowercase characters from position start to end are replaced by the corresponding uppercase ones. If end is not given, all lowercase characters from start to the end of string are replaced.
Example:
(\%i1) supcase("english",1,2);
(\%o1) English

tokens  [Function]
tokens (string)
tokens (string, test)
Returns a list of tokens, which have been extracted from string. The tokens are substrings whose characters satisfy a certain test function. If test is not given, constituent is used as the default test. \{constituent, alphacharp, digitcharp, lowercasep, uppercasep, charp, characterp, alphanumericp\} is the set of test functions. (The Lisp-version of tokens is written by Paul Graham. ANSI Common Lisp, 1996, page 67.)
Examples:
(\%i1) tokens("24 October 2005");
(\%o1) [24, October, 2005]
84.5 Octets and Utilities for Cryptography

**base64 (arg)**

[Function]

Returns the base64-representation of arg as a string. The argument arg may be a string, a non-negative integer or a list of octets.

Examples:

```lisp
(%i1) base64: base64("foo bar baz");
(%o1) Zm9vIGJhciBiYXo=
(%i2) string: base64_decode(base64);
(%o2) foo bar baz
(%i3) obase: 16.$
(%i4) integer: base64_decode(base64, 'number);
(%o4) 666f6f206261722062617a
(%i5) octets: base64_decode(base64, 'list);
(%o5) [66, 6F, 6F, 20, 62, 61, 72, 20, 62, 61, 7A]
(%i6) ibase: 16.$
(%i7) base64(octets);
(%o7) Zm9vIGJhciBiYXo=
```

Note that if arg contains umlauts (resp. octets larger than 127) the resulting base64-string is platform dependend. However the decoded string will be equal to the original.

**base64_decode**

[Function]

base64_decode (base64-string)

base64_decode (base64-string, return-type)

By default base64_decode decodes the base64-string back to the original string.

The optional argument return-type allows base64_decode to alternatively return the corresponding number or list of octets. return-type may be string, number or list.

Example: See [base64], page 1093.

**crc24sum**

[Function]

crc24sum (octets)

crc24sum (octets, return-type)

By default crc24sum returns the CRC24 checksum of an octet-list as a string.

The optional argument return-type allows crc24sum to alternatively return the corresponding number or list of octets. return-type may be string, number or list.

Example:

```text
-----BEGIN PGP SIGNATURE-----
Version: GnuPG v2.0.22 (GNU/Linux)
iQEcBAEBAgAGBQJvDCTzAAoJEG/1Mgf2DWAqCSYH/AhVFwhu1D89C3/QFcgVvZTM
wn0YZBUURJAL/cT+InqkLepp3hEeRcUg4p+Tm6aw3R4CdJ7G3FLxExBH/5KnDHi
```
(\%i1) ibase : obase : 16.$
(\%i2) sig64 : sconcat(
   "iQEcbABAgAGBQJVdCTzAAoJEG/1Mgf2DWAqCSYH/AhVFwhu1D98C3/QFcgVvZTM",
   "wn0YzBUURJAL/cT+iNkLcEp3hEbREcugwp+Tm6aw3R4CdJ7G3FLxExBH/5KnDHi",
   "rBQu+I7+3ySK2hprrQ6WxJ59uZSa4YmfNteR8up0zGkauJeWkS4pjRM+aWve",
   "vajlKZCIK52P080DG7Q2dpshh4fgTeNwqCuCiBhQ73t8g1IaLdhDN6EzJVjGiZam",
   "/spqT/sTo6sw8yD0jvU+QvN6/mSMjC/YxjhRM4Qt9EMR1AZ4ukBF5uG1S7mXOH
WdiwkSPZ3gIBhM9SuC076gLWZUNs6NqTeE3UzMjADAFhH3jYk1T7mysCvdtIkms=
  =WmeC
-----END PGP SIGNATURE-----
(\%i3) octets: base64_decode(sig64, 'list)$
(\%i4) crc24: crc24sum(octets, 'list);
(\%o4) [5A, 67, 82]
(\%i5) base64(crc24);
(\%o5) WmeC

\textbf{md5sum}

\textbf{md5sum} (arg)
\textbf{md5sum} (arg, return-type)

Returns the MD5 checksum of a string, a non-negative integer or a list of octets. The
default return value is a string containing 32 hex characters.

The optional argument \texttt{return-type} allows \textbf{md5sum} to alternatively return the
corresponding number or list of octets. \texttt{return-type} may be \texttt{string}, \texttt{number} or \texttt{list}.

Examples:

(\%i1) ibase: obase: 16.$
(\%i2) msg: "foo bar baz"$
(\%i3) string: md5sum(msg);
(\%o3) ab07acbb1e496801937adfa772424bf7
(\%i4) integer: md5sum(msg, 'number);
(\%o4) 0ab07acbb1e496801937adfa772424bf7
(\%i5) octets: md5sum(msg, 'list);
(\%o5) [0AB,7,0AC,0BB,1E,49,68,1,93,7A,0DF,0A7,72,42,4B,0F7]
(\%i6) sdowncase( printf(false, "~~{~2,'0x~^:~}, octets") );

Note that in case arg contains German umlauts or other non-ASCII characters (resp. octets larger than 127) the MD5 checksum is platform dependend.

\textbf{mgf1_sha1}

\textbf{mgf1_sha1} (seed, len)
\textbf{mgf1_sha1} (seed, len, return-type)

Returns a pseudo random number of variable length. By default the returned value
is a number with a length of \texttt{len} octets.
The optional argument return-type allows `mgf1_sha1` to alternatively return the corresponding list of len octets. return-type may be number or list.

The computation of the returned value is described in RFC 3447, appendix B.2.1 MGF1. SHA1 ist used as hash function, i.e. the randomness of the computed number relies on the randomness of SHA1 hashes.

Example:

```
(%i1) ibase: obase: 16.$
(%i2) number: mgf1_sha1(4711., 8);
(%o2) 0e0252e5a2a42fea1
(%i3) octets: mgf1_sha1(4711., 8, 'list);
(%o3) [0E0,25,2E,5A,2A,42,0FE,0A1]
```

**number_to_octets (number)**  
Returns an octet-representation of number as a list of octets. The number must be a non-negative integer.

Example:

```
(%i1) ibase : obase : 16.$
(%i2) octets: [0ca,0fe,0ba,0be]$  
(%i3) number: octets_to_number(octets);
(%o3) 0cafebabe
(%i4) number_to_octets(number);
(%o4) [0CA, 0FE, 0BA, 0BE]
```

**octets_to_number (octets)**  
Returns a number by concatenating the octets in the list of octets.

Example: See [number_to_octets], page 1095.

**octets_to_oid (octets)**  
Computes an object identifier (OID) from the list of octets.

Example: RSA encryption OID

```
(%i1) ibase : obase : 16.$
(%i2) oid: octets_to_oid([2A,86,48,86,0F7,0D,1,1,1]);
(%o2) 1.2.840.113549.1.1.1
(%i3) oid_to_octets(oid);
(%o3) [2A, 86, 48, 86, 0F7, 0D, 1, 1, 1]
```

**octets_to_string**  
```
octets_to_string (octets)
octets_to_string (octets, encoding)
```
Decodes the list of octets into a string according to current system defaults. When decoding octets corresponding to Non-US-ASCII characters the result depends on the platform, application and underlying Lisp.

Example: Using system defaults (Maxima compiled with GCL, which uses no format definition and simply passes through the UTF-8-octets encoded by the GNU/Linux terminal).

```
(%i1) octets: string_to_octets("abc");
```
octets_to_string(octets);
(\%i2) abc
(\%i3) unicode(20AC);
(\%o4) €
(\%i5) octets: string_to_octets(\%);
(\%o5) [0E2, 82, 0AC]
(\%i6) octets_to_string(octets);
(\%o6) €
(\%i7) utf8_to_unicode(octets);
(\%o7) 20AC

In case the external format of the Lisp reader is equal to UTF-8 the optional argument \texttt{encoding} allows to set the encoding for the octet to string conversion. If necessary see [adjust_external_format], page 1082, for changing the external format.

Some names of supported encodings (see corresponding Lisp manual for more):
- CCL, CLISP, SBCL: utf-8, ucs-2be, ucs-4be, iso-8859-1, cp1252, cp850
- CMUCL: utf-8, utf-16-be, utf-32-be, iso8859-1, cp1252
- ECL: utf-8, ucs-2be, ucs-4be, iso-8859-1, windows-cp1252, dos-cp850

Example (continued): Using the optional encoding argument (Maxima compiled with SBCL, GNU/Linux terminal).

oid_to_octets (oid-string)
Converts an object identifier (OID) to a list of \texttt{octets}.
Example: See [octets_to_oid], page 1095.

sha1sum (arg)
sha1sum (arg, return-type)
Returns the SHA1 fingerprint of a string, a non-negative integer or a list of octets.
The default return value is a string containing 40 hex characters.

The optional argument \texttt{return-type} allows \texttt{sha1sum} to alternatively return the corresponding number or list of octets. \texttt{return-type} may be \texttt{string}, \texttt{number} or \texttt{list}.

Example:
Note that in case arg contains German umlauts or other non-ASCII characters (resp. octets larger than 127) the SHA1 fingerprint is platform dependend.

**sha256sum**  

**Function**

```
sha256sum (arg)
sha256sum (arg, return-type)
```

Returns the SHA256 fingerprint of a string, a non-negative integer or a list of octets. The default return value is a string containing 64 hex characters.

The optional argument return-type allows sha256sum to alternatively return the corresponding number or list of octets (see [sha1sum], page 1096).

Example:

```
(%i1) string: sha256sum("foo bar baz");
(%o1) dbd318c1c462aee872f41109a4dfd3048871a03dedd0fe0e757ced57dad6f2d7
```

Note that in case arg contains German umlauts or other non-ASCII characters (resp. octets larger than 127) the SHA256 fingerprint is platform dependend.

**string_to_octets**  

**Function**

```
string_to_octets (string)
string_to_octets (string, encoding)
```

Encodes a string into a list of octets according to current system defaults. When encoding strings containing Non-US-ASCII characters the result depends on the platform, application and underlying Lisp.

In case the external format of the Lisp reader is equal to UTF-8 the optional argument encoding allows to set the encoding for the string to octet conversion. If necessary see [adjust_external_format], page 1082, for changing the external format.

See [octets_to_string], page 1095, for examples and some more information.
85 to_poly_solve

85.1 Functions and Variables for to_poly_solve

The packages to_poly and to_poly_solve are experimental; the specifications of the functions in these packages might change or the some of the functions in these packages might be merged into other Maxima functions.

Barton Willis (Professor of Mathematics, University of Nebraska at Kearney) wrote the to_poly and to_poly_solve packages and the English language user documentation for these packages.

%and

The operator %and is a simplifying nonshort-circuited logical conjunction. Maxima simplifies an %and expression to either true, false, or a logically equivalent, but simplified, expression. The operator %and is associative, commutative, and idempotent. Thus when %and returns a noun form, the arguments of %and form a non-redundant sorted list; for example

(%i1) a %and (a %and b);
(%o1) a %and b

If one argument to a conjunction is the explicit the negation of another argument, %and returns false:

(%i2) a %and (not a);
(%o2) false

If any member of the conjunction is false, the conjunction simplifies to false even if other members are manifestly non-boolean; for example

(%i3) 42 %and false;
(%o3) false

Any argument of an %and expression that is an inequation (that is, an inequality or equation), is simplified using the Fourier elimination package. The Fourier elimination simplifier has a pre-processor that converts some, but not all, nonlinear inequations into linear inequations; for example the Fourier elimination code simplifies abs(x) + 1 > 0 to true, so

(%i4) (x < 1) %and (abs(x) + 1 > 0);
(%o4) x < 1

Notes

• The option variable prederror does not alter the simplification %and expressions.
• To avoid operator precedence errors, compound expressions involving the operators %and, %or, and not should be fully parenthesized.
• The Maxima operators and and or are both short-circuited. Thus and isn’t associative or commutative.

Limitations The conjunction %and simplifies inequations locally, not globally. This means that conjunctions such as

(%i5) (x < 1) %and (x > 1);
do not simplify to false. Also, the Fourier elimination code ignores the fact database;

```
(%i5) (x > 1) %and (x < 1)
```

Finally, nonlinear inequations that aren’t easily converted into an equivalent linear inequation aren’t simplified.

There is no support for distributing \( \%\text{and} \) over \( \%\text{or} \); neither is there support for distributing a logical negation over \( \%\text{and} \).

To use `load(to_poly_solve)`

Related functions \( \%\text{or}, \%\text{if}, \%\text{and}, \%\text{or}, \%\text{not} \)

Status The operator \( \%\text{and} \) is experimental; the specifications of this function might change and its functionality might be merged into other Maxima functions.

\( \%\text{if} \) (bool, a, b)

The operator \( \%\text{if} \) is a simplifying conditional. The conditional \( \text{bool} \) should be boolean-valued. When the conditional is true, return the second argument; when the conditional is false, return the third; in all other cases, return a noun form.

Maxima inequations (either an inequality or an equality) are not boolean-valued; for example, Maxima does not simplify 5 < 6 to true, and it does not simplify 5 = 6 to false; however, in the context of a conditional to an \( \%\text{if} \) statement, Maxima automatically attempts to determine the truth value of an inequation. Examples:

```
(%i1) f : %if(x # 1, 2, 8);
(%i2) [subst(x = -1,f), subst(x=1,f)];
```

If the conditional involves an inequation, Maxima simplifies it using the Fourier elimination package.

Notes
- If the conditional is manifestly non-boolean, Maxima returns a noun form:

```
(%i3) %if(42,1,2);
(%i3) %if(42, 1, 2)
```
- The Maxima operator \( \text{if} \) is nary, the operator \( \%\text{if} \) isn’t nary.

Limitations The Fourier elimination code only simplifies nonlinear inequations that are readily convertible to an equivalent linear inequation.

To use: `load(to_poly_solve)`

Status: The operator \( \%\text{if} \) is experimental; its specifications might change and its functionality might be merged into other Maxima functions.

\( \%\text{or} \)

The operator \( \%\text{or} \) is a simplifying nonshort-circuited logical disjunction. Maxima simplifies an \( \%\text{or} \) expression to either true, false, or a logically equivalent, but simplified,
expression. The operator \%or is associative, commutative, and idempotent. Thus when \%or returns a noun form, the arguments of \%or form a non-redundant sorted list; for example

(\%i1) a \%or (a \%or b);
(\%o1) a \%or b

If one member of the disjunction is the explicit the negation of another member, \%or returns true:

(\%i2) a \%or (not a);
(\%o2) true

If any member of the disjunction is true, the disjunction simplifies to true even if other members of the disjunction are manifestly non-boolean; for example

(\%i3) 42 \%or true;
(\%o3) true

Any argument of an \%or expression that is an inequation (that is, an inequality or equation), is simplified using the Fourier elimination package. The Fourier elimination code simplifies \( \text{abs}(x) + 1 > 0 \) to true, so we have

(\%i4) (x < 1) \%or (abs(x) + 1 > 0);
(\%o4) true

Notes

- The option variable \texttt{prederror} does not alter the simplification of \%or expressions.

- You should parenthesize compound expressions involving the operators \%and, \%or, and not; the binding powers of these operators might not match your expectations.

- The Maxima operators \texttt{and} and \texttt{or} are both short-circuited. Thus or isn’t associative or commutative.

Limitations The conjunction \%or simplifies inequations locally, not globally. This means that conjunctions such as

(\%i1) (x < 1) \%or (x >= 1);
(\%o1) (x > 1) \%or (x >= 1)

do not simplify to true. Further, the Fourier elimination code ignores the fact database;

(\%i2) assume(x > 5);
(\%o2) [x > 5]
(\%i3) (x > 1) \%and (x > 2);
(\%o3) (x > 1) \%and (x > 2)

Finally, nonlinear inequations that aren’t easily converted into an equivalent linear inequation aren’t simplified.

The algorithm that looks for terms that cannot both be false is weak; also there is no support for distributing %or over %and; neither is there support for distributing a logical negation over %or.

To use load(to_poly_solve)
Related functions %or, %if, and, or, not

Status The operator %or is experimental; the specifications of this function might change and its functionality might be merged into other Maxima functions.

complex_number_p (x)  [Function]
The predicate complex_number_p returns true if its argument is either a + %i * b, a, %i b, or %i, where a and b are either rational or floating point numbers (including big floating point); for all other inputs, complex_number_p returns false; for example

(%i1) map('complex_number_p,[2/3, 2 + 1.5 * %i, %i]);
(%o1) [true, true, true]
(%i2) complex_number_p((2+%i)/(5-%i));
(%o2) false
(%i3) complex_number_p(cos(5 - 2 * %i));
(%o3) false

Related functions isreal_p
To use load(to_poly_solve)

Status The operator complex_number_p is experimental; its specifications might change and its functionality might be merged into other Maxima functions.

compose_functions (l)  [Function]
The function call compose_functions(l) returns a lambda form that is the composition of the functions in the list l. The functions are applied from right to left; for example

(%i1) compose_functions([cos, exp]);
(%o1) lambda([%g151], cos(%e ))
(%i2) %(x);
(%o2) cos(%e )

When the function list is empty, return the identity function:

(%i3) compose_functions([]);
(%o3) lambda([%g152], %g152)
(%i4) %(x);
(%o4) x

Notes
• When Maxima determines that a list member isn’t a symbol or a lambda form, funmake (not compose_functions) signals an error:

(%i5) compose_functions([a < b]);
funmake: first argument must be a symbol, subscripted symbol, string, or lambda expression; found: a < b
#0: compose_functions(l=[a < b])(to_poly_solve.mac line 40)
-- an error. To debug this try: debugmode(true);

• To avoid name conflicts, the independent variable is determined by the function new_variable.

(%i6) compose_functions([%g0]);
Although the independent variables are different, Maxima is able to deduce that these lambda forms are semantically equal:

\[
\text{(\%i8) is(equal(\%o6,\%o7));}
\]
\[
\text{true}
\]

\text{To use load(to\_poly\_solve)}

\text{Status} The function \text{compose\_functions} is experimental; its specifications might change and its functionality might be merged into other Maxima functions.

d\text{float (x)} \quad [\text{Function}]

The function \text{dfloat} is a similar to \text{float}, but the function \text{dfloat} applies \text{rectform} when \text{float} fails to evaluate to an IEEE double floating point number; thus

\[
\text{(\%i1) float(4.5^(1 + \%i));}
\]
\[
\text{\%i + 1}
\]
\[
\text{(\%o1) 4.5}
\]
\[
\text{(\%i2) dfloat(4.5^(1 + \%i));}
\]
\[
4.48998802962884 \%i + .3000124893895671
\]

\text{Notes}

- The rectangular form of an expression might be poorly suited for numerical evaluation—for example, the rectangular form might needlessly involve the difference of floating point numbers (subtractive cancellation).
- The identifier \text{float} is both an option variable (default value false) and a function name.

\text{Related functions \text{float}, \text{bfloat}}

\text{To use load(to\_poly\_solve)}

\text{Status} The function \text{dfloat} is experimental; its specifications might change and its functionality might be merged into other Maxima functions.

elim (l, x) \quad [\text{Function}]

The function \text{elim} eliminates the variables in the set or list \text{x} from the equations in the set or list \text{l}. Each member of \text{x} must be a symbol; the members of \text{l} can either be equations, or expressions that are assumed to equal zero.

The function \text{elim} returns a list of two lists; the first is the list of expressions with the variables eliminated; the second is the list of pivots; thus, the second list is a list of expressions that \text{elim} used to eliminate the variables.

Here is an example of eliminating between linear equations:

\[
\text{(\%i11) elim(set(x + y + z = 1, x - y - z = 8, x - z = 1), set(x,y));}
\]
\[
[[2 z - 7], [y + 7, z - x + 1]]
\]

Eliminating \text{x} and \text{y} yields the single equation \text{2 z - 7 = 0}; the equations \text{y + 7 = 0} and \text{z - z + 1 = 1} were used as pivots. Eliminating all three variables from these equations, triangularizes the linear system:

\[
\text{(\%i12) elim(set(x + y + z = 1, x - y - z = 8, x - z = 1),}
\]
set(x,y,z));
(%o2) [[], [2 z - 7, y + 7, z - x + 1]]

Of course, the equations needn't be linear:
(%i3) elim(set(x^2 - 2 * y^3 = 1, x - y = 5), [x,y]);
3 2
(%o3) [[], [2 y - y - 10 y - 24, y - x + 5]]

The user doesn't control the order the variables are eliminated. Instead, the algorithm uses a heuristic to attempt to choose the best pivot and the best elimination order.

Notes

- Unlike the related function eliminate, the function elim does not invoke solve when the number of equations equals the number of variables.
- The function elim works by applying resultants; the option variable resultant determines which algorithm Maxima uses. Using sqfr, Maxima factors each resultant and suppresses multiple zeros.
- The elim will triangularize a nonlinear set of polynomial equations; the solution set of the triangularized set can be larger than that solution set of the untriangularized set. Thus, the triangularized equations can have spurious solutions.

Related functions elim_allbut, eliminate_using, eliminate

Option variables resultant

To use load(to_poly)

Status The function elim is experimental; its specifications might change and its functionality might be merged into other Maxima functions.

elim_allbut (l, x)

This function is similar to elim, except that it eliminates all the variables in the list of equations l except for those variables that in in the list x
(%i1) elim_allbut([x+y = 1, x - 5*y = 1],[]);
(%o1) [[], [y, y + x - 1]]
(%i2) elim_allbut([x+y = 1, x - 5*y = 1],[x]);
(%o2) [[x - 1, [y + x - 1]]

To use load(to_poly)

Option variables resultant

Related functions elim, eliminate_using, eliminate

Status The function elim_allbut is experimental; its specifications might change and its functionality might be merged into other Maxima functions.

eliminate_using (l, e, x)

Using e as the pivot, eliminate the symbol x from the list or set of equations in l. The function eliminate_using returns a set.
(%i1) eq : [x^2 - y^2 - z^3 , x*y - z^2 - 5, x - y + z];
3 2 2 2
(%o1) [- z - y + x , - z + x y - 5, z - y + x]
(%i2) eliminate_using(eq,first(eq),z);
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(2) \{y + (1 - 3 x) y + 3 x y - x - x,
    4 3 2 2 4
y - x y + 13 x y - 75 x y + x + 125\}

(%i3) eliminate_using(eq, second(eq), z);

(%o3) \{y - 3 x y + x + 5, y - x y + 13 x y - 75 x y + x + 125\}

(%i4) eliminate_using(eq, third(eq), z);

(%o4) \{y - 3 x y + x + 5, y + (1 - 3 x) y + 3 x y - x - x\}

Option variables resultant

Related functions elim, eliminate, elim_allbut

To use load(to_poly)

Status The function eliminate_using is experimental; its specifications might change and its functionality might be merged into other Maxima functions.

fourier_elim ([eq1, eq2, ...], [var1, var, ...])

Fourier elimination is the analog of Gauss elimination for linear inequations (equations or inequalities). The function call fourier_elim([eq1, eq2, ...], [var1, var2, ...]) does Fourier elimination on a list of linear inequations [eq1, eq2, ...] with respect to the variables [var1, var2, ...]; for example

(%i1) fourier_elim([y-x < 5, x - y < 7, 10 < y], [x,y]);

(%o1) \[y - 5 < x, x < y + 7, 10 < y\]

(%i2) fourier_elim([y-x < 5, x - y < 7, 10 < y], [y,x]);

(%o2) \[max(10, x - 7) < y, y < x + 5, 5 < x\]

Eliminating first with respect to x and second with respect to y yields lower and upper bounds for x that depend on y, and lower and upper bounds for y that are numbers. Eliminating in the other order gives x dependent lower and upper bounds for y, and numerical lower and upper bounds for x.

When necessary, fourier_elim returns a disjunction of lists of inequations:

(%i3) fourier_elim([x # 6], [x]);

(%o3) \[x < 6\] or \[6 < x\]

When the solution set is empty, fourier_elim returns emptyset, and when the solution set is all reals, fourier_elim returns universalset; for example

(%i4) fourier_elim([x < 1, x > 1], [x]);

(%o4) emptyset

(%i5) fourier_elim([minf < x, x < inf], [x]);

(%o5) universalset

For nonlinear inequations, fourier_elim returns a (somewhat) simplified list of inequations:

(%i6) fourier_elim([x^3 - 1 > 0], [x]);

(%o6) \[1 < x, x + x + 1 > 0\] or \[x < 1, -(x + x + 1) > 0\]

(%i7) fourier_elim([cos(x) < 1/2], [x]);
Instead of a list of inequations, the first argument to `fourier_elim` may be a logical disjunction or conjunction:

\[
(\%i8) \text{fourier_elim}((x + y < 5) \text{ and } (x - y > 8), [x, y]);
\]
\[
(\%o8) \left[ y + 8 < x, x < 5 - y, y < -\right]
\]

\[
(\%i9) \text{fourier_elim}(((x + y < 5) \text{ and } x < 1) \text{ or } (x - y > 8), [x, y]);
\]
\[
(\%o9) \left[ y + 8 < x \right] \text{ or } [x < \text{min}(1, 5 - y)]
\]

The function `fourier_elim` supports the inequation operators `<`, `<=`, `>`, `>=`, `#`, and `=`.

The Fourier elimination code has a preprocessor that converts some nonlinear inequations that involve the absolute value, minimum, and maximum functions into linear in equations. Additionally, the preprocessor handles some expressions that are the product or quotient of linear terms:

\[
(\%i10) \text{fourier_elim}([\text{max}(x, y) > 6, x \# 8, \text{abs}(y-1) > 12], [x, y]);
\]
\[
(\%o10) \left[ 6 < x, x < 8, y < -11 \right] \text{ or } [8 < x, y < -11]
\]
\[
\text{or } [x < 8, 13 < y] \text{ or } [x = y, 13 < y] \text{ or } [8 < x, x < y, 13 < y]
\]
\[
\text{or } [y < x, 13 < y]
\]

\[
(\%i11) \text{fourier_elim}([\frac{x+6}{x-9} <= 6], [x]);
\]
\[
(\%o11) \left[ x = 12 \right] \text{ or } [12 < x] \text{ or } [x < 9]
\]

\[
(\%i12) \text{fourier_elim}([x^2 - 1 \# 0], [x]);
\]
\[
(\%o12) \left[ -1 < x, x < 1 \right] \text{ or } [1 < x] \text{ or } [x < -1]
\]

To use `load(fourier_elim)`

isreal_p (e) [Function]

The predicate `isreal_p` returns true when Maxima is able to determine that `e` is real-valued on the entire real line; it returns false when Maxima is able to determine that `e` isn’t real-valued on some nonempty subset of the real line; and it returns a noun form for all other cases.

\[
(\%i1) \text{map(’isreal_p, [-1, 0, %i, %pi])};
\]
\[
(\%o1) \left[ \text{true, true, false, true} \right]
\]

Maxima variables are assumed to be real; thus

\[
(\%i2) \text{isreal_p(x)};
\]
\[
(\%o2) \text{true}
\]

The function `isreal_p` examines the fact database:

\[
(\%i3) \text{declare(z, complex)}$
\]

\[
(\%i4) \text{isreal_p(z)}$
\]
\[
(\%o4) \text{isreal_p(z)}
\]

Limitations Too often, `isreal_p` returns a noun form when it should be able to return false; a simple example: the logarithm function isn’t real-valued on the entire real line, so `isreal_p(log(x))` should return false; however

\[
(\%i5) \text{isreal_p(log(x))};
\]
Chapter 85: to_poly_solve

To use load(to_poly_solve)

Related functions complex_number_p

Status The function isreal_p is experimental; its specifications might change and its functionality might be merged into other Maxima functions.

new_variable (type) [Function]

Return a unique symbol of the form %[z,n,r,c,g]k, where k is an integer. The allowed values for type are integer, natural_number, real, natural_number, and general. (By natural number, we mean the nonnegative integers; thus zero is a natural number. Some, but not all, definitions of natural number exclude zero.)

When type isn’t one of the allowed values, type defaults to general. For integers, natural numbers, and complex numbers, Maxima automatically appends this information to the fact database.

```
(%i1) map('new_variable,
       ['integer, 'natural_number, 'real, 'complex, 'general]);
(%o1) [%z144, %n145, %r146, %c147, %g148]
(%i2) nicedummies(%);
(%o2) [%z0, %n0, %r0, %c0, %g0]
(%i3) featurep(%z0, 'integer);
(%o3) true
(%i4) featurep(%n0, 'integer);
(%o4) true
(%i5) is(%n0 >= 0);
(%o5) true
(%i6) featurep(%c0, 'complex);
(%o6) true
```

Note Generally, the argument to new_variable should be quoted. The quote will protect against errors similar to

```
(%i7) integer : 12$
(%i8) new_variable(integer);
(%o8) %g149
(%i9) new_variable('integer);
(%o9) %z150
```

Related functions nicedummies

To use load(to_poly_solve)

Status The function new_variable is experimental; its specifications might change and its functionality might be merged into other Maxima functions.

nicedummies [Function]

Starting with zero, the function nicedummies re-indexes the variables in an expression that were introduced by new_variable;

```
(%i11) new_variable('integer) + 52 * new_variable('integer);
(%o11) 52 %z136 + %z135
```
\texttt{(\%i2) new\_variable('integer) - new\_variable('integer);} \\
\texttt{(\%o2) \hspace{1cm} \%z137 - \%z138} \\
\texttt{(\%i3) nicedummies(\%);} \\
\texttt{(\%o3) \hspace{1cm} \%z0 - \%z1} \\
\textbf{Related functions} \textit{new\_variable} \\
\textbf{To use} \texttt{load(to\_poly\_solve)} \\
\textbf{Status} The function \textit{nicedummies} is experimental; its specifications might change and its functionality might be merged into other Maxima functions. \\

\textbf{parg (x)} \hspace{1cm} \textbf{[Function]} \\
The function \textit{parg} is a simplifying version of the complex argument function \textit{carg}; thus \\
\texttt{(\%i1) map('parg,[1,1+%i,%i, -1 + %i, -1]);} \\
\texttt{(\%o1) \hspace{1cm} \%pi \hspace{1cm} \%pi \hspace{1cm} \%pi} \\
\texttt{[0, ---, ---, -----, \%pi]} \\
\texttt{4 \hspace{1cm} 2 \hspace{1cm} 4} \\
Generally, for a non-constant input, \textit{parg} returns a noun form; thus \\
\texttt{(\%i2) parg(x + %i * sqrt(x));} \\
\texttt{(\%o2) \hspace{1cm} parg(x + %i sqrt(x))} \\
When \textit{sign} can determine that the input is a positive or negative real number, \textit{parg} will return a non-noun form for a non-constant input. Here are two examples: \\
\texttt{(\%i3) parg(abs(x));} \\
\texttt{(\%o3) 0} \\
\texttt{(\%i4) parg(-x^2-1);} \\
\texttt{(\%o4) \hspace{1cm} \%pi} \\
\textbf{Note} The \textit{sign} function mostly ignores the variables that are declared to be complex (\texttt{declare(x,complex)}); for variables that are declared to be complex, the \textit{parg} can return incorrect values; for example \\
\texttt{(\%i1) declare(x,complex)$} \\
\texttt{(\%i2) parg(x^2 + 1);} \\
\texttt{(\%o2) 0} \\
\textbf{Related function} \textit{carg, isreal.p} \\
\textbf{To use} \texttt{load(to\_poly\_solve)} \\
\textbf{Status} The function \textit{parg} is experimental; its specifications might change and its functionality might be merged into other Maxima functions. \\

\textbf{real\_imagpart\_to\_conjugate (e)} \hspace{1cm} \textbf{[Function]} \\
The function \textit{real\_imagpart\_to\_conjugate} replaces all occurrences of \textit{realpart} and \textit{imagpart} to algebraically equivalent expressions involving the \textit{conjugate}. \\
\texttt{(\%i1) declare(x, complex)$} \\
\texttt{(\%i2) real\_imagpart\_to\_conjugate(realpart(x) + \textit{imagpart}\hspace{1cm} (x) = 3);} \\
\texttt{conjugate(x) + x \hspace{1cm} \%i \hspace{1cm} (x - conjugate(x))}
\begin{verbatim}
(%o2) \frac{\log(2) \cdot \%i \cdot \%pi}{2} - \log(x - \%i) + \frac{\log(2) \cdot \%i \cdot \%pi}{4} = 3

To use load(to_poly_solve)

Status The function real_imagpart_to_conjugate is experimental; its specifications
might change and its functionality might be merged into other Maxima functions.

rectform_log_if_constant (e) [Function]
The function rectform_log_if_constant converts all terms of the form log(c)
to rectform(log(c)), where c is either a declared constant expression or explicitly
declared constant.

(%i1) rectform_log_if_constant(log(1-%i) - log(x - %i));
(%o1) log(2) \cdot \%i \cdot \%pi
\quad - \log(x - \%i) + \frac{\log(2) \cdot \%i \cdot \%pi}{2} - \frac{\log(2) \cdot \%i \cdot \%pi}{4}

(%i2) declare(a,constant, b,constant)$

(%i3) rectform_log_if_constant(log(a + %i*b));
(%o3) \frac{\log(b + a)}{2} \quad + \%i \cdot \text{atan2}(b, a)

To use load(to_poly_solve)

Status The function rectform_log_if_constant is experimental; the specifications
of this function might change and its functionality might be merged
into other Maxima functions.

simp_inequality (e) [Function]
The function simp_inequality applies some simplifications to conjunctions and
disjunctions of inequations.

Limitations The function simp_inequality is limited in at least two ways; first, the
simplifications are local; thus

(%i1) simp_inequality((x > minf) \%and (x < 0));
(%o1) (x > 1) \%and (x < 1)

And second, simp_inequality doesn't consult the fact database:

(%i2) assume(x > 0)$

(%i3) simp_inequality(x > 0);
(%o3) x > 0

To use load(fourier_elim)

Status The function simp_inequality is experimental; its specifications might change
and its functionality might be merged into other Maxima functions.

standardize_inverse_trig (e) [Function]
This function applies the identities \cot(x) = \atan(1/x), \acsc(x) = \asin(1/x), and
similarly for \asec, \acoth, \acsch and \asech to an expression. See Abramowitz and
Stegun, Eqs. 4.4.6 through 4.4.8 and 4.6.4 through 4.6.6.
\end{verbatim}
To use load(to_poly_solve)

Status The function standardize_inverse_trig is experimental; its specifications might change and its functionality might be merged into other Maxima functions.

\textbf{subst_parallel (l, e)} \hspace{1cm} \text{[Function]}

When \textit{l} is a single equation or a list of equations, substitute the right hand side of each equation for the left hand side. The substitutions are made in parallel; for example

\begin{verbatim}(\%i1) load(to_poly_solve)$
(\%i2) subst_parallel([x=y,y=x], [x,y]);
(\%o2) \[y, x]\n\end{verbatim}

Compare this to substitutions made serially:

\begin{verbatim}(\%i3) subst([x=y,y=x],[x,y]);
(\%o3) \[x, x]\n\end{verbatim}

The function subst_parallel is similar to sublis except that subst_parallel allows for substitution of nonatoms; for example

\begin{verbatim}(\%i4) subst_parallel([x^2 = a, y = b], x^2 * y);
(\%o4) a * b
(\%i5) sublis([x^2 = a, y = b], x^2 * y);
\end{verbatim}

\texttt{sublis: left-hand side of equation must be a symbol; found: x}
\hspace{1cm} -- an error. To debug this try: debugmode(true);

The substitutions made by subst_parallel are literal, not semantic; thus subst_parallel does not recognize that \(x \cdot y\) is a subexpression of \(x^2 \cdot y\)

\begin{verbatim}(\%i6) subst_parallel([x \cdot y = a], x^2 \cdot y);
(\%o6) x \cdot y
\end{verbatim}

\texttt{sublis: left-hand side of equation must be a symbol; found: x}
\hspace{1cm} -- an error. To debug this try: debugmode(true);

The function subst_parallel completes all substitutions before simplifications. This allows for substitutions into conditional expressions where errors might occur if the simplifications were made earlier:

\begin{verbatim}(\%i7) subst_parallel([x = 0], \%if(x < 1, 5, log(x)));
(\%o7) 5
(\%i8) subst([x = 0], \%if(x < 1, 5, log(x)));
\end{verbatim}

\texttt{log: encountered log(0).}
\hspace{1cm} -- an error. To debug this try: debugmode(true);

Related functions subst, sublis, ratsubst

To use load(to_poly_solve_extra.lisp)

Status The function subst_parallel is experimental; the specifications of this function might change and its functionality might be merged into other Maxima functions.
to_poly (e, l)  [Function]
The function to_poly attempts to convert the equation e into a polynomial system along with inequality constraints; the solutions to the polynomial system that satisfy the constraints are solutions to the equation e. Informally, to_poly attempts to polynomialize the equation e; an example might clarify:

(%i1) load(to_poly_solve)$
(%i2) to_poly(sqrt(x) = 3, [x]);
   2
(%o2) [ [%g130 - 3, x = %g130 ],
       %pi
     [ - --- < parg(%g130), parg(%g130) <= ---] , []
     2 2

The conditions -%pi/2<parg(%g130),parg(%g130)<=%pi/2 tell us that %g130 is in the range of the square root function. When this is true, the solution set to sqrt(x) = 3 is the same as the solution set to %g130-3,x=%g130^2.

To polynomialize trigonometric expressions, it is necessary to introduce a non algebraic substitution; these non algebraic substitutions are returned in the third list returned by to_poly; for example

(%i3) to_poly(cos(x),[x]);
   2 %i x
(%o3) [ [%g131 + 1], [2 %g131 # 0], [%g131 = %e ] ]

Constant terms aren’t polynomialized unless the number one is a member of the variable list; for example

(%i4) to_poly(x = sqrt(5),[x]);
   2
(%o4) [ [ x - sqrt(5) ], , ]

(%i5) to_poly(x = sqrt(5),[1,x]);
   2
(%o5) [ [ x - %g132, 5 = %g132 ],
       %pi
     [ - --- < parg(%g132), parg(%g132) <= ---] , []
     2 2

To generate a polynomial with sqrt(5)+sqrt(7) as one of its roots, use the commands

(%i6) first(elim_allbut(first(to_poly(x = sqrt(5) + sqrt(7),
   4 2
first(to_poly(x = sqrt(5) + sqrt(7),
   [1,x])), [x]));
   2
(%o6) [ x - 24 x + 4]

Related functions to_poly_solve
To use load(to_poly)

Status: The function to_poly is experimental; its specifications might change and its functionality might be merged into other Maxima functions.

to_poly_solve (e, l, [options])  [Function]
The function to_poly_solve tries to solve the equations e for the variables l. The equation(s) e can either be a single expression or a set or list of expressions; similarly,
\( l \) can either be a single symbol or a list of set of symbols. When a member of \( e \) isn't explicitly an equation, for example \( x^2 - 1 \), the solver assumes that the expression vanishes.

The basic strategy of \texttt{to_poly_solve} is to convert the input into a polynomial form and to call \texttt{algsys} on the polynomial system. Internally \texttt{to_poly_solve} defaults \texttt{algexact} to true. To change the default for \texttt{algexact}, append \texttt{'algexact=false} to the \texttt{to_poly_solve} argument list.

When \texttt{to_poly_solve} is able to determine the solution set, each member of the solution set is a list in a \texttt{%union} object:

\begin{verbatim}
(%i1) load(to_poly_solve)$
(%i2) to_poly_solve(x*(x-1) = 0, x);
(%o2) %union([x = 0], [x = 1])
\end{verbatim}

When \texttt{to_poly_solve} is unable to determine the solution set, a \texttt{%solve} nounform is returned (in this case, a warning is printed)

\begin{verbatim}
(%i3) to_poly_solve(x^k + 2* x + 1 = 0, x);
Nonalgebraic argument given to 'to_poly'
unable to solve
k
(%o3) %solve([x + 2 x + 1 = 0], [x])
\end{verbatim}

Subsitution into a \texttt{%solve} nounform can sometimes result in the solution

\begin{verbatim}
(%i4) subst(k = 2, %);
(%o4) %union([x = - 1])
\end{verbatim}

Especially for trigonometric equations, the solver sometimes needs to introduce an arbitrary integer. These arbitrary integers have the form \texttt{%zXXX}, where \texttt{XXX} is an integer; for example

\begin{verbatim}
(%i5) to_poly_solve(sin(x) = 0, x);
(%o5) %union([x = 2 %pi %z33 + %pi], [x = 2 %pi %z35])
\end{verbatim}

To re-index these variables to zero, use \texttt{nicedummies}:

\begin{verbatim}
(%i6) nicedummies(%);
(%o6) %union([x = 2 %pi %z0 + %pi], [x = 2 %pi %z1])
\end{verbatim}

Occasionally, the solver introduces an arbitrary complex number of the form \texttt{%cXXX} or an arbitrary real number of the form \texttt{%rXXX}. The function \texttt{nicedummies} will re-index these identifiers to zero.

The solution set sometimes involves simplifing versions of various of logical operators including \texttt{%and}, \texttt{%or}, or \texttt{%if} for conjunction, disjuntion, and implication, respectively; for example

\begin{verbatim}
(%i7) sol : to_poly_solve(abs(x) = a, x);
(%o7) %union(%if(isnonnegative_p(a), [x = - a], %union()),
     %if(isnonnegative_p(a), [x = a], %union()))
(%i8) subst(a = 42, sol);
(%o8) %union([x = - 42], [x = 42])
(%i9) subst(a = -42, sol);
\end{verbatim}
The empty set is represented by \%union().

The function to_poly_solve is able to solve some, but not all, equations involving rational powers, some nonrational powers, absolute values, trigonometric functions, and minimum and maximum. Also, some it can solve some equations that are solvable in in terms of the Lambert W function; some examples:

\begin{verbatim}
(%o1) load(to_poly_solve)$

(%i2) to_poly_solve(set(max(x,y) = 5, x+y = 2), set(x,y));
(%o2) %union([x = -3, y = 5], [x = 5, y = -3])

(%i3) to_poly_solve(abs(1-abs(1-x)) = 10,x);
(%o3) %union([x = -10], [x = 12])

(%i4) to_poly_solve(set(sqrt(x) + sqrt(y) = 5, x + y = 10),
set(x,y));
3/2 3/2
[5 %i - 10 5 %i + 10]

(%i5) to_poly_solve(cos(x) * sin(x) = 1/2,x,
'simpfuncts = ['expand, 'nicedummies]);
(%pi)
(%o5) %union([x = %pi %z0 + --])
4

(%i6) to_poly_solve(x^(2*a) + x^a + 1,x); 2 2
2 %i %pi %z81
-------------
1/a a

(%i7) to_poly_solve(x * exp(x) = a, x);
(%o7) %union([x = lambert_w(a)])
\end{verbatim}

For linear inequalities, to_poly_solve automatically does Fourier elimination:

\begin{verbatim}
(%i8) to_poly_solve([x + y < 1, x - y >= 8], [x,y]);
\end{verbatim}
Each optional argument to \texttt{to_poly_solve} must be an equation; generally, the order of these options does not matter.

- \texttt{simpfun} = \texttt{l}, where \texttt{l} is a list of functions. Apply the composition of the members of \texttt{l} to each solution.

\begin{verbatim}
(%i11) to_poly_solve(x^2=%i,x);

1/4 1/4
(%o11) \%union([x = - (- 1) ], [x = (- 1 )])

(%i12) to_poly_solve(x^2=%i,x, 'simpfuncs = [\text{rectform}]);

1 \%i
(%o12) \%union([x = - \frac{\text{\frac{\%i}{\sqrt{2}}} + \frac{\%i}{\sqrt{2}}}{\text{\sqrt{2}}}], [x = \frac{\%i}{\sqrt{2}} + \frac{\%i}{\sqrt{2}}])

(%i13) to_poly_solve(x^2=1,x);

(%o13) \%union([x = - 1], [x = 1])

(%i14) to_poly_solve(x^2=1,x, 'simpfuncs = [\text{polarform}]);

%pi
(%o14) \%union([x = 1], [x = \%e ^\frac{\text{\frac{\pi}{\text{\sqrt{2}}}}}{\text{\sqrt{2}}}], [x = \frac{\%i}{\sqrt{2}} + \frac{\%i}{\sqrt{2}}])

(%i15) to_poly_solve(x^2 = %i,x, 'simpfuncs = [\text{lambda([s],s^2})]);

(%o15) \%union([x = %i])

(%i16) to_poly_solve(x^3 +x + 1 = 0,x,
'simpfun = [\text{dfloat}]);

(%o16) \%union([x = - .6823278038280178],
x = .3411639019140089 - 1.16154139997251 \%i,
x = 1.16154139997251 \%i + .3411639019140089])

(%i17) to_poly_solve([x^2+y^2=2^2,(x-1)^2+(y-1)^2=2^2],[x,y],
 'use_grobner = true);

sqrt(7) - 1
(%o17) \%union([x = - \frac{\text{\frac{\sqrt{7}}{2} - 1}}{\text{\sqrt{7} + 1}}, y = \frac{\sqrt{2}}{2}])

(%o17) \%union([x = - \frac{\text{\frac{\sqrt{7} + 1}}{\text{\sqrt{7} + 1}}}, y = \frac{\sqrt{2}}{2}])
\end{verbatim}
\[
\begin{align*}
(\%i8) & \quad \text{to\_poly\_solve}([x^2+y^2=2^2,(x-1)^2+(y-1)^2=2^2],[x,y]); \\
(\%o8) & \quad \%\text{union}() \\
\end{align*}
\]

- **maxdepth** = \(k\), where \(k\) is a positive integer. This function controls the maximum recursion depth for the solver. The default value for **maxdepth** is five. When the recursions depth is exceeded, the solver signals an error:

\[
(\%i9) \quad \text{to\_poly\_solve}(%\cos(x)=x,x, \text{maxdepth}=2);
\]

Unable to solve

\[
(\%o9) \quad \%\text{solve([cos(x)=x], [x], \text{maxdepth}=2)}
\]

- **parameters** = \(l\), where \(l\) is a list of symbols. The solver attempts to return a solution that is valid for all members of the list \(l\); for example:

\[
(\%i10) \quad \text{to\_poly\_solve}(a \times x = x, x);
\]

\[
(\%o10) \quad \%\text{union([x = 0])}
\]

\[
(\%i11) \quad \text{to\_poly\_solve}(a \times x = x, x, \text{parameters = [a]});
\]

\[
(\%o11) \quad \%\text{union(\%if(a - 1 = 0, [x = %c111], \%union()),}
\]

\[
\quad \%\text{if(a - 1 \# 0, [x = 0], \%union()))}
\]

In \(\%o2\), the solver introduced a dummy variable; to re-index the these dummy variables, use the function **nicedummies**:

\[
(\%i12) \quad \text{nicedummies(\%)};
\]

\[
(\%o12) \quad \%\text{union(\%if(a - 1 = 0, [x = %c0], \%union()),}
\]

\[
\quad \%\text{if(a - 1 \# 0, [x = 0], \%union()))}
\]

The **to\_poly\_solve** uses data stored in the hashed array **one_to_one_reduce** to solve equations of the form \(f(a) = f(b)\). The assignment **one_to_one_reduce**['f','f'] : \(\lambda([a,b], a=b)\) tells **to\_poly\_solve** that the solution set of \(f(a) = f(b)\) equals the solution set of \(a = b\); for example

\[
(\%i13) \quad \text{one\_to\_one\_reduce['f','f'] : \lambda([a,b], a=b)}$
\]

\[
(\%i14) \quad \text{to\_poly\_solve}(f(x^2-1) = f(0),x);
\]

\[
(\%o14) \quad \%\text{union([x = - 1], [x = 1])}
\]

More generally, the assignment **one_to_one_reduce**['f','g'] : \(\lambda([a,b], w(a, b) = 0\) tells **to\_poly\_solve** that the solution set of \(f(a) = f(b)\) equals the solution set of \(w(a,b) = 0\); for example

\[
(\%i15) \quad \text{one\_to\_one\_reduce['f','g'] : \lambda([a,b], a = 1 + b/2)}$
\]

\[
(\%i16) \quad \text{to\_poly\_solve}(f(x) - g(x),x);
\]

\[
(\%o16) \quad \%\text{union([x = 2])}
\]

Additionally, the function **to\_poly\_solve** uses data stored in the hashed array **function_inverse** to solve equations of the form \(f(a) = b\). The assignment **function_inverse**['f'] : \(\lambda([s], g(s))\) informs **to\_poly\_solve** that the solution set to \(f(x) = b\) equals the solution set to \(x = g(b)\); two examples:

\[
(\%i17) \quad \text{function\_inverse['Q'] : \lambda([s], P(s))}$
(%i18) to_poly_solve(Q(x-1) = 2009,x);
(%o18) %union([x = P(2009) + 1])
(%i19) function_inverse['G] : lambda([s], s+new_variable(integer));
(%o19) lambda([s], s + new_variable(integer))
(%i20) to_poly_solve(G(x - a) = b,x);
(%o20) %union([x = b + a + %z125])

Notes
• The solve variables needn’t be symbols; when fullratsubst is able to appropriately make substitutions, the solve variables can be nonsymbols:
  (%i1) to_poly_solve([x^2 + y^2 + x * y = 5, x * y = 8],
                  [x^2 + y^2, x * y]);
  (%o1) %union([x y = 8, y + x = - 3])
• For equations that involve complex conjugates, the solver automatically appends the conjugate equations; for example
  (%i1) declare(x,complex)$
  (%i2) to_poly_solve(x + (5 + %i) * conjugate(x) = 1, x);
  (%o2) %union([x = - --------------------])
         25 %i - 125
  (%i3) declare(y,complex)$
  (%i4) to_poly_solve(set(conjugate(x) - y = 42 + %i,
                       x + conjugate(y) = 0), set(x,y));
  (%o4) %union([x = - -------, y = - -------])
         2 2
• For an equation that involves the absolute value function, the to_poly_solve consults the fact database to decide if the argument to the absolute value is complex valued. When
  (%i1) to_poly_solve(abs(x) = 6, x);
  (%o1) %union([x = - 6], [x = 6])
  (%i2) declare(z,complex)$
  (%i3) to_poly_solve(abs(z) = 6, z);
  (%o3) %union(%if((%c11 # 0) %and (%c11 conjugate(%c11) - 36 =
                       0), [z = %c11], %union()))

This is the only situation that the solver consults the fact database. If a solve variable is declared to be an integer, for example, to_poly_solve ignores this declaration.

Relevant option variables algexact, resultant, algebraic

Related functions to_poly

To use load(to_poly_solve)
**Status:** The function `to_poly_solve` is experimental; its specifications might change and its functionality might be merged into other Maxima functions.

\[
\%union (\text{soln}_1, \text{soln}_2, \text{soln}_3, \ldots) \quad \text{[Operator]}
\]

\[
\%union () \quad \text{[Operator]}
\]

\[\%union(\text{soln}_1, \text{soln}_2, \text{soln}_3, \ldots)\] represents the union of its arguments, each of which represents a solution set, as determined by `to_poly_solve`. `\%union()` represents the empty set.

In many cases, a solution is a list of equations \([x = \ldots, y = \ldots, z = \ldots]\) where \(x, y,\) and \(z\) are one or more unknowns. In such cases, `to_poly_solve` returns a `\%union` expression containing one or more such lists.

The solution set sometimes involves simplifying versions of various of logical operators including `\%and`, `\%or`, or `\%if` for conjunction, disjunction, and implication, respectively.

**Examples:**

\[\%union(\ldots)\] represents the union of its arguments, each of which represents a solution set, as determined by `to_poly_solve`. In many cases, a solution is a list of equations.

\[
\begin{align*}
\text{(\%i1)} & \text{ load ("to_poly_solve") } \\
\text{(\%i2)} & \text{ to_poly_solve ([sqrt(x^2 - y^2), x + y], [x, y]); } \\
\text{(\%o2)} & \text{ \%union([x = 0, y = 0], [x = \%c13, y = - \%c13])}
\end{align*}
\]

`\%union()` represents the empty set.

\[
\begin{align*}
\text{(\%i1)} & \text{ load ("to_poly_solve") } \\
\text{(\%i2)} & \text{ to_poly_solve (abs(x) = -1, x); } \\
\text{(\%o2)} & \text{ \%union()}
\end{align*}
\]

The solution set sometimes involves simplifying versions of various of logical operators.

\[
\begin{align*}
\text{(\%i1)} & \text{ load ("to_poly_solve") } \\
\text{(\%i2)} & \text{ sol : to_poly_solve (abs(x) = a, x); } \\
\text{(\%o2)} & \text{ \%union(\%if(isnonnegative_p(a), [x = - a], \%union()), } \\
& \qquad \text{\%if(isnonnegative_p(a), [x = a], \%union()))}
\end{align*}
\]

\[
\begin{align*}
\text{(\%i3)} & \text{ subst (a = 42, sol); } \\
\text{(\%o3)} & \text{ \%union([x = - 42], [x = 42])}
\end{align*}
\]

\[
\begin{align*}
\text{(\%i4)} & \text{ subst (a = -42, sol); } \\
\text{(\%o4)} & \text{ \%union()}
\end{align*}
\]
86 unit

86.1 Introduction to Units

The unit package enables the user to convert between arbitrary units and work with dimensions in equations. The functioning of this package is radically different from the original Maxima units package - whereas the original was a basic list of definitions, this package uses rulesets to allow the user to chose, on a per dimension basis, what unit final answers should be rendered in. It will separate units instead of intermixing them in the display, allowing the user to readily identify the units associated with a particular answer. It will allow a user to simplify an expression to its fundamental Base Units, as well as providing fine control over simplifying to derived units. Dimensional analysis is possible, and a variety of tools are available to manage conversion and simplification options. In addition to customizable automatic conversion, units also provides a traditional manual conversion option.

Note - when unit conversions are inexact Maxima will make approximations resulting in fractions. This is a consequence of the techniques used to simplify units. The messages warning of this type of substitution are disabled by default in the case of units (normally they are on) since this situation occurs frequently and the warnings clutter the output. (The existing state of ratprint is restored after unit conversions, so user changes to that setting will be preserved otherwise.) If the user needs this information for units, they can set unitverbose: on to reactivate the printing of warnings from the unit conversion process.

unit is included in Maxima in the share/contrib/unit directory. It obeys normal Maxima package loading conventions:

    (%i1) load("unit")$

Redefining necessary functions...
WARNING: DEFUN/DEFFUNCTION: redefining function TOLEVEL-MACSYMA-EVAL ...
WARNING: DEFUN/DEFFUNCTION: redefining function MSETCHK ...
WARNING: DEFUN/DEFFUNCTION: redefining function KILL1 ...
WARNING: DEFUN/DEFFUNCTION: redefining function NFORMAT ...
Initializing unit arrays...
Done.

The WARNING messages are expected and not a cause for concern - they indicate the unit package is redefining functions already defined in Maxima proper. This is necessary in order to properly handle units. The user should be aware that if other changes have been made to these functions by other packages those changes will be overwritten by this loading process.

The unit.mac file also loads a lisp file unit-functions.lisp which contains the lisp functions needed for the package.
Clifford Yapp is the primary author. He has received valuable assistance from Barton Willis of the University of Nebraska at Kearney (UNK), Robert Dodier, and other intrepid folk of the Maxima mailing list.

There are probably lots of bugs. Let me know. float and numer don’t do what is expected.

TODO : dimension functionality, handling of temperature, showabbr and friends. Show examples with addition of quantities containing units.

86.2 Functions and Variables for Units

setunits (list)  [Function]

By default, the unit package does not use any derived dimensions, but will convert all units to the seven fundamental dimensions using MKS units.

(%i2) N;
   kg m
    ----
      2
    s

(%i3) dyn;
   1 kg m
(------) (----)
100000 2
    s

(%i4) g;
   1 (kg)
(----) (kg)
1000

(%i5) centigram*inch/minutes^2;
   127 kg m
(-------------) (----)
1800000000000 2
    s

In some cases this is the desired behavior. If the user wishes to use other units, this is achieved with the setunits command:

(%i6) setunits([centigram,inch,minute]);
   done

(%i7) N;
   1800000000000 %in cg
(-------------) (----)
127 2
   %min

(%i8) dyn;
   18000000 %in cg
(----) (----)
127 2
   %min
The setting of units is quite flexible. For example, if we want to get back to kilograms, meters, and seconds as defaults for those dimensions we can do:

\%(i11) setunits([kg,m,s]);
\%(o11) done
\%(i12) centigram*inch/minutes^2;
\%(o12) 127 \frac{kg \cdot m}{1800000000000 \cdot s^2}

Derived units are also handled by this command:

\%(i17) setunits(N);
\%(o17) done
\%(i18) N;
\%(o18) N
\%(i19) dyn;
\%(o19) \frac{1}{100000} (N)
\%(i20) kg*m/s^2;
\%(o20) N
\%(i21) centigram*inch/minutes^2;
\%(o21) \frac{127}{1800000000000} (N)

Notice that the unit package recognized the non MKS combination of mass, length, and inverse time squared as a force, and converted it to Newtons. This is how Maxima works in general. If, for example, we prefer dyne to Newtons, we simply do the following:

\%(i22) setunits(dyn);
\%(o22) done
\%(i23) kg*m/s^2;
\%(o23) \frac{100000}{(N)}
\%(i24) centigram*inch/minutes^2;
\%(o24) \frac{127}{1800000000000} (dyn)

To discontinue simplifying to any force, we use the uforget command:

\%(i26) uforget(dyn);
\%(o26) false
(%i27) kg*m/s^2;

(%o27) kg m

2
s

(%i28) centigram*inch/minutes^2;

127 kg m

1800000000000 2
s

This would have worked equally well with uforget(N) or uforget(%force).
See also uforget. To use this function write first load("unit").

uforget (list) [Function]
By default, the unit package converts all units to the seven fundamental dimensions using MKS units. This behavior can be changed with the setunits command. After that, the user can restore the default behavior for a particular dimension by means of the uforget command:

(%i13) setunits([centigram,inch,minute]);
(%o13) done
(%i14) centigram*inch/minutes^2;

127 kg m

1800000000000 2
s

uforget operates on dimensions, not units, so any unit of a particular dimension will work. The dimension itself is also a legal argument.

See also setunits. To use this function write first load("unit").

convert (expr, list) [Function]
When resetting the global environment is overkill, there is the convert command, which allows one time conversions. It can accept either a single argument or a list of units to use in conversion. When a convert operation is done, the normal global evaluation system is bypassed, in order to avoid the desired result being converted again. As a consequence, for inexact calculations "rat" warnings will be visible if the global environment controlling this behavior (ratprint) is true. This is also useful for spot-checking the accuracy of a global conversion. Another feature is convert will allow a user to do Base Dimension conversions even if the global environment is set to simplify to a Derived Dimension.
(%i2) kg*m/s^2;

(%o2) \frac{kg \ m}{2 \ s^2}

(%i3) convert(kg*m/s^2,[g,km,s]);

(%o3) \frac{g \ km}{2 \ s}

(%i4) convert(kg*m/s^2,[g,inch,minute]);

'rat' replaced 39.37007874015748 by 5000/127 = 39.37007874015748

(%o4) \frac{18000000000 \text{g in}}{127 \text{ s}^2 \text{ min}}

(%i5) convert(kg*m/s^2,[N]);

(%o5) N

(%i6) convert(kg*m^2/s^2,[N]);

(%o6) m \ N

(%i7) setunits([N,J]);

(%o7) \text{done}

(%i8) convert(kg*m^2/s^2,[N]);

(%o8) m \ N

(%i9) convert(kg*m^2/s^2,[N,inch]);

'rat' replaced 39.37007874015748 by 5000/127 = 39.37007874015748

(%o9) \frac{5000 \text{in N}}{127 \text{ s}^2 \text{ min}}

(%i10) convert(kg*m^2/s^2,[J]);

(%o10) J

(%i11) kg*m^2/s^2;

(%o11) J

(%i12) setunits([g,inch,s]);

(%o12) \text{done}

(%i13) kg*m/s^2;

(%o13) N

(%i14) uforget(N);

(%o14) false

(%i15) kg*m/s^2;

(%o15) \frac{5000000 \text{g in}}{127 \text{ s}^2}
(%i16) convert(kg*m/s^2, [g, inch, s]);

'rat' replaced 39.37007874015748 by 5000/127 = 39.37007874015748

5000000 %in g

(%o16) (-------) (-----)
     127    2

s

See also setunits and uforget. To use this function write first load("unit").

usersetunits

[Optional variable]

Default value: none

If a user wishes to have a default unit behavior other than that described, they can make use of maxima-init.mac and the usersetunits variable. The unit package will check on startup to see if this variable has been assigned a list. If it has, it will use setunits on that list and take the units from that list to be defaults. uforget will revert to the behavior defined by usersetunits over its own defaults. For example, if we have a maxima-init.mac file containing:

usersetunits : [N,J];

we would see the following behavior:

(%i1) load("unit")$

********************************************************************************
* Units version 0.50 *
* Definitions based on the NIST Reference on *
* Constants, Units, and Uncertainty *
* Conversion factors from various sources including *
* NIST and the GNU units package *
********************************************************************************

Redefining necessary functions...
WARNING: DEFUN/DEFMACRO: redefining function
   TOPLEVEL-MACSYMA-EVAL ...
WARNING: DEFUN/DEFMACRO: redefining function MSETCHK ...
WARNING: DEFUN/DEFMACRO: redefining function KILL1 ...
WARNING: DEFUN/DEFMACRO: redefining function NFORMAT ...
Initializing unit arrays...
Done.
User defaults found...
User defaults initialized.

(%i2) kg*m/s^2;
(%o2) N

(%i3) kg*m^2/s^2;
(%o3) J

(%i4) kg*m^3/s^2;
(%o4) J m

(%i5) kg*m*km/s^2;
(%o5) 1000 (J)
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(%i6) setunits([dyn,eV]); done
(%i7) kg*ms^-2;
(%o7) (100000) (dyn)
(%i8) kg*ms^-2/s^-2;
(%o8) (6241509596477042688) (eV)
(%i9) kg*ms^-3/s^-2;
(%o9) (6241509596477042688) (eV m)
(%i10) kg*ms*km/s^-2;
(%o10) (6241509596477042688000) (eV)
(%i11) uforget([dyn,eV]); [false, false]
(%i12) kg*ms^-2;
(%o12) N
(%i13) kg*ms^-2/s^-2;
(%o13) J
(%i14) kg*ms^-3/s^-2;
(%o14) J m
(%i15) kg*ms*km/s^-2;
(%o15) (1000) (J)

Without usersetunits, the initial inputs would have been converted to MKS, and uforget would have resulted in a return to MKS rules. Instead, the user preferences are respected in both cases. Notice these can still be overridden if desired. To completely eliminate this simplification - i.e. to have the user defaults reset to factory defaults - the dontusedimension command can be used. uforget can restore user settings again, but only if usedimension frees it for use. Alternately, kill(usersetunits) will completely remove all knowledge of the user defaults from the session. Here are some examples of how these various options work.

(%i12) kg*ms^-2;
(%o12) N
(%i13) kg*ms^-2/s^-2;
(%o13) J
(%i14) setunits([dyn,eV]); done
(%i15) kg*ms^-2;
(%o15) (100000) (dyn)
(%i16) kg*ms^-2/s^-2;
(%o16) (6241509596477042688) (eV)
(%i17) uforget([dyn,eV]); [false, false]
(%i18) kg*ms^-2;
(%o18) N
(%i19) kg*ms^-2/s^-2;
(%o19) J
(%i20) dontusedimension(N); [%force]
(%i11) dontusedimension(J);
(%o11) [%energy, %force]
(%i12) kg*m/s^2;
(%o12) \frac{\text{kg m}}{\text{s}^2}
(%i13) kg*m^2/s^2;
(%o13) \frac{\text{kg m}^2}{\text{s}^2}
(%i14) setunits([dyn,eV]);
(%o14) done
(%i15) kg*m/s^2;
(%o15) \frac{\text{kg m}}{\text{s}^2}
(%i16) kg*m^2/s^2;
(%o16) \frac{\text{kg m}^2}{\text{s}^2}
(%i17) uforget([dyn,eV]);
(%o17) [false, false]
(%i18) kg*m/s^2;
(%o18) \frac{\text{kg m}}{\text{s}^2}
(%i19) kg*m^2/s^2;
(%o19) \frac{\text{kg m}^2}{\text{s}^2}
(%i20) usedimension(N);
Done. To have Maxima simplify to this dimension, use setunits([unit]) to select a unit.
(%o20) true
(%i21) usedimension(J);
Done. To have Maxima simplify to this dimension, use setunits([unit]) to select a unit.
(%o21) true
Chapter 86: unit

Unfortunately this wide variety of options is a little confusing at first, but once the user grows used to them they should find they have very full control over their working environment.

**metricexpandall** (x)  
Rebuilds global unit lists automatically creating all desired metric units. x is a numerical argument which is used to specify how many metric prefixes the user wishes defined. The arguments are as follows, with each higher number defining all lower numbers' units:
0 - none. Only base units
1 - kilo, centi, milli
(default) 2 - giga, mega, kilo, hecto, deka, deci, centi, milli, micro, nano
3 - peta, tera, giga, mega, kilo, hecto, deka, deci, centi, milli, micro, nano, pico, femto
4 - all

Normally, Maxima will not define the full expansion since this results in a very large number of units, but `metricexpandall` can be used to rebuild the list in a more or less complete fashion. The relevant variable in the `unit.mac` file is `%unitexpand`.

`%unitexpand` [Variable]
Default value: 2

This is the value supplied to `metricexpandall` during the initial loading of `unit`. 
87 zeilberger

87.1 Introduction to zeilberger

zeilberger is a implementation of Zeilberger’s algorithm for definite hypergeometric summation, and also Gosper’s algorithm for indefinite hypergeometric summation.

zeilberger makes use of the "filtering" optimization method developed by Axel Riese.

zeilberger was developed by Fabrizio Caruso.

load ("zeilberger") loads this package.

87.1.1 The indefinite summation problem

zeilberger implements Gosper’s algorithm for indefinite hypergeometric summation. Given a hypergeometric term $F_k$ in $k$ we want to find its hypergeometric anti-difference, that is, a hypergeometric term $f_k$ such that

$$F_k = f_{k+1} - f_k.$$ 

87.1.2 The definite summation problem

zeilberger implements Zeilberger’s algorithm for definite hypergeometric summation. Given a proper hypergeometric term (in $n$ and $k$) $F_{n,k}$ and a positive integer $d$ we want to find a $d$-th order linear recurrence with polynomial coefficients (in $n$) for $F_{n,k}$ and a rational function $R$ in $n$ and $k$ such that

$$a_0 F_{n,k} + \ldots + a_d F_{n+d}, \quad k = \Delta_K (R(n,k) F_{n,k}),$$

where $\Delta_k$ is the $k$-forward difference operator, i.e., $\Delta_k(t_k) \equiv t_{k+1} - t_k$.

87.1.3 Verbosity levels

There are also verbose versions of the commands which are called by adding one of the following prefixes:

Summary Just a summary at the end is shown

Verbose Some information in the intermediate steps

VeryVerbose More information

Extra Even more information including information on the linear system in Zeilberger’s algorithm

For example:

87.2 Functions and Variables for zeilberger

**AntiDifference \((F_k, k)\)**

Returns the hypergeometric anti-difference of \(F_k\), if it exists.
Otherwise \texttt{AntiDifference} returns \texttt{no_hyp_antidifference}.

**Gosper \((F_k, k)\)**

Returns the rational certificate \(R(k)\) for \(F_k\), that is, a rational function such that 
\[F_k = R(k+1) F_{k+1} - R(k) F_k\]
if it exists. Otherwise, \texttt{Gosper} returns \texttt{no_hyp_sol}.

**GosperSum \((F_k, k, a, b)\)**

Returns the summation of \(F_k\) from \(k = a\) to \(k = b\) if \(F_k\) has a hypergeometric anti-difference. Otherwise, \texttt{GosperSum} returns \texttt{nongosper_summable}.

Examples:

\begin{verbatim}
(%i1) load ("zeilberger")$
(%i2) GosperSum ((-1)^k*k / (4*k^2 - 1), k, 1, n);
Dependent equations eliminated: (1)
    3 n + 1
(n + -) (- 1)
    2 1
- ------------------ - -
    2 4
2 (4 (n + 1) - 1)
(%o2) - ------------------ - -
    2 4
2 (4 (n + 1) - 1)
(%i3) GosperSum (1 / (4*k^2 - 1), k, 1, n);
    2 1
- n - -
    2 4
4 (n + 1) - 1
(%o3) -------------- + -
    2 2
4 (n + 1) - 1
(%i4) GosperSum (x^k, k, 1, n);
    x
n + 1
------ - ----- -
x - 1 x - 1
(%o4) ------ - ----- -
x - 1 x - 1
(%i5) GosperSum ((-1)^k*a! / (k!*(a - k)!), k, 1, n);
a! (n + 1) (- 1)
----------- - -
a (- n + a - 1)! (n + 1)! a (a - 1)!
(%o5) - ------------------ - ------------------ -
a (- n + a - 1)! (n + 1)! a (a - 1)!
(%i6) GosperSum (k*k!, k, 1, n);
Dependent equations eliminated: (1)
(%o6) (n + 1)! - 1
(%i7) GosperSum (((k + 1)*k! / (k + 1)!, k, 1, n);
(n + 1) (n + 2) (n + 1)!
----------- - 1
(n + 2)!
(%o7) ------------------ - 1
(n + 2)!
\end{verbatim}
(\%i8) GosperSum (1 / ((a - k)!*k!)), k, 1, n);
(\%o8) NON_GOSPER_SUMMABLE

\textbf{parGosper} \((F_{n,k}, k, n, d)\) \hspace{1em} [Function]
Attempts to find a \(d\)-th order recurrence for \(F_{n,k}\).
The algorithm yields a sequence \([s_1, s_2, ..., s_m]\) of solutions. Each solution has the form

\[ [R(n,k), [a_0, a_1, \ldots, a_d]] \]

\texttt{parGosper} returns \([]\) if it fails to find a recurrence.

\textbf{Zeilberger} \((F_{n,k}, k, n)\) \hspace{1em} [Function]
Attempts to compute the indefinite hypergeometric summation of \(F_{n,k}\).
\texttt{Zeilberger} first invokes \texttt{Gosper}, and if that fails to find a solution, then invokes \texttt{parGosper} with order 1, 2, 3, ..., up to \texttt{MAX_ORD}. If Zeilberger finds a solution before reaching \texttt{MAX_ORD}, it stops and returns the solution.
The algorithm yields a sequence \([s_1, s_2, ..., s_m]\) of solutions. Each solution has the form

\[ [R(n,k), [a_0, a_1, \ldots, a_d]] \]

\texttt{Zeilberger} returns \([]\) if it fails to find a solution.
\texttt{Zeilberger} invokes \texttt{Gosper} only if \texttt{Gosper_in_Zeilberger} is \texttt{true}.

\section*{87.3 General global variables}

\textbf{MAX_ORD} \hspace{1em} [Global variable]
Default value: 5
\texttt{MAX_ORD} is the maximum recurrence order attempted by \texttt{Zeilberger}.

\textbf{simplified_output} \hspace{1em} [Global variable]
Default value: \texttt{false}
When \texttt{simplified_output} is \texttt{true}, functions in the \texttt{zeilberger} package attempt further simplification of the solution.

\textbf{linear_solver} \hspace{1em} [Global variable]
Default value: \texttt{linsolve}
\texttt{linear_solver} names the solver which is used to solve the system of equations in Zeilberger’s algorithm.

\textbf{warnings} \hspace{1em} [Global variable]
Default value: \texttt{true}
When \texttt{warnings} is \texttt{true}, functions in the \texttt{zeilberger} package print warning messages during execution.
Gosper_in_Zeilberger

[Global variable]

Default value: true

When Gosper_in_Zeilberger is true, the Zeilberger function calls Gosper before calling parGosper. Otherwise, Zeilberger goes immediately to parGosper.

trivial_solutions

[Global variable]

Default value: true

When trivial_solutions is true, Zeilberger returns solutions which have certificate equal to zero, or all coefficients equal to zero.

87.4 Variables related to the modular test

mod_test

[Global variable]

Default value: false

When mod_test is true, parGosper executes a modular test for discarding systems with no solutions.

modular_linear_solver

[Global variable]

Default value: linsolve

modular_linear_solver names the linear solver used by the modular test in parGosper.

ev_point

[Global variable]

Default value: big_primes[10]

ev_point is the value at which the variable n is evaluated when executing the modular test in parGosper.

mod_big_prime

[Global variable]

Default value: big_primes[1]

mod_big_prime is the modulus used by the modular test in parGosper.

mod_threshold

[Global variable]

Default value: 4

mod_threshold is the greatest order for which the modular test in parGosper is attempted.
88 Error and warning messages

This chapter provides detailed information about the meaning of some error messages or on how to recover from errors.

88.1 Error messages

88.1.1 apply: no such "list" element

One common cause for this error message is that square brackets operator ([ ]) was used trying to access a list element that whose element number was < 1 or > length(list).

88.1.2 argument must be a non-atomic expression

This normally means that a list, a set or something else that consists of more than one element was expected. One possible cause for this error message is a construct of the following type:

```
(%i1) l: [1,2,3];
(%o1) [1, 2, 3]
(%%i2) append(l,4);
append: argument must be a non-atomic expression; found 4
   -- an error. To debug this try: debugmode(true);
```

The correct way to append variables or numbers to a list is to wrap them in a single-element list first:

```
(%i1) l: [1,2,3];
(%o1) [1, 2, 3]
(%%i2) append(l,[4]);
(%%o2) [1, 2, 3, 4]
```

88.1.3 assignment: cannot assign to <function name>

Maxima supports several assignment operators. When trying to define a function := has to be used.

88.1.4 expt: undefined: 0 to a negative exponent.

This message notifies about a classical division by zero error.

88.1.5 incorrect syntax: , is not a prefix operator

This might be caused by a command starting with a comma (,) or by one comma being directly followed by another one.

88.1.6 incorrect syntax: Illegal use of delimiter )

Common reasons for this error appearing are a closing parenthesis without an opening one or a closing parenthesis directly preceded by a comma.
88.1.7 loadfile: failed to load <filename>
This error message normally indicates that the file exists, but can not be read. If the file is present and readable there is another possible for this error message: Maxima can compile packages to native binary files in order to make them run faster. If after compiling the file something in the system has changed in a way that makes it incompatible with the binary the file cannot be loaded any more. Maxima normally puts binary files it creates from its own packages in a folder named binary within the folder whose name it is printed after typing:

```
(%i1) maxima_userdir;
(%o1) /home/gunter/.maxima
```
If this directory is missing maxima will recreate it again as soon as it has to compile a package.

88.1.8 makelist: second argument must evaluate to a number
makelist expects the second argument to be the name of the variable whose value is to be stepped. This time instead of the name of a still-undefined variable maxima has found something else, possibly a list or the name of a list.

88.1.9 Only symbols can be bound
The most probable cause for this error is that there was an attempt to either use a number or a variable whose numerical value is known as a loop counter.

88.1.10 operators of arguments must all be the same
One possible reason for this error message to appear is a try to use append in order to add an equation to a list:

```
(%i1) l: [a=1,b=2,c=3];
(%o1) [a = 1, b = 2, c = 3]
(%i2) append(l,d=5);
    append: operators of arguments must all be the same.
```
-- an error. To debug this try: debugmode(true);
In order to add an equation to a list it has to be wrapped in a single-element list first:

```
(%i1) l: [a=1,b=2,c=3];
(%o1) [a = 1, b = 2, c = 3]
(%i2) append(l,[d=5]);
(%o2) [a = 1, b = 2, c = 3, d = 5]
```

88.1.11 Out of memory
Lisp typically handles several types of memory containing at least one stack and a heap that contains user objects. To avoid running out of memory several approaches might be useful:

- If possible, the best solution normally is to use an algorithm that is more memory-efficient.
- Compiling a function might drastically reduce the amount of memory it needs.
- Arrays of a fixed type might be more memory-efficient than lists.
• If maxima is run by sbcl sbcl’s memory limit might be set to a value that is too low to solve the current problem. In this case the command-line option `--dynamic-space-size <n>` allows to tell sbcl to reserve n megabytes for the heap. It is to note, though, that sbcl has to handle several distinct types of memory and therefore might be able to only reserve about half of the available physical memory. Also note that 32-bit processes might only be able to access 2GB of physical memory.

88.1.12 part: fell off the end

`part()` was used to access the nth item in something that has less than n items.

88.1.13 undefined variable (draw or plot)

A function could not be plotted since it still contained a variable maxima doesn’t know the value of.

In order to find out which variable this could be it is sometimes helpful to temporarily replace the name of the drawing command (draw2d, plot2d or similar) by a random name (for example `ddraw2d`) that doesn’t coincide with the name of an existing function to make maxima print out what parameters the drawing command sees.

```
(%i1) load("draw")$
(%i2) f(x):=sin(omega*t);
(%o2) f(x) := sin(omega t)
(%i3) draw2d(
    explicit(
        f(x),
        x,1,10
    )
);
```

`draw2d (explicit): non defined variable`  
-- an error. To debug this try: `debugmode(true)`;

```
(%i4) ddraw2d(
    explicit(
        f(x),
        x,1,10
    )
);
```

```
(%o4) ddraw2d(explicit(sin(omega t), x, 1, 10))
```

88.1.14 VTK is not installed, which is required for Scene

This might either mean that VTK is actually not installed - or cannot be found by maxima - or that maxima has no write access to the directory whose name is output if the following maxima command is entered:

```
(%i1) maxima_tempdir;
(%o1) /tmp
```

88.2 Warning messages
88.2.1 Encountered undefined variable \texttt{x} in translation

A function was compiled but the type of the variable \texttt{x} was not known. This means that the compiled command contains additional code that makes it retain all the flexibility maxima provides in respect to this variable. If \texttt{x} isn’t meant as a variable name but just a named option to a command prepending the named option by a single quote (’) should resolve this issue.

88.2.2 Rat: replaced \texttt{x} by \texttt{y} = \texttt{z}

Floating-point numbers provide a maximum number of digits that is typically high, but still limited. Good examples that this limitation might be too low even for harmless-looking examples include Wilkinson’s Polynomial (\url{https://en.wikipedia.org/wiki/Wilkinson%27s_polynomial}), The Rump polynomial and the fact that an exact 1/10 cannot be expressed as a binary floating-point number. In places where the floating-point error might add up or hinder terms from cancelling each other out maxima therefore by default replaces them with exact fractions. See also \texttt{ratprint}, \texttt{ratepsilon}, \texttt{bftorat}, \texttt{fpprintprec} and \texttt{rationalize}.
Appendix A Function and Variable Index

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